Pseudo-metrizability of quotient spaces *

by

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1. Introduction. In [1] we obtained a necessary and sufficient condition that the image Y of a pseudo-metrizable space X under a quotient map f be pseudo-metrizable. (See Theorem 2, below.) In this note the principal result is the following.

THEOREM 1. *Let f be a quotient map from a pseudo-metrizable space X onto a space Y. Then Y is pseudo-metrizable if and only if there exists a pseudo-metric space (Z, δ) containing X as a subspace and a continuous extension $g\colon Z \to Y$ of f such that the function ϱ defined by $\varrho(y, w) = \delta(g^{-1}[y], g^{-1}[w])$, for $y, w \in Y$, is a pseudo-metric compatible with the topology of Y. Moroever, it may be required that the space Z belong to the smallest class of topological spaces which contains X and is closed under the formation of one point adjunctions* (1) *and (finite or infinite) topological sums.*

We note that, as a continuous extension of a quotient map onto Y, g must also be a quotient map onto Y.

As a by-product of our results we obtain a proof of Theorem 2 without having to use quotient uniformities (as was necessary in [1]). In Section 3 we obtain some results on the pseudo-metrizability of orbit spaces.

The notation in this paper is in most cases the same as in [1]. In particular, if (W, d) is a pseudo-metric space and if A, B are subsets of W, then $d(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}$; if $A \subset W$ and $\varepsilon > 0$, then $N_\varepsilon[A] = \{w \in W \mid d(w, A) < \varepsilon\}$. The word *map* means continuous function. A map $f\colon X \to Y$ is a quotient map if and only if Y has the quotient topology relative to f.

2. Proof of Theorem 1. In [1] we proved the following

THEOREM 2. *Let f be a quotient map from a pseudo-metrizable space X onto a space Y. Then Y is pseudo-metrizable if and only if the topology on X can be defined by a pseudo-metric d such that X, d, Y, f satisfy*

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(1) If X_1, X_2 are topological spaces, and if $x_1 \in X_1$, $x_2 \in X_2$, then the space formed by identifying x_1 and x_2 in the topological sum of X_1 and X_2 is called a *one point adjunction* of X_1 and X_2.

(C) *for each member G of some base of Y there exists a family $\{\varepsilon(y)|\ y \in G\}$ of positive real numbers such that*

(i) $$N_{\varepsilon(y)}[f^{-1}[y]] \subset f^{-1}[G] \quad \text{if} \quad y \in G\,,$$

(ii) $$d(f^{-1}[y], f^{-1}[z]) \geqslant \varepsilon(y) - \varepsilon(z) \quad \text{if} \quad y, z \in G\,.$$

Theorem 1 follows immediately from Theorem 2 and the following:

THEOREM 3. *Let f be a quotient map from a pseudo-metric space (X, d) onto a space Y such that X, d, Y, f satisfy* (C). *Then there exists a pseudo-metric space (Z, δ) containing X as a subspace (isometrically if (X, d) is bounded or if δ is allowed to take the value ∞), and there exists a continuous extension $g\colon Z \to Y$ of f such that the function ϱ defined by $\varrho(y, w) = \delta(g^{-1}[y], g^{-1}[w])$, for $y, w \in Y$, is a pseudo-metric compatible with the topology of Y. It may be required that the space Z belong to the smallest class of topological spaces which contains X and is closed under the formation of one point adjunctions and (finite or infinite) topological sums.*

We remarked that the proof in [1] of the "if" part of Theorem 2 (of this paper) leaned heavily on some facts about quotient uniformities. However, this part of Theorem 2 follows trivially from Theorem 3, and, as will be seen, the proof of Theorem 3 makes no use of uniformities.

Proof. Our argument allows pseudo-metrics to take the value ∞. The theorem is true as stated for finite valued pseudo-metrics, however, since any pseudo-metric d for X such that X, d, Y, f satisfy (C) can clearly be replaced by a bounded pseudo-metric d' such that X, d', Y, f satisfy (C). Moreover, if (X, d) is of diameter r, and if (Z, δ) is as in the theorem (with δ possibly taking the value ∞), then replace δ by $\delta' = \min\{d, r\}$. It follows easily that (Z, δ') contains (X, d) isometrically, and that $\varrho'(y, w) = \delta'(g^{-1}[y], g^{-1}[w])$ defines a finite valued pseudo-metric compatible with the topology of Y.

