ANR divisors and absolute neighborhood contractibility

by

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1. Introduction. Let B be a compact subset of an ANR Y. The purpose of this paper is to determine conditions which will guarantee that the quotient space Y/B is an ANR. It is known [4] that the question of whether or not Y/B is an ANR depends only on B and not on Y. Precisely,

1.1. THEOREM. *Let B be a compact metric space. If there exists an* ANR Y_0 *containing B such that Y_0/B is an* ANR, *then for every* ANR Y *containing B, Y/B is an* ANR.

If B is an arbitrary closed subset (not necessarily compact) of an ANR Y, then Y/B may be non-metrizable, in which case it cannot be an ANR. However, we can still ask if Y/B is an *absolute neighborhood extensor* for metric pairs (abbreviated ANE). As before the question of whether or not Y/B is an ANE depends only on B and not on Y [4]:

1.2. THEOREM. *Let B be a metric space, and let* ANR(B) *denote the collection of all* ANR*'s that contain B as a closed subset. If there exists a $Y_0 \in$* ANR(B) *such that Y_0/B is an* ANE, *then for every $Y \in$* ANR(B), *Y/B is an* ANE.

Playing a central role in our discussion will be the following

1.3. DEFINITION. A space B is called an ANR *divisor* if it is metrizable and if Y/B is an ANE for every $Y \in$ ANR(B).

We will obtain a number of sufficient conditions for a space to be an ANR divisor. Special emphasis is placed on a certain class of ANR divisors, the compact absolutely neighborhood contractible spaces. Many examples of ANR divisors can be built up from these spaces.

2. Deformation neighborhood bases. [1] Let B be a closed subset of an ANR Y. In this section we will obtain a condition involving the neighborhoods of B in Y which will guarantee that Y/B is an ANR. We first state the following result, which is an immediate consequence of [4], Lemma 3.4:

[1] Some of the results in this section are taken from the author's doctoral dissertation, written at the University of Maryland under the direction of Professor G. R. Lehner. The author was a NASA Fellow.

2.1. LEMMA. *Let B be a closed subset of an* ANR Y. *Then* Y/B *is an* ANE *if and only if* Y/B *is strongly locally contractible* [2] *at the point* $p(B)$, *where* $p\colon Y\to Y/B$ *is the natural projection.*

In view of 2.1, we must find a condition which will guarantee that Y/B is strongly locally contractible at $p(B)$. For this purpose we introduce the notation of a deformation neighborhood basis.

2.2. DEFINITION. Let (Y, B) be a pair. [3] A sequence $\{(U_n, h_n)|\ n\geqslant 1\}$ is called a *deformation neighborhood basis* for B in Y if

DNB-1) Each U_n is a neighborhood [4] of B in Y.

DNB-2) $\overline{U}_{n+1}\subset U_n$ for all n.

DNB-3) Every neighborhood V of B in Y contains some U_n.

DNB-4) $h_1\colon \overline{U}_1\times I\to Y$ is a deformation such that

$$h_1(\overline{U}_1\times 1)\subset \overline{U}_2,$$

and $h_n\colon \overline{U}_n\times I\to \overline{U}_{n-1}$ is a deformation such that

$$h_n(\overline{U}_n\times 1)\subset \overline{U}_{n+1},\qquad n>1.$$

DNB-5) $h_n(\overline{U}_m\times I)\subset \overline{U}_{m-1}$ if $m>n$.

2.3. LEMMA. *Let (Y, B) be a pair. If B has a deformation neighborhood basis in Y, then Y/B is strongly locally contractible at the point $[B]=p(B)$, where $p\colon Y\to Y/B$ is the natural projection.*

Proof. Let $\{(U_n, h_n)|\ n\geqslant 1\}$ be a deformation neighborhood basis for B. For each n and for all $s\in I$, define $h_n^s\colon \overline{U}_n\to Y$ by $h_n^s(x)=h_n(x, s)$. We define a map [5] $h\colon \overline{U}_1\times[0,\infty)\to Y$ as follows: Let $x\in\overline{U}_1$ and suppose that $t\in[k, k+1]$, where k is a non-negative integer. Define $h(x, t)$ by

$$h(x,t)=\begin{cases} h_1(x,t) & \text{if}\quad k=0,\\ h_{k+1}^{t-k}\circ h_k^1\circ h_{k-1}^1\circ\ldots\circ h_2^1\circ h_1^1(x) & \text{if}\quad k\geqslant 1.\end{cases}$$

Since the range of h_n^1 is contained in the domain of h_{n+1}^s for all n and s, the composition is meaningful, and it is easily verified that h is single-valued. Also, h is continuous, since it is continuous on each set of the form $\overline{U}_1\times[k, k+1]$.

