A note on pretopologies

by

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Introduction. A pretopology p on a set S can be defined by means of a generalized interior operator I_p on S, that is, a set function which has all of the properties of a topological interior operator except idempotency. Repeated application of I_p yields a chain of pretopologies called the "decomposition series for p" which terminates with the finest topology $\lambda(p)$ coarser than p. The primary goal of this paper is to give an alternate description of the decomposition series in terms of a primordial uniform-like structure called a "diagonal filter." In the process, we define the notion of "symmetry" for pretopologies, a concept closely related to the "weakly uniformizable convergence structure" discussed in [3].

1. Pretopologies and diagonal filters. Let S be a set, $F(S)$ the set of all filters on S, and $\mathcal{T}(S)$ the set of all subsets of S. For each x in S, let $\dot{x}$ denote the ultrafilter generated by $\{x\}$.

Definition 1. A *convergence structure* q on S is a mapping from $F(S)$ into $\mathcal{T}(S)$ which satisfies the following conditions:

(1) $\mathcal{F}, \mathcal{G}$ in $F(S)$ and $\mathcal{F} \subset \mathcal{G}$ implies $q(\mathcal{F}) \subset q(\mathcal{G})$;

(2) $x \in q(\dot{x})$, all x in S;

(3) $x \in q(\mathcal{F})$ implies $x \in q(\mathcal{F} \cap \dot{x})$.

If q is a convergence structure and $x \in q(\mathcal{F})$, then the filter $\mathcal{F}$ is said to q-converge to x. Let $\mathcal{V}_q(x)$ be the filter obtained by intersecting all of the filters that q-converge to x; $\mathcal{V}_q(x)$ is called the *q-neighborhood filter at x.*

Definition 2. A convergence structure q is called a *pretopology* if $\mathcal{V}_q(x)$ q-converges to x for each x in S.

Then term "pretopology" was introduced by G. Choquet [1]; other discussions of this concept can be found in [2] and [3].

Let $P(S)$ be the set of all pretopologies on S, partially ordered as follows: $p \leqslant q$ means $\mathcal{V}_p(x) \leqslant \mathcal{V}_q(x)$, all x in S. With this ordering $P(S)$ is a complete lattice which contains the lattice of all topologies on S (as a subset, not as a sublattice).

DEFINITION 3. A *diagonal filter* $\mathfrak{D}$ on S is a filter on $S\times S$ with the property that each member D of $\mathfrak{D}$ contains the diagonal $\Delta = \{(x, x): x \in S\}$.

Before investigating the relationship between pretopologies and diagonal filters, it will be convenient to introduce some additional notation. If $\mathfrak{D}$ is a diagonal filter, let $\mathfrak{D}^{-1} = \{D^{-1}: D \in \mathfrak{D}\}$ (where $D^{-1} = \{(x, y): (y, x) \in D\}$). The filter $\mathfrak{D}$ is *symmetric* if $\mathfrak{D} = \mathfrak{D}^{-1}$. Given a diagonal filter $\mathfrak{D}$ and $x \in S$, we denote by $\mathfrak{D}[x]$ the filter on S generated by $\{D[x]: D \in \mathfrak{D}\}$ (where $D[x] = \{y \in S: (x, y) \in D\}$). If $\mathfrak{D}$ is a uniformity, then $\mathfrak{D}[x]$ is the filter of neighborhoods at x in the uniform topology. The symbol $\dot{\Delta}$ denotes the diagonal filter consisting of all sets in $S\times S$ that include Δ. If $\mathcal{F}$ and $\mathcal{G}$ are filters on S, then let $\mathcal{F}\times\mathcal{G}$ be the filter on $S\times S$ generated by the filter base $\{F\times G: F \in \mathcal{F},\ G \in \mathcal{G}\}$. Given diagonal filters $\mathfrak{U}$ and $\mathfrak{V}$, we denote by $\mathfrak{U}\cdot\mathfrak{V}$ the filter on $S\times S$ generated by all compositions of the form UV, for $U \in \mathfrak{U}$, $V \in \mathfrak{V}$.