We now proceed with the proof. Z, δ and g are constructed in precisely the same way that X_∞, d_∞ and f_∞ were constructed in [1]. However, for the sake of completeness, we repeat that construction here.

For each ordered pair (u, v) of points of X such that $f(u) = f(v)$, let φ be a 1-1 isometry from X onto a space $\varphi[X]$ such that the intersection of $\varphi[X]$ with X is v and such that $\varphi(u) = v$. Let Φ be the set of isometries chosen in this way. For each $X \cup \varphi[X]$ define a pseudo-metric d_φ by

$$d_\varphi(x, y) = \begin{cases} d(x, y) & \text{if} \quad x, y \in X\,, \\ d(\varphi^{-1}[x], \varphi^{-1}[y]) & \text{if} \quad x, y \in \varphi[X]\,, \\ d(x, v) + d(u, \varphi^{-1}[y]) & \text{if} \quad x \in X,\ y \in \varphi[X]\,, \\ d(\varphi^{-1}[x], u) + d(v, y) & \text{if} \quad x \in \varphi[X],\ y \in X\,, \end{cases}$$

and extend f to the map $f_\varphi\colon X \cup \varphi[X] \to Y$ defined by

$$f_\varphi(x) = \begin{cases} f(x) & \text{if} \quad x \in X\,, \\ f \circ \varphi^{-1}[x] & \text{if} \quad x \in \varphi(X)\,. \end{cases}$$

Next regard any two of the spaces $X \cup \varphi[X]$, $X \cup \psi[X]$, with φ, ψ distinct members of Φ, as disjoint, and let

$$X_1 = \bigcup \{X \cup \varphi[X]|\ \varphi \in \Phi\}\,.$$

Define a pseudo-metric d_1 for X_1 by

$$d_1(x, y) = \begin{cases} d_\varphi(x, y) & \text{if} \quad x, y \in X \cup \varphi(X)\,,\ \varphi \in \Phi\,, \\ \infty & \text{otherwise} \end{cases}$$

and extend all f_φ's simultaneously to the function

$$f_1 = \bigcup \{f_\varphi|\ \varphi \in \Phi\}\colon X_1 \to Y\,.$$

We regard X_1 as disjoint from X, although it may help to keep in mind that X_1 contains Φ copies of X, on each of which f_1, d_1 behave exactly like f, d, respectively.

Obtain a sequence of mutually disjoint pseudo-metric spaces $(X, d) = (X_0, d_0), (X_1, d_1), \ldots, (X_n, d_n), \ldots$ and maps $f_0 = f\colon X \to Y, f_1\colon X_1 \to Y, \ldots, f_n\colon X_n \to Y, \ldots$ by constructing Φ_n, X_{n+1}, d_{n+1}, f_{n+1}, from X_n, d_n, f_n in the same way that Φ, X_1, d_1, f_1 were constructed from X, d, f. If $m < n$, then X_n contains $\Phi_m \times \ldots \times \Phi_{n-1}$ copies of X_m, on each of which f^n, d_n behave exactly like f_m, d_m, respectively. Now define

$$Z = \bigcup \{X_n|\ n = 0, 1, \ldots\}\,, \qquad g = \bigcup \{f_n|\ n = 0, 1, \ldots\}\,,$$

$$\delta(x, y) = \begin{cases} d_n(x, y) & \text{if} \quad x, y \in X_n\,, \\ \infty & \text{if} \quad x \in X_m,\ y \in X_n,\ m \neq n\,. \end{cases}$$

It is trivial that g is a quotient map. It now remains to prove that ϱ, as defined in the statement of the theorem, is a pseudo-metric compatible with the topology of Y.

To prove that ϱ is a pseudo-metric, it is sufficient to check the triangle inequality. So let $y_1, y_2, y_3 \in Y$, and let $\varepsilon > 0$. Then there exist $x_1 \in g^{-1}[y_1]$, $x_2, x_2' \in g^{-1}[y_2]$, and $x_3 \in g^{-1}[y_3]$ such that

$$\delta(x_1, x_2) \leqslant \delta(g^{-1}[y_1], g^{-1}[y_2]) + \varepsilon\,,$$

and

$$\delta(x_2', x_3) \leqslant \delta(g^{-1}[y_2], g^{-1}[y_3]) + \varepsilon\,.$$