The map h has the following properties:

(1) $h(\overline{U}_n\times[0,\infty))\subset\overline{U_{[n/2]}}$ for all $n>1$, where $[n/2]$ denotes the greatest integer less than or equal to $n/2$,

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[2] A space X is *strongly locally contractible at a point* x if for every neighborhood U of x there is a neighborhood V of x and a contraction k_t of V in U such that $k_t(x)=x$ for all t. It is easily shown that X is strongly locally contractible at x if and only if there is a neighborhood V of x and a contraction k_t of V in X such that $k_t(x)=x$ for all t.

[3] By a *pair* (Y, B) we mean a space Y and a closed subset B.

[4] All neighborhoods are open.

[5] A map is a continuous function.

(2) $h(\overline{U}_1\times t)\subset\overline{U}_{[t]}$ for all $t\in[1,\infty)$, and

(3) $h(B\times[0,\infty))\subset B$.

To verify (1), let $(x, t)\in\overline{U}_n\times[0,\infty)$ be given. Choose a non-negative integer k such that $t\in[k, k+1]$. If $k\geqslant[n/2]$, then by DNB-4 and the definition of h, $h(x,t)\in\overline{U}_k\subset\overline{U}_{[n/2]}$. If $k<[n/2]$, then $n-k-1\geqslant[n/2]$. Apply DNB-5 in the definition of h $k+1$ times to conclude that $h(x,t)\in\overline{U_{n-k-1}}$. Therefore $h(x,t)\in\overline{U_{[n/2]}}$. This proves (1). (2) follows from the fact that h_{k+1} is a deformation over $\overline{U}_k$ (see the definition of h). (3) follows from (1) and the equation $B=\bigcap_{n=1}^{\infty}\overline{U}_n$.

Let g be a homeomorphism from $[0, 1)$ onto $[0,\infty)$ and let $p\colon Y\to Y/B$ be the natural projection. Define a map $J\colon p(U_1)\times I\to Y/B$ by

$$J(x,t)=\begin{cases} p\big(h(p^{-1}(x),\ g(t))\big) & \text{if}\quad x\in p(U_1),\ t<1,\\ [B] & \text{if}\quad x\in p(U_1),\ t=1.\end{cases}$$

It follows from (3) that J is single-valued and that $J([B]\times I)=[B]$. By DNB-3, the collection $\{p(U_n)|\ n\geqslant 1\}$ is a basis for neighborhoods of $[B]$ in Y/B. In view of this, the continuity of J on $p(U_1)\times 1$ follows from (2), and the continuity of J on $[B]\times I$ follows from (1). J is obviously continuous everywhere else. Observe that $J|p(U_1)\times 1=[B]$, and for each $x\in p(U_1)$, $J(x,0)=p\big(h(p^{-1}(x),0)\big)=pp^{-1}(x)=x$. This completes the proof.

Combining 2.3 and 2.1, we have

2.4. THEOREM. *Let B be a closed subset of an* ANR Y. *If B has a deformation neighborhood basis in Y, then Y/B is an* ANE.

By 2.4 and 1.2, we have

2.5. COROLLARY. *Let B be a closed subset of an* ANR Y. *If B has a deformation neighborhood basis in Y, then B is an* ANR *divisor.*

3. Absolute neighborhood contractibility. One of the most important classes of ANR divisors is the class of compact absolutely neighborhood contractible spaces.

3.1. DEFINITION. Let (Y, B) be a pair. B is said to be *neighborhood contractible in* Y if B is contractible in every neighborhood U of B in Y. A metric space B is said to be *absolutely neighborhood contractible* if it is neighborhood contractible in every $Y\in\mathrm{ANR}(B)$.

By this definition, to verify that B is absolutely neighborhood contractible, we must show that B is contractible in every neighborhood of every $Y\in\mathrm{ANR}(B)$. Actually, to show that B is absolutely neighborhood contractible we need only verify that it satisfies somewhat weaker conditions:

3.2. THEOREM. *Let B be a metric space. The following statements are equivalent:*

(a) *There exists a $Y \in \mathrm{ANR}(B)$ such that B is neighborhood contractible in Y.*

(b) *For every $Y \in \mathrm{ANR}(B)$, B is contractible in Y.*

(c) *B is absolutely neighborhood contractible.*

Proof. (a)$\to$(b). Given $Z \in \mathrm{ANR}(B)$, the map $i\colon B \to B$ is extendable to $g\colon U \to Z$, for some neighborhood U of B in Y. By (a), B is contractible in U under a homotopy h_t; the homotopy gh_t contracts B in Z.