PROPOSITION 1. *Let $\{\mathfrak{D}_a\}$ be a collection of diagonal filters.*

(1) $(\bigcap \{\mathfrak{D}_a\})[x] = \bigcap \{\mathfrak{D}_a[x]\}$;

(2) $(\bigcup \{\mathfrak{D}_a\})[x] = \bigcup \{\mathfrak{D}_a[x]\}$.

A diagonal filter $\mathfrak{D}$ is said to be *compatible* with a pretopology p if $\mathfrak{D}[x] = \mathfrak{V}_p(x)$ for all x in S. It follows from Proposition 1 that if $\{\mathfrak{D}_a\}$ is a collection of diagonal filters, each compatible with the same pretopology p, that $\bigcap \mathfrak{D}_a$ and $\bigcup \mathfrak{D}_a$ are also compatible with p.

PROPOSITION 2. *To each pretopology p there corresponds an equivalence class $[p]$ of compatible diagonal filters. For any $p \in P(S)$, $[p]$ contains both a least element and a greatest element; the latter is given by*

$$\mathfrak{W}_p = \bigcap \{\dot{x}\times\mathfrak{V}_p(x): x \in S\}\ .$$

DEFINITION 4. A pretopology p is *symmetric* if $[p]$ contains a symmetric diagonal filter.

THEOREM 1. *The following statements about a pretopology p are equivalent*:

(1) *p is symmetric*;

(2) *$\mathfrak{W}_p \cap \mathfrak{W}_p^{-1}$ is compatible with p*;

(3) *$x \in \bigcap \mathfrak{V}_p(y)$ if and only if $y \in \bigcap \mathfrak{V}_p(x)$*;

(4) *p is the infimum in $P(S)$ of a set of completely regular topologies.*

Proof. (1) and (2) are obviously equivalent.

(1)$\Rightarrow$(3). Choose $\mathfrak{D} \in [p]$ such that $\mathfrak{D} = \mathfrak{D}^{-1}$. Let $y \in \bigcap \mathfrak{V}_p(x)$. Then, for each symmetric set $D \in \mathfrak{D}$, we have $y \in D[x]$, which implies $x \in D^{-1}[y]$. But this means that $x \in \bigcap \mathfrak{D}^{-1}[y] = \bigcap \mathfrak{V}_p(x)$.

(3)$\Rightarrow$(4). If $\mathcal{F}$ is an ultrafilter which p-converges to x, then form the diagonal filter $\mathfrak{U}_{x,\mathcal{F}} = \dot{\Delta} \cap [(\mathcal{F} \cap \dot{x})\times(\mathcal{F} \cap \dot{x})]$. It can be shown that $\mathfrak{U}_{x,\mathcal{F}}$ is a uniformity for S; let $\tau_{x,\mathcal{F}}$ be the topology compatible with this uniformity. For any ultrafilter $\mathcal{F}$, $\tau_{x,\mathcal{F}}$ is finer than p; when $\mathcal{F} = \dot{y}$ is a principal ultrafilter, one needs (2) to establish this result. A second application of (2) enables us to deduce that

$$p = \inf\{\tau_{x,\mathcal{F}}: x \in S\ ,\ \mathcal{F} \text{ an ultrafilter on } S, \text{ and } x \in q(\mathcal{F})\}\ .$$

(4)$\Rightarrow$(1). Let $\{\tau_a\}$ be a set of completely regular topologies such that $p = \inf\{\tau_a\}$. With each τ_a, associate a compatible uniformity $\mathfrak{U}_a$, and let $\mathfrak{D} = \bigcap\{\mathfrak{U}_a\}$. Then $\mathfrak{D}$, being an intersection of symmetric filters, is itself symmetric, and $\mathfrak{D} \in [p]$ by Proposition 1.

