A note on pretopologies
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Introduction. A pretopology $\mathcal{P}$ on a set $S$ can be defined by means of a generalized interior operator $I_\mathcal{P}$ on $S$, that is, a set function which has all of the properties of a topological interior operator except idempotency. Repeated application of $I_\mathcal{P}$ yields a chain of pretopologies called the “decomposition series for $\mathcal{P}$” which terminates with the finest topology $\lambda(\mathcal{P})$ coarser than $\mathcal{P}$. The primary goal of this paper is to give an alternate description of the decomposition series in terms of a primordial uniform-like structure called a “diagonal filter.” In the process, we define the notion of “symmetry” for pretopologies, a concept closely related to the “weakly uniformizable convergence structure” discussed in [3].

1. Pretopologies and diagonal filters. Let $S$ be a set, $\mathcal{F}(S)$ the set of all filters on $S$, and $\mathcal{F}(\mathcal{F})$ the set of all subsets of $S$. For each $\mathcal{F} \in \mathcal{F}(S)$, let $\hat{\mathcal{F}}$ denote the ultrafilter generated by $\mathcal{F}$.

DEFINITION 1. A convergence structure $\mathcal{Q}$ on $S$ is a mapping from $\mathcal{F}(S)$ into $\mathcal{F}(S)$ which satisfies the following conditions:

1. $\mathcal{F}, \mathcal{G} \in \mathcal{F}(S)$ and $\mathcal{F} \subseteq \mathcal{G}$ implies $\mathcal{Q}(\mathcal{F}) \subseteq \mathcal{Q}(\mathcal{G})$;
2. $\mathcal{F} \neq \mathcal{G}$ implies all $\mathcal{F} \not\in \mathcal{F}$;
3. $\mathcal{F} \neq \mathcal{G}$ implies $\mathcal{Q}(\mathcal{F}) \not\in \mathcal{Q}(\mathcal{G})$.

If $\mathcal{Q}$ is a convergence structure and $\mathcal{Q}(\mathcal{F})$, then the filter $\mathcal{F}$ is said to $\mathcal{Q}$-converge to $\mathcal{F}$. Let $\mathcal{U}_\mathcal{Q}(\mathcal{F})$ be the filter obtained by intersecting all of the filters that $\mathcal{Q}$-converge to $\mathcal{F}$; $\mathcal{U}_\mathcal{Q}(\mathcal{F})$ is called the $\mathcal{Q}$-neighborhood filter at $\mathcal{F}$.

DEFINITION 2. A convergence structure $\mathcal{Q}$ is called a pretopology if $\mathcal{U}_\mathcal{Q}(\mathcal{F})$ $\mathcal{Q}$-converges to $\mathcal{F}$ for each $\mathcal{F} \in \mathcal{F}(S)$.

Then term “pretopology” was introduced by G. Choquet [1]; other discussions of this concept can be found in [2] and [3].

Let $\mathcal{F}(S)$ be the set of all pretopologies on $S$, partially ordered as follows: $\mathcal{P} \subseteq \mathcal{Q}$ means $\mathcal{U}_\mathcal{Q}(\mathcal{F}) \subseteq \mathcal{U}_\mathcal{Q}(\mathcal{F})$, all $\mathcal{F} \in S$. With this ordering $\mathcal{F}(S)$ is a complete lattice which contains the lattice of all topologies on $S$ (as a subset, not as a sublattice).
DEFINITION 3. A diagonal filter $D$ on $S$ is a filter on $S \times S$ with the property that each member $D$ of $D$ contains the diagonal $\Delta = \{(x, x): x \in S\}$.

Before investigating the relationship between pretopologies and diagonal filters, it will be convenient to introduce some additional notation. If $D$ is a diagonal filter, let $D^{-1} = \{D^{-1}: D \in D\}$ (where $D^{-1} = \{(x, y): (y, x) \in D\}$). The filter $D$ is symmetric if $D = D^{-1}$. Given a diagonal filter $D$ and $x \in S$, we denote by $D[x]$ the filter on $S$ generated by $\{D[x]: D \in D\}$ (where $D[x] = \{y \in S: (x, y) \in D\}$). If $D$ is a uniformity, then $D[x]$ is the filter of neighborhoods at $x$ in the uniform topology. The symbol $\Delta$ denotes the diagonal filter consisting of all sets in $S \times S$ that include $\Delta$.