(c)$\to$(a) is trivial.

(b)$\to$(c). Let U be a neighborhood of B in Y. Then U is an ANR, and by (b) B is contractible in U. Therefore B is absolutely neighborhood contractible.

Our goal in this section is to show that every compact absolutely neighborhood contractible space is an ANR divisor. This will be accomplished in several steps, the first of which is a characterization of absolute neighborhood contractibility. First we state for future reference the Homotopy Extension Theorem and one of its corollaries.

3.3. THEOREM. *Let (X, A) be a metric pair and let f be a map from X into an ANR Y. If $h_t\colon A \to Y$ is a homotopy such that $h_0 = f|A$, then h_t can be extended to a homotopy $H_t\colon X \to Y$ such that $H_0 = f$.* [2]

Since any constant mapping on A can be extended to X, it follows that

3.4. COROLLARY. *If (X, A) is a metric pair and if f is a nullhomotopic map from A into an ANR Y, then f has an extension $F\colon X \to Y$.*

3.5. THEOREM. *A metric space B is absolutely neighborhood contractible if and only if for every $Y \in \mathrm{ANR}(B)$ there is a neighborhood V of B in Y such that for every metric pair (X, A), each map $f\colon A \to \overline{V}$ has an extension $F\colon X \to Y$.*

Proof. Suppose first that B is absolutely neighborhood contractible and let $Y \in \mathrm{ANR}(B)$. Let k_t be a contraction of B over Y to a point b_0. Define a map $g\colon Y \times \{0\} \cup B \times I \cup Y \times \{1\} \to Y$ by

$$g(y, 0) = y \quad \text{for all } y \in Y,$$
$$g(b, t) = k_t(b) \quad \text{for all } b \in B,\ 0 \leqslant t \leqslant 1,$$
$$g(y, 1) = b_0 \quad \text{for all } y \in Y.$$

Since Y is an ANR, g has an extension $G\colon W \to Y$, where W is some open set in $Y \times I$. Let V be a neighborhood of B in Y such that $\overline{V} \times I \subset W$. $G|\overline{V} \times I$ contracts $\overline{V}$ over Y to b_0. Therefore any map into $\overline{V}$ is nullhomotopic over Y, and by 3.4, extendable over Y.

Conversely, if $Y \in \mathrm{ANR}(B)$ and if V has the property stated in the theorem, then the map $f\colon \overline{V} \times \{0\} \cup \overline{V} \times \{1\} \to \overline{V}$ defined by $f(v, 0) = v$, $f(v, 1) = b_0$ has an extension $F\colon \overline{V} \times I \to Y$. Therefore $\overline{V}$ is contractible over Y; in particular, B is contractible over Y. The result now follows from 3.2(b).

3.6. LEMMA. *Let B be a compact absolutely neighborhood contractible metric space, and let $Y \in \mathrm{ANR}(B)$. Then B has a deformation neighborhood basis in Y.*

Proof. By 3.5, there is a neighborhood U_1 if B in Y such that any map from a closed subset of a metric space into $\overline{U}_1$ has an extension over Y. We may choose U_1 such that $d(x, B) < 1$ for all $x \in U_1$, where d is some metric on Y. By repeated application of 3.5, we can obtain a sequence of neighborhoods $\{U_n|\ n > 1\}$ of B such that

(1) $\overline{U}_n \subset U_{n-1}$,

(2) every map from a closed subset of a metric space into $\overline{U}_n$ has an extension over U_{n-1},

(3) $d(x, B) < 1/n$ for all $x \in U_n$.

It follows at once that the sequence $\{U_n|\ n \geqslant 1\}$ satisfies DNB1-3.

Choose a point $b_0 \in B$. For each positive integer n, define a map $f_n\colon (\overline{U}_n - U_{n+1}) \cup \overline{U_{n+2}} \to \overline{U_{n+2}}$ by $f_n(x) = b_0$ if $x \in \overline{U}_n - U_{n+1}$ and $f_n(x) = x$ if $x \in \overline{U_{n+2}}$. By (2), f_n extends to a map $F_n\colon \overline{U}_n \to U_{n+1}$. Define a map $g_n\colon \overline{U_{n+1}} \times \{0\} \cup \overline{U_{n+2}} \times I \cup \overline{U_{n+1}} \times \{1\} \to \overline{U_{n+1}}$ by

$$g_n(x, 0) = x \quad \text{if} \quad x \in \overline{U_{n+1}},$$
$$g_n(x, t) = x \quad \text{if} \quad x \in \overline{U_{n+2}},\ 0 \leqslant t \leqslant 1,$$
$$g_n(x, 1) = F_n(x) \quad \text{if} \quad x \in \overline{U_{n+1}}.$$