If either $\varrho(y_1, y_2) = \infty$ or $\varrho(y_2, y_3) = \infty$, then trivially $\varrho(y_1, y_3) \leqslant \varrho(y_1, y_2) + \varrho(y_2, y_3)$. So we may assume that $\delta(x_1, x_2)$, $\delta(x_2', x_3)$ are finite, and that there exist integers m and n such that $x_1, x_2 \in X_m$, and $x_2', x_3 \in X_n$. If $m < n$, then there exist copies of x_1, x_2 in X_n which are also in $g^{-1}[y_1]$,

$g^{-1}[y_2]$, respectively, and which are such that the distance between the copies is also $\delta(x_1, x_2)$. Argue similarly with x_2' and x_3 if $m > n$. Hence we may assume $x_1, x_2, x_2', x_3 \in X_n$. Let $\varphi \in \Phi_n$ be the isometry of X_n onto $\varphi(X_n)$ such that $\varphi(x_2) = x_2'$. Then $\varphi(x_1) \in g^{-1}[y_1]$, and now regarding x_1, x_2, x_2', x_3 as points of $X_n \cup \varphi(X_n) \subset X_{n+1}$, we have

$$\begin{aligned}\varrho(y_1, y_3) &= \delta(g^{-1}[y_1], g^{-1}[y_3]) \\ &\leqslant \delta(\varphi(x_1), x_3) \\ &= \delta(\varphi^{-1}[\varphi(x_1)], x_2) + \delta(x_2', x_3) \\ &= \delta(x_1, x_2) + \delta(x_2', x_3) \\ &\leqslant \delta(g^{-1}[y_1], g^{-1}[y_2]) + \delta(g^{-1}[y_2], g^{-1}[y_3]) + 2\varepsilon \\ &= \varrho(y_1, y_2) + \varrho(y_2, y_3) + 2\varepsilon .\end{aligned}$$

But ε was arbitrary. So $\varrho(y_1, y_3) \leqslant \varrho(y_1, y_2) + \varrho(y_2, y_3)$.

The continuity of g: $(Z, \delta) \to (Y, \varrho)$ is a trivial consequence of the fact that g is a distance depressing function. We conclude the proof by showing that the topology defined by ϱ is larger than the given quotient topology on Y. Let G be an open set in the base for Y prescribed by (C), and let $y \in G$. By proposition 5 of [1], there exists $\varepsilon > 0$ such that $N_\varepsilon[g^{-1}[y]] \subset g^{-1}[G]$. It follows that the ϱ-sphere about y of radius ε is contained in G. For suppose $\varrho(w, y) < \varepsilon$. Then $\delta(g^{-1}[w], g^{-1}[y]) < \varepsilon$ and there exists $z \in g^{-1}[w]$ such that $\delta(z, g^{-1}[y]) < \varepsilon$. Hence $z \in g^{-1}[G]$ and $w \in G$.

This concludes the proof of Theorem 3.

There are cases in which the space (Z, δ) of Theorem 3 can be chosen equal to (X, d). This may be done, for example, if X is a topological group with a right invariant pseudo-metric d, Y is the quotient space X/H of left cosets relative to a subgroup H of X, and f: $X \to Y$ is the natural projection. (For a proof see [2], page 36, or Theorem 4 below.) We extend this example somewhat in Theorem 4 in the next section.

3. Pseudo-metrizability of orbit spaces. Let (X, d) be a pseudo-metric space and let Π be a group of homeomorphisms of X. Let $\Pi(x) = \{\varphi(x) |\ \varphi \in \Pi\}$, for each $x \in X$. Then the collection $\{\Pi(x) |\ x \in X\}$ is a decomposition of X which we denote by X/Π and which we give the quotient topology relative to the natural projection f taking x to $\Pi(x)$ if $x \in X$. X/Π with this topology is called the *orbit space of X relative to Π*. For example, if X is a topological group, and Π a group of right translations determined by a subgroup H of X, then the space X/Π is the space X/H of left cosets. It is natural to ask when X/Π is pseudo-metrizable. The next two theorems give a partial answer.