If $F$ and $G$ are filters on $S$, then let $F \times G$ be the filter on $S \times S$ generated by the base $\{F \times G: F \in F, G \in G\}$. Given diagonal filters $\mathcal{U}$ and $\mathcal{V}$, we denote by $\mathcal{U} \cdot \mathcal{V}$ the filter on $S \times S$ generated by all compositions of the form $UV$, for $U \in \mathcal{U}$, $V \in \mathcal{V}$.

PROPOSITION 1. Let $(\mathcal{U}_a)$ be a collection of diagonal filters.
   (1) $\bigcap (\mathcal{U}_a)[x] = \bigcap \{\mathcal{U}_a[x]: x \in S\}$.
   (2) $\bigcup (\mathcal{U}_a)[x] = \bigcup \{\mathcal{U}_a[x]: x \in S\}$.

A diagonal filter $D$ is said to be compatible with a pretopology $p$ if $D[x] = \mathcal{U}_p(x)$ for all $x$ in $S$. It follows from Proposition 1 that if $(\mathcal{U}_a)$ is a collection of diagonal filters, each compatible with the same pretopology $p$, then $\bigcap \mathcal{U}_a$ and $\bigcup \mathcal{U}_a$ are also compatible with $p$.

PROPOSITION 2. To each pretopology $p$ there corresponds an equivalence class $[p]$ of compatible diagonal filters. For any $p \in P(S)$, $[p]$ contains both a least element and a greatest element; the latter is given by $\mathcal{U}_p = \bigcap \{x \in \mathcal{U}_p(x): x \in S\}$.

DEFINITION 4. A pretopology $p$ is symmetric if $[p]$ contains a symmetric diagonal filter.

THEOREM 1. The following statements about a pretopology $p$ are equivalent:
   (1) $p$ is symmetric;
   (2) $\mathcal{U}_p \subseteq [p]$ is compatible with $p$;
   (3) $\mathcal{U}_p \subseteq \mathcal{U}_p(y)$ if and only if $y \in \bigcap \mathcal{U}_p(x)$;
   (4) $p$ is the infimum in $P(S)$ of a set of completely regular topologies.

Proof. (1) and (2) are obviously equivalent.

0. Choose $D \in [p]$ such that $D = D^{-1}$. Let $y \in \bigcap \mathcal{U}_p(x)$. Then, for each symmetric set $D \in D$, we have $y \in D[x]$, which implies $x \in D^{-1}[y]$. But this means that $x \in \bigcap D^{-1}[y] = \bigcap \mathcal{U}_p(x)$.

1. Let $\mathcal{F}$ be an ultrafilter which $p$-converges to $x$, then form the diagonal filter $\mathcal{U}_p \mathcal{F} = \mathcal{F} \cap (\mathcal{F} \circ \Delta)$. It can be shown that $\mathcal{U}_p \mathcal{F}$ is a uniformity for $S$; let $\tau_{p, \mathcal{F}}$ be the topology compatible with this uniformity. For any ultrafilter $\mathcal{G}$, $\tau_{p, \mathcal{G}}$ is finer than $\tau_{p, \mathcal{F}}$; if $\mathcal{F} = \mathcal{G}$ is a principal ultrafilter, one needs (2) to establish this result. A second application of (2) enables us to deduce that $p = \inf \tau_{p, \mathcal{F}}$, $x \in S$, $\mathcal{F}$ an ultrafilter on $S$, and $x \in q(\mathcal{F})$.

4. Let $(\mathcal{F})$ be a set of completely regular topologies such that $p = \inf \tau_{p, \mathcal{F}}$. With each $\tau_{p, \mathcal{F}}$, associate a compatible uniformity $\mathcal{U}_{\mathcal{F}}$, and let $\mathcal{D}$ be any $\mathcal{U}_{\mathcal{F}}$. Then $\mathcal{D}$, being an intersection of symmetric filters, is itself symmetric, and $D \in [p]$ by Proposition 1.

2. The decomposition series for a pretopology. Starting with a pretopology $p$ on a set $S$, let $I_p$ be the set function on $S$ defined by $I_p(A) = \{x \in A: x \in \mathcal{U}_p(x)\}$ for each $A \subseteq S$. Except for idempotency, $I_p$ satisfies the conditions for being a topological interior operator. The collection $(U \subseteq S): I_p(U) = U$ is the finest topology coarser than $p$, and $p = \lambda(p)$ if and only if $I_p$ is idempotent. These results, in slightly modified form, are proved in [4].

We shall now give a recursive definition of a generalized interior operator for each ordinal number $\alpha \geq 1$.