By (2), g_n extends to a map $G_n\colon \overline{U_{n+1}} \times I \to U_n$. Finally, define a map $k_n\colon \overline{U}_n \times \{0\} \cup \overline{U_{n+1}} \times I \cup \overline{U}_n \times \{1\} \to \overline{U}_n$ by

$$k_n(x, 0) = x \quad \text{if} \quad x \in \overline{U}_n,$$
$$k_n(x, t) = G_n(x, t) \quad \text{if} \quad x \in \overline{U_{n+1}},\ 0 \leqslant t \leqslant 1,$$
$$k_n(x, 1) = F_n(x) \quad \text{if} \quad x \in \overline{U}_n.$$

By (2), k_n extends to a map $h_n\colon \overline{U}_n \times I \to U_{n-1}$ if $n > 1$; k_1 extends to $h_1\colon \overline{U}_1 \times I \to Y$. It is straightforward to verify that the sequence $\{h_n|\ n \geqslant 1\}$ satisfies DNB4-5. Therefore $\{(U_n, h_n)|\ n \geqslant 1\}$ is a deformation neighborhood basis for B in Y.

By combining 3.6 and 2.5 we obtain the main result of this section.

3.7. THEOREM. *If a compact metric space B is absolutely neighborhood contractible, then B is an* ANR *divisor.*

4. Homotopy characterization of absolute neighborhood contractibility. It is well known that if B is a contractible subset of an ANR Y, then the projection $p\colon Y \to Y/B$ is a homotopy equivalence. This conclusion is still valid if B is compact and absolutely neighborhood contractible.

4.1. THEOREM. *Let B be a compact metric space. The following statements are equivalent:*

(a) *B is absolutely neighborhood contractible.*

(b) *For every $Y \in \mathrm{ANR}(B)$, the natural projection $p\colon Y \to Y/B$ is a homotopy equivalence.*

(c) *For every $Y \in \mathrm{ANR}(B)$, p has a left homotopy inverse.*

Proof (a)→(b). Suppose first that B is absolutely neighborhood contractible, and let $Y \in \mathrm{ANR}(B)$. By 3.7, Y/B is an ANR; therefore there is a neighborhood U of $[B] = p(B)$ in Y/B and a contraction j_t of $\overline{U}$ to $[B]$ in Y/B such that $j_t([B]) = [B]$ for all $t \in I$. The Homotopy Extension Theorem can be applied to yield a homotopy $J_t\colon Y/B \to Y/B$ extending j_t and such that J_0 is the identity on Y/B. Since J_1 extends j_1, we have

$$J_1(U) = [B]\,. \tag{1}$$

Since $p^{-1}(U)$ is open in Y, $p^{-1}(U)$ is an ANR and therefore $p^{-1}(U) \in \mathrm{ANR}(B)$. Since B is absolutely neighborhood contractible, by 3.5 there is a neighborhood V of B in $p^{-1}(U)$ such that, for any metric pair (X, A), every map $f\colon A \to \overline{V}$ has an extension $F\colon X \to p^{-1}(U)$. Let k_t be a contraction of B to a point b_0 over V. Define a map $f\colon \overline{V} \times \{0\} \cup B \times I \cup \overline{V} \times \{1\} \to \overline{V}$ by

$$\begin{aligned} f(y, 0) &= y && \text{if} \quad y \in \overline{V}\,, \\ f(b, t) &= k_t(b) && \text{if} \quad b \in B\,,\ 0 \leqslant t \leqslant 1\,, \\ f(y, 1) &= b_0 && \text{if} \quad y \in \overline{V}\,. \end{aligned}$$

Since the image of f is contained in $\overline{V}$, f can be extended to a map $F\colon \overline{V} \times I \to p^{-1}(U)$. Since Y is an ANR, the Homotopy Extension Theorem can be applied to yield a homotopy $K_t\colon Y \to Y$ such that $K_t(y) = F(y, t)$ for all $y \in \overline{V}$ and $0 \leqslant t \leqslant 1$, and such that K_0 is the identity on Y. Since K_t extends k_t, we have

$$K_1(B) = b_0 \tag{2}$$

and

$$K_t(B) \subset p^{-1}(U)\,. \tag{3}$$

Let i and j be the identity maps on Y and Y/B, respectively, and let $\varphi = K_1 p^{-1}\colon Y/B \to Y$. By (2), φ is single valued, and since p is an identification, [6] φ is continuous. We will show that φ is a homotopy inverse of p.