2. The decomposition series for a pretopology. Starting with a pretopology p on a set S, let I_p be the set function on S defined by $I_p(A) = \{x \in A: A \in \mathfrak{V}_p(x)\}$ for each $A \subset S$. Except for idempotency, I_p satisfies the conditions for being a topological interior operator. The collection $\{U \subset S: I_p(U) = U\}$ is a topology for S denoted by $\lambda(p)$; $\lambda(p)$ is the finest topology coarser than p, and $p = \lambda(p)$ if and only if I_p is idempotent. These results, in slightly modified form, are proved in [4].

We shall now give a recursive definition of a generalized interior operator for each ordinal number $\alpha \geqslant 1$.

DEFINITION 5. Let $I_p^1 = I_p$. If α is an ordinal number with an immediate predecessor $\alpha-1$, let $I_p^\alpha(A) = I_p(I_p^{\alpha-1}(A))$ for each $A \subset S$. If α is a limit ordinal (that is, an infinite ordinal with no immediate predecessor) then let $I_p^\alpha(A) = \bigcap\{I_p^\beta(A): \beta < \alpha\}$.

DEFINITION 6. For each ordinal number α, let p^α be the pretopology whose neighborhood filter at each point x in S is given by $\mathfrak{V}_p^\alpha(x) = \{A \subset S: x \in I_p(A)\}$.

For each ordinal number α, I_p satisfies the following conditions:

(1) $I_p^\alpha(A) \subset A$, all $A \subset S$;

(2) $A \subset B$ implies $I_p^\alpha(A) \subset I_p^\alpha(B)$;

(3) $I_p^\alpha(A \cap B) = I_p^\alpha(A) \cap I_p^\alpha(B)$;

(4) $I_p^\alpha(S) = S$.

Let γ_p be the smallest of the ordinal numbers α such that $I_p^\alpha(I_p^\alpha(A)) = I_p^\alpha(A)$, all $A \subset S$.

PROPOSITION 2. (a) *If $1 \leqslant \alpha < \beta \leqslant \gamma_p$, then $p^\alpha > p^\beta$.*

(b) $p^{\gamma_p} = \lambda(p)$.

Proof. (a) If is clear that $p^\alpha \geqslant p^\beta$. Since $\alpha < \beta \leqslant \gamma_p$, there is $A \subset S$ such that $I_p^\beta(A) \subset I_p^\alpha(A)$, but $I_p^\alpha(A) \neq I_p^\beta(A)$. If $x \in I_p^\alpha(A)$ and $x \notin I_p^\beta(A)$, then A belongs to $\mathfrak{V}_p^\alpha(x)$, but not to $\mathfrak{V}_p^\beta(x)$, and the pretopologies p^α and p^β are distinct.

(b) Since $I_p^{\gamma_p}$ is idempotent, it follows from the remarks following Definition 6 that p^{γ_p} is a topology; by definition of $\lambda(p)$, this topology must be coarser than $\lambda(p)$. But if $I_p(U) = U$, the $I_p^\alpha(U) = U$ for all ordinal numbers α; thus the topologies coincide.

Definition 7. The collection $\{p^\alpha\colon 1 \leqslant \alpha \leqslant \gamma_p\}$ is called the *decomposition series for p*. γ_p is called the *length* of this series.

The length of the decomposition series can be regarded as a criterion for describing quantitatively how non-topological a given pretopology is. In the example that follows, we show that decomposition series can have arbitrary length; that is, for any ordinal number δ there is a pretopology p such that $\gamma_p = \delta$.

Example. Let δ be a fixed ordinal number greater than 0, and let S be the set of all ordinal numbers less than δ (including 0). We define a pretopology p on S by specifying convergence on ultrafilters as follows:

(1) $p(\dot\beta) = \{\beta, \beta+1\}$, all $0 \leqslant \beta < \delta$;

(2) If α is a limit ordinal, then any ultrafilter finer than the filter $\mathcal{F}_\alpha$ generated by sets of the form $\{\gamma\colon 0 \leqslant \gamma \leqslant \beta\}$, for $\beta < \alpha$ order converges to α.