DEFINITION 5. Let $I_\alpha = I_{I_{\alpha-1}}$. If $\alpha$ is an ordinal number with an immediate predecessor $\alpha-1$, let $I_{\alpha} = I_{I_{\alpha-1}}(I_{\alpha-1}(A))$ for each $A \subseteq S$. If $\alpha$ is a limit ordinal (that is, an infinite ordinal with no immediate predecessor) then $I_{\alpha} = \bigcap \{I_{\alpha}(A): \beta < \alpha\}$.

DEFINITION 6. For each ordinal number $\alpha$, let $p^\alpha$ be the pretopology whose neighborhood filter at each point $x$ is given by $\mathcal{U}_p(x) = \{A \subseteq S: x \in I_{p}(A)\}$.

For each ordinal number $\alpha$, $I_{\alpha}$ satisfies the following conditions:
   (1) $I_{\alpha}(A) \subseteq A$, all $A \subseteq S$;
   (2) $A \subseteq S$ implies $I_{\alpha}(A) \subseteq I_{\alpha}(B)$;
   (3) $I_{\alpha}(A \cap \beta) = I_{\alpha}(A) \cap I_{\alpha}(B)$;
   (4) $I_{\alpha}(S) = S$.

Let $\tau_{p^\alpha}$ be the smallest of the ordinal numbers $\alpha$ such that $I_{\alpha}(I_{\alpha}(A)) = I_{\alpha}(A)$, all $A \subseteq S$.

PROPOSITION 2. (a) $1 < \alpha < \beta < \gamma$, then $p^\alpha > p^\beta$.
   (b) $p^\alpha = \lambda(p)$.

Proof. (a) If $\alpha < \beta$, there is $A \subseteq S$ such that $I_{\alpha}(A) \subseteq I_{\alpha}(A)$, but $I_{\beta}(I_{\alpha}(A)) = I_{\alpha}(A)$. If $x \in I_{\alpha}(A)$ and $x \in I_{\beta}(A)$, then $A$ belongs to $\mathcal{U}_p(x)$, but not to $\mathcal{U}_p(x)$, and the pretopologies $p^\alpha$ and $p^\beta$ are distinct.

(b) Since $I_{p^\alpha}$ is idempotent, it follows from the remarks following Definition 6 that $p^\alpha$ is a topology; by definition of $\lambda(p)$, this topology
must be coarser than \( \lambda(p) \). But if \( I_p(U) = U \), the \( I^*_p(U) = U \) for all ordinal numbers \( \alpha \); thus the topologies coincide.

**Definition 7.** The collection \( \{ \lambda^\alpha : 1 \leq \alpha \leq \gamma \} \) is called the decomposition series for \( p \). \( \gamma_p \) is called the length of this series.

The length of the decomposition series can be regarded as a criterion for describing quantitatively how non-topological a given pretopology is. In the example that follows, we show that decomposition series can have arbitrary length; that is, for any ordinal number \( \delta \) there is a pretopology \( p \) such that \( \gamma_p = \delta \).

**Example.** Let \( \delta \) be a fixed ordinal number greater than 0, and let \( S \) be the set of all ordinal numbers less than \( \delta \) (including 0). We define a pretopology \( p \) on \( S \) by specifying convergence on ultrafilters as follows:

1. \( p(\beta) = (\beta, \beta + 1) \), all \( 0 < \beta < \delta \);
2. If \( \alpha \) is a limit ordinal, then any ultrafilter finer than the filter \( F_\alpha \) generated by sets of the form \( \gamma : 0 < \gamma < \beta \), for \( \beta < \alpha \) order converges to \( \alpha \).
3. \( p(F) = \emptyset \) (i.e., \( F \) diverges) for all other ultrafilters \( F \). If \( S \) is a finite set, then \( \gamma_p = \delta - 1 \) if \( S \) is infinite, then \( \gamma_p = \delta \).

### 3. The decomposition series in terms of diagonal filters.

Recall that \( \omega_p = \bigcap \{ u \times \omega_p : u \in S \} \) is the largest diagonal filter compatible with \( p \).

**Definition 8.** Let \( \omega_p^\alpha = \omega_p \). If \( \alpha \) is an ordinal number with an immediate predecessor \( \alpha - 1 \), let \( \omega_p^\alpha = \omega_p^{\alpha - 1} \setminus \omega_p \). If \( \alpha \) is a limit ordinal, let \( \omega_p^\alpha = \bigcap \{ \omega_p^\beta : \beta < \alpha \} \).

**Lemma 1.** Suppose \( U \in \omega_p \) and \( U(A) \subseteq I_p^*(V) \) for some ordinal number \( \alpha \) and for subsets \( A \) and \( V \) of \( S \). Then there is \( W \in \omega_p^\alpha \) such that \( W \subseteq U(A) \cap V \).