THEOREM 4. *Let Π be a group of isometries of a pseudo-metric space (X, d). Then the formula $\varrho(A, B) = d(A, B)$, for $A, B \in X/\Pi$, defines a pseudo-metric compatible with the topology of the orbit space X/Π.*

Proof. The proof is virtually the same as the proof of the corresponding theorem for topological groups. We first show that ϱ satisfies the triangle inequality. Let $A, B, C \in X/\Pi$, and let $a \in A$, $b, b' \in B$, and $c \in C$ be such that $d(a, b) < \varrho(A, B) + \varepsilon/2$ and $d(b', c) < \varrho(B, C) + \varepsilon/2$. Choose an isometry $\varphi \in \Pi$ such that $\varphi(b') = b$. Then $\varphi(c) \in C$,

$$d(b, \varphi(c)) = d(b', c) < \varrho(B, C) + \varepsilon/2 ,$$

and

$$\varrho(A, C) \leqslant d(a, \varphi(c)) < \varrho(A, B) + \varrho(B, C) + \varepsilon .$$

Since ε is arbitrary, the triangle inequality follows.

The topology on X/Π defined by ϱ makes the natural projection f: $X \to X/\Pi$ a distance depressing, and therefore continuous, function. So there remains only to show that the topology defined by ϱ is larger than the quotient topology. Let Γ be open in X/Π, $a \in A \in \Gamma$ and $\varepsilon = d(a, X - \bigcup \Gamma)$. Then $\varepsilon > 0$, and the ϱ-sphere of radius ε about the *point* A is contained in Γ. To prove the latter statement it is clearly sufficient to prove that the d-neighborhood $N_\varepsilon[A]$ of the *set* A is contained in $\bigcup \Gamma$. So let $b \in A$, $x \in X - \bigcup \Gamma$, and let $\varphi \in \Pi$ be such that $\varphi(b) = a$. Then $d(b, x) = d(a, \varphi(x)) \geqslant d(a, X - \bigcup \Gamma) = \varepsilon$. It follows that $d(A, X - \bigcup \Gamma) \geqslant \varepsilon$, and hence that $N_\varepsilon[A] \subset \bigcup \Gamma$.

In the next theorem we obtain a slightly sharper sufficient condition for the pseudo-metrizability of X/Π. However, it will no longer be true that we can choose $(Z, \delta) = (X, d)$.

THEOREM 5. *Let Π be a group of homeomorphisms of a pseudo-metric space (X, d), and suppose that there exists $K > 0$ such that $d(\varphi(x), \varphi(y)) \leqslant K d(x, y)$ for all $\varphi \in \Pi$ and all $x, y \in X$. Then X/Π is pseudo-metrizable.*

Remark. Since Π is a group, it follows that $K \geqslant 1$.

Proof. Let f: $X \to X/\Pi$ be the natural projection. We will show that $X, d, X/\Pi, f$ satisfy (C).

LEMMA. *Let $A, B \in X/\Pi$, $b \in B$, and $\varepsilon > 0$. Then there exists $a \in A$ such that $d(a, b) \leqslant K d(A, B) + \varepsilon$.*

Proof. Let $a_0 \in A$, $b_0 \in B$ be such that $d(a_0, b_0) \leqslant d(A, B) + \varepsilon/K$. There exists $\varphi \in \Pi$ such that $\varphi(b_0) = b$. Let $a = \varphi(a_0)$. Then

$$d(a, b) = d(\varphi(a_0), \varphi(b_0)) \leqslant K d(a_0, b_0) \leqslant K d(A, B) + \varepsilon .$$

Returning to the proof of Theorem 5, let Γ be an open subset of X/Π. We will find a set $\{\varepsilon(A) |\ A \in \Gamma\}$ of positive reals such that

(i) $$N_{\varepsilon(A)}[A] \subset \bigcup \Gamma \quad \text{if} \quad A \in \Gamma ,$$

(ii) $$d(A, B) \geqslant \varepsilon(A) - \varepsilon(B) \quad \text{if} \quad A, B \in \Gamma.$$

It then follows from Theorem 2 that X/Π is pseudo-metrizable.

Define $\varepsilon(A) = \eta(A)/K$, where

$$\eta(A) = \sup\{\varepsilon \mid \varepsilon > 0 \text{ and } N_\varepsilon[A] \subset \bigcup \Gamma\} \quad \text{if} \quad A \in \Gamma.$$

We first prove that $\eta(A) > 0$ for all $A \in \Gamma$. Let $a \in A$ and let $\varepsilon = (1/K)d(a, X - \bigcup\Gamma)$. Clearly $\varepsilon > 0$. To show $\eta(A) > 0$, it is sufficient to show that $N_\varepsilon[A] \subset \bigcup\Gamma$, or, equivalently, that $d(A, X - \bigcup\Gamma) \geqslant \varepsilon$. So let $x \in A$, $u \in X - \bigcup\Gamma$, and let $\varphi \in \Pi$ be such that $\varphi(x) = a$. Then

$$Kd(x, u) \geqslant d(\varphi(x), \varphi(u)) = d(a, \varphi(u)) \geqslant d(a, X - \bigcup\Gamma) = K\varepsilon,$$

so $d(x, u) \geqslant \varepsilon$, and consequently $d(A, X - \bigcup\Gamma) \geqslant \varepsilon$.