(3) $p(\mathcal{F}) = \emptyset$ (i.e., $\mathcal{F}$ diverges) for all other ultrafilters $\mathcal{F}$. If S is a finite set, then $\gamma_p = \delta-1$; if S is infinite, then $\gamma_p = \delta$.

3. The decomposition series in terms of diagonal filters.

Recall that $\mathfrak{W}_p = \bigcap \{\dot x \times \mathfrak{V}_p(x)\colon x \in S\}$ is the largest diagonal filter compatible with p.

Definition 8. Let $\mathfrak{W}_p^1 = \mathfrak{W}_p$. If α is an ordinal number with an immediate predecessor $\alpha-1$, let $\mathfrak{W}_p^\alpha = \mathfrak{W}_p^{\alpha-1}\cdot\mathfrak{W}_p$. If α is a limit ordinal, let $\mathfrak{W}_p^\alpha = \bigcap \{\mathfrak{W}_p^\beta\colon \beta < \alpha\}$.

Lemma 1. *Suppose $U \in \mathfrak{W}_p$ and $U[A] \subset I_p^\alpha(V)$ for some ordinal number α and for subsets A and V of S. Then there is $W \in \mathfrak{W}_p^\alpha$ such that $WU[A] \subset V$.*

Proof. (Transfinite induction on α.) If $U[A] \subset I_p(V)$, then let $W = \bigcup \{\{z\} \times V_z\colon z \in S\}$, where $V_z = V$ for z in $U[A]$ and $V_z = S$ otherwise. If z is in $WU[A]$, then there is y in S and x in A such that (x, y) is in U and (y, z) is in W. $y \in U[A]$ implies $z \in V_y = V$. It is a simple matter to verify that $W \in \mathfrak{W}_p$.

Next, assume that α is a limit ordinal. Then $U[A] \subset I_p^\alpha(V) = \bigcap \{I_p^\beta(V)\colon \beta < \alpha\}$. By the induction hypothesis there is, for each $\beta < \alpha$, $W_\beta \in \mathfrak{W}^\beta$ such that $W_\beta U[A] \subset V$. If $W = \bigcup \{W_\beta\colon \beta < \alpha\}$, then $W \in \mathfrak{W}^\beta$, and $WU[A] = \bigcup \{W_\beta U[A]\colon \beta < \alpha\} \subset V$.

Finally, assume that α is an ordinal number with an immediate predecessor $\alpha-1$. Let $y \in U[A]$. Then $y \in I_p^\alpha(V)$ implies that $I_p^{\alpha-1}(V) \in \mathfrak{V}_p(y)$. Let $T \in \mathfrak{W}$ be defined by $T = \bigcup \{\{z\} \times V_z\colon z \in S\}$, where $V_z = I_p^{\alpha-1}(V)$, for $z \in U[A]$, and $V_z = S$, otherwise. Then $T[y] \subset I_p^{\alpha-1}(V)$, all $y \in U[A]$, and it follows from the induction hypothesis that there is $W_1 \in \mathfrak{W}^{\alpha-1}$ such that $W_1 T[y] \subset V$, all $y \in U[A]$. Let $W = W_1 T$; then $WU[A] \subset V$ follows immediately.

Lemma 2. *If $U \in \mathfrak{W}_p$ and $V \in \mathfrak{W}_p^\alpha$, then $U[x] \subset I_p^\alpha(VU[x])$, all $x \in S$.*

Proof. (Transfinite induction on α.) When $\alpha = 1$, we have $U \in \mathfrak{W}_p$ and $V \in \mathfrak{W}_p$. Let $z \in U[x]$; then $V[z] \in \mathfrak{V}_p(z)$. But $V[z] \subset VU[x]$, and hence $VU[x] \in \mathfrak{V}_p(z)$, which implies $z \in I_p(VU[x])$.