**Proof.** (Transfinite induction on \( \alpha \).) Suppose \( U \subseteq I_p^*(V) \), then let \( W = \bigcup \{ \alpha \times V : \alpha \in \alpha \} \), where \( V \) is \( p \)-for all \( U(A) \cap V \) and \( V = S \) otherwise. If \( \alpha \) is in \( W \subseteq U(A) \cap V \), then there is \( y \in V \) and \( \beta \in A \) such that \( \beta < \alpha \) implies \( y \in U \). It is a simple matter to verify that \( W \in \omega_p^\alpha \).

Next, assume that \( \alpha \) is a limit ordinal. Then \( U(A) \cap I_p^*(V) = \bigcap \{ I_p^*(V) : \beta < \alpha \} \). By the induction hypothesis there is, for each \( \beta < \alpha \), \( W \subseteq \omega_p^\beta \) such that \( W \subseteq U(A) \cap V \). If \( W = \bigcup \{ W \subseteq \omega_p^\beta \), \( \beta < \alpha \), then \( W \in \omega_p^\alpha \), and \( \omega_p(A) \cap V \subseteq \bigcup \{ \omega_p(A) : \beta < \alpha \} \subseteq V \).

Finally, assume that \( \alpha \) is an ordinal number with an immediate predecessor \( \alpha - 1 \). Let \( y \in U(A) \), then \( y \in L_p^\alpha(V) \) implies that \( L_p^\alpha(V) \subseteq \omega_p^\alpha \). Let \( T \subseteq U \) be defined by \( T = \bigcup \{ \alpha \times V : \alpha \in \alpha \} \), where \( V = L_p^\alpha(V) \), for \( \alpha \in U(A) \), and \( V = S \) otherwise. Then \( T(A) \subseteq L_p^\alpha(V) \), all \( y \in U(A) \), and it follows from the induction hypothesis that there is \( W \subseteq \omega_p^\alpha \) such that \( W \subseteq U(A) \cap V \).

**Theorem 2.** For each ordinal number \( \alpha \), where \( 1 < \alpha < \gamma_p \), and each \( x \in S \), \( \omega_p^\alpha(x) = \omega_p(x) \). (In other words, \( \omega_p^\alpha = \{ \alpha \}_p \) for \( p \).

**Proof.** Let \( \alpha \) be any ordinal number with an immediate predecessor \( \alpha - 1 \). Let \( W \subseteq \omega_p \); then there are \( U \subseteq \omega_p \) and \( T \subseteq \omega_p^{\alpha - 1} \) such that \( T \subseteq U \subseteq \omega_p \).

By Lemma 2, \( U(A) \subseteq I_p^\alpha(U(A)) \), all \( x \in S \), and hence \( I_p^\alpha(U(A)) \subseteq \omega_p(x) \), which implies that \( I_p^\alpha(U(A)) \subseteq \omega_p(x) \). Thus \( W \subseteq \omega_p(x) \). On the other hand, if \( V \subseteq \omega_p(x) \), then \( I_p^\alpha(V) \subseteq \omega_p(x) \), and so there is \( W \subseteq \omega_p(x) \). By Lemma 1, there is \( W \subseteq \omega_p \) such that \( W \subseteq U \subseteq \omega_p \). Finally, if \( \alpha \) is a limit ordinal and \( \beta < \alpha \) and \( \beta < \alpha \), then \( \omega_p^\beta(x) = \omega_p^\gamma(x) = \omega_p^\gamma(x) = \omega_p(x) \). Thus the proof is complete.

**Concluding remarks.** Following the recent development of quasi-uniformities (for example, see [4]) diagonal filters seem to be the next logical step in the process of generalizing the notion of a uniformity. Diagonal filters also provide some insights in the theory of pretopologies; for instance, given a pretopology \( p \), it is easy to see that \( \lambda(p) \) is completely regular and only if \( \omega_p^\alpha \) for some ordinal number \( \alpha \).

If we define a "diagonal structure" to be the pair \( (S, D) \), where \( D \) is a diagonal filter on \( S \), then we can easily define such terms as Cauchy structure, completeness, and total boundedness for diagonal structures by analogy to the definitions currently in use for quasi-uniform spaces. This leads to other interesting questions: for example, can a meaningful \( \ast \)
completion theorem be proved for diagonal structures? (The latter question was recently answered in the affirmative for quasi-uniform spaces by R. Stoltenberg [5].) 

References


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