It is trivial that $N_{\eta(A)}[A] \subset \bigcup\Gamma$ for all $A \in \Gamma$; thus (i) is true since $\varepsilon(A) \leqslant \eta(A)$ for all $A \in \Gamma$. It remains to prove (ii). Let $A, B \in \Gamma$. By the definition of $\varepsilon(A)$ and $\varepsilon(B)$, it is sufficient to prove that $Kd(A, B) \geqslant \eta(A) - \eta(B)$, i.e., that $\eta(B) \geqslant \eta(A) - Kd(A, B)$; and to do this it is sufficient to show that the following implication is true:

$$d(x, B) < \eta(A) - Kd(A, B) \Rightarrow x \in \bigcup\Gamma.$$

Suppose $d(x, B) < \eta(A) - Kd(A, B)$. Then there exists $\varepsilon > 0$ such that

$$d(x, B) < \eta(A) - Kd(A, B) - \varepsilon.$$

Now let $\delta > 0$ be arbitrary. Then there exists $b \in B$ such that

$$d(x, b) \leqslant d(x, B) + \delta,$$

and by the lemma there exists $a \in A$ such that

$$d(a, b) \leqslant Kd(A, B) + \delta.$$

It follows that

$$d(x, A) \leqslant d(x, a) \leqslant d(x, B) + Kd(A, B) + 2\delta < \eta(A) - \varepsilon + 2\delta.$$

Hence $d(x, A) \leqslant \eta(A) - \varepsilon < \eta(A)$ and $x \in \bigcup\Gamma$.

This concludes the proof of Theorem 5.

References

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The first order properties of Dedekind finite integers

by

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1. Introduction. It is well known that mathematics is often simplified by the introduction of ideal elements. In the past it has been said that even when their existence is entirely fictitious (points at infinity in geometry, for example), theorems about the original structure which are proved with their aid may be interpreted as relative consistency results. More recently, our firm belief in set theory has led us to take ideal elements which are constructed in set theory as bonafide mathematical objects. In this paper such notions are applied to the Dedekind finite cardinals Δ (cf. [4]). In theorem 1 we show that just as the finite cardinals $\mathcal{E}$ can be extended to the ring of rational integers $\mathcal{E}^*$, Δ can be extended to the ring of Dedekind finite integers Δ^*. Of course all of this is going on in a set theory $\mathfrak{S}^0$ which does not include the axiom of choice. Next, a series of lemmas shows that every function defined on $\mathcal{E}^*$ can be extended to a function defined on Δ^*. Since this extension procedure depends in an essential way on the methods of [4], we must require that $\mathfrak{S}^0$ contains the axiom of choice for sets of finite sets. This does not force $\Delta^* = \mathcal{E}^*$ as is shown in [4]. In order to study the first order properties of Δ^* we define a language L which contains equations between terms, which are built up by composition of function symbols, as atomic formula. L is interpreted in $\mathcal{E}^*$ by letting the function symbols denote functions on $\mathcal{E}^*$, and interpreted in Δ^* by letting the function symbols denote extensions to Δ^* of functions defined on $\mathcal{E}^*$. The bulk of our work is concerned with giving necessary and sufficient conditions that a sentence $\mathfrak{A}$ which holds in $\mathcal{E}^*$ will also hold in Δ^*. Our main sufficiency result is given by corollary 2, which says in essence that if $\mathfrak{A}$ is equivalent in $\mathcal{E}^*$ to a Horn sentence, then $\mathfrak{A}$ will also hold in Δ^*. This theorem easily follows by a routine transcription of [4], theorem 8. The more interesting part of our paper is concerned with necessity. We use metamathematical tools. In lemma 5 we show that in the Fraenkel–Mostowski model $\mathfrak{W}^+$ (cf. [11]), Δ^* is isomorphic to a direct limit of reduced powers of $\mathcal{E}^*$. In lemmas 6

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