When α is a limit ordinal, we have

$$U \in \mathfrak{W}_p \quad \text{and} \quad V \in \mathfrak{W}_p^\alpha = \bigcap \{\mathfrak{W}_p^\beta(VU[x]\colon \beta < \alpha\}\,.$$

By the induction hypothesis,

$$U[x] \in \bigcap \{I_p^\beta(VU[x]\colon \beta < \alpha\} = I_p(VU[x])\,.$$

If α has an immediate predecessor $\alpha-1$, then let $z \in U[x]$. Since $V \in \mathfrak{W}_p^\alpha$, we can assume that $V \supset V_1 W$, where $V_1 \in \mathfrak{W}_p^{\alpha-1}$ and $W \in \mathfrak{W}_p$. By the induction hypothesis, we have

$$W[z] \subset I_p^{\alpha-1}(V_1 W[z]) \subset I_p^{\alpha-1}(V[z]) \subset I_p^{\alpha-1}(VU[x])\,.$$

Hence

$$I_p^{\alpha-1}(VU[x]) \in \mathfrak{V}_q(z)\,, \quad \text{and} \quad z \in I_q^\alpha(VU[x])\,.$$

Theorem 2. *For each ordinal number α, where $1 \leqslant \alpha \leqslant \gamma_p$, and each $x \in S$, $\mathfrak{W}_p^\alpha[x] = \mathfrak{V}_p^\alpha(x)$. (In other words, $\mathfrak{W}_p^\alpha \in [p^\alpha]$ for each α.)*

Proof. Let α be any ordinal number with an immediate predecessor $\alpha-1$. Let $W \in \mathfrak{W}^\alpha$; then there are $U \in \mathfrak{W}$ and $T \in \mathfrak{W}^{\alpha-1}$ such that $TU \subset W$. By Lemma 2, $U[x] \subset I_p^{\alpha-1}(TU[x])$, all $x \in S$, and hence $I_p^{\alpha-1}(TU[x]) \in \mathfrak{V}_p(x)$, which implies that $TU[x] \in \mathfrak{V}_p^\alpha(x)$. Thus $W[x] \in \mathfrak{V}_p^\alpha(x)$. On the other hand, if $V \in \mathfrak{V}_p^\alpha(x)$, then $I_p^{\alpha-1}(V) \in \mathfrak{V}_p(x)$, and so there is $U \in \mathfrak{W}$ such that $U[x] \subset I_p^{\alpha-1}(V)$. By Lemma 1, there is $W \in \mathfrak{W}^{\alpha-1}$ such that $WU[x] \subset V$. But $WU \in \mathfrak{W}^\alpha$; thus $V \in \mathfrak{W}^\alpha[x]$. Finally, if α is a limit ordinal and β any non-limit ordinal less than α, then we have $\mathfrak{V}_p^\alpha(x) = \bigcap \mathfrak{V}_p^\beta(x) = \bigcap \mathfrak{W}^\beta[x] = \mathfrak{W}^\alpha[x]$. Thus the proof is complete.

Concluding remarks. Following the recent development of quasi-uniformities (for example, see [4]) diagonal filters seem to be the next logical step in the process of generalizing the notion of a uniformity. Diagonal filters also provide some insights in the theory of pretopologies; for instance, given a pretopology p, it is easy to see that $\lambda(p)$ is completely regular if and only if $\mathfrak{W}_p^{-1} \geqslant \mathfrak{W}_p^\alpha$ for some ordinal number α.

If we define a "diagonal structure" to be the pair $(S, \mathfrak{D})$, where $\mathfrak{D}$ is a diagonal filter on S, then we can easily define such terms as Cauchy filter, completeness, and total boundedness for diagonal structures by analogy to the definitions currently in use for quasi-uniform spaces. This leads to other interesting questions: for example, can a meaningful completion theorem be proved for diagonal structures? (The latter question was recently answered in the affirmative for quasi-uniform spaces by R. Stoltenberg [5].)

References

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