Combining (1) and (3), we see that $J_1 p K_t p^{-1}$ is a single-valued (and continuous) homotopy between the maps $J_1 p K_0 p^{-1}$ and $J_1 p K_1 p^{-1}$ on Y/B. Therefore we can write

$$j = J_0 \sim J_1 = J_1 p K_0 p^{-1} \sim J_1 p K_1 p^{-1} = J_1 p\varphi \sim p\varphi\colon\ Y/B \to Y/B\,.$$

Also,

$$i = K_0 \sim K_1 = K_1 p^{-1} p = \varphi p\colon\ Y \to Y\,.$$

Therefore p is a homotopy equivalence with homotopy inverse φ.

(b)→(c) is trivial.

(c)→(a). Let $Y \in \mathrm{ANR}(B)$ and let $q\colon Y/B \to Y$ be a left homotopy inverse of p. Then $qp\colon Y \to Y$ is homotopic to the identity of Y, and $qp(B)$ is a single point. Therefore B is contractible in Y, and B is absolutely neighborhood contractible by 3.2(b).

By 3.2(b), every inclusion of an absolutely neighborhood contractible space into an ANR is nullhomotopic. More generally,

4.2. THEOREM. *A metric space B is absolutely neighborhood contractible if and only if every map from B into an ANR is nullhomotopic.*

Proof. Suppose that B is absolutely neighborhood contractible, and let f be a map from B into an ANR Y. Choose some $X \in \mathrm{ANR}(B)$. Since Y is an ANR, f has a neighborhood extension $F\colon U \to Y$. Since B is absolutely neighborhood contractible, the inclusion $i\colon B \to U$ is nullhomotopic; and therefore $f = Fi$ is nullhomotopic. The converse is trivial.

It is known that a space B is homotopically trivial if and only if for every ANR Y, every map from Y into B is nullhomotopic. Therefore 4.2 shows that, among metric spaces, absolute neighborhood contractibility and homotopic triviality are dual concepts. Unlike compact absolutely neighborhood contractible spaces, not all compact homotopically trivial metric spaces are ANR divisors; in particular it follows that there exist compact acyclic (using singular homology) spaces which are not ANR divisors. It is an open question if every compact acyclic (using Čech homology) metric space is an ANR divisor. Since it is easily seen that every absolutely neighborhood contractible space is Čech-acyclic, an affirmative answer to this question would generalize 3.7.

An immediate consequence of 4.2 is

4.3. COROLLARY. *If a metric space B is homotopically dominated by an absolutely neighborhood contractible space A, then B is absolutely neighborhood contractible. In particular, absolute neighborhood contractibility is an invariant of homotopy type among metric spaces, and every retract*

[6] A surjection $g\colon X \to Z$ having the property that $U \subset Z$ is open if and only if $g^{-1}(U) \subset X$ is open is called an *identification*.

of an absolutely neighborhood contractible space is absolutely neighborhood contractible.

We can establish a similar result for ANR divisors.

4.4. THEOREM. *If a metric space B is homotopically dominated by an* ANR *divisor A, then B is an* ANR *divisor.*

Proof. Let $X \in \mathrm{ANR}(A)$ and $Y \in \mathrm{ANR}(B)$, and let $p\colon X \to X/A$ and $q\colon Y \to Y/B$ be the natural projections. To prove that Y/B is an ANE it is sufficient by 2.1 to show that Y/B is strongly locally contractible at $q(B)$.

Let $f\colon B \to A$ and $g\colon A \to B$ be maps such that the identity on B is homotopic to gf under a deformation α_t. Since Y is an ANR, there exists a neighborhood N of A in X and a map $\varphi\colon N \to Y$ extending g. Since N is open in X, N is an ANR, and since A is an ANR divisor, N/A is an ANE. Therefore by 2.1 there is a neighborhood W of $p(A)$ in X/A and a strong contraction h_t of W to $p(A)$ over N/A. Since $p^{-1}(W)$ is open in X, $p^{-1}(W)$ is an ANR; therefore there exists a neighborhood U of B in Y and a map $\psi\colon U \to p^{-1}(W)$ extending f. Define a map $\lambda\colon U \times \{0\} \cup B \times I \cup U \times \{1\} \to Y$ by

$$\lambda(u, 0) = u \qquad \text{for all } u \in U,$$
$$\lambda(b, t) = \alpha_t(b) \qquad \text{for all } b \in B,\ 0 \leqslant t \leqslant 1,$$
$$\lambda(u, 1) = \varphi\psi(u) \qquad \text{for all } u \in U.$$

Since Y is an ANR, λ is extendable to a map $J\colon E \to Y$, where E is a neighborhood of $U \times \{0\} \cup B \times I \cup U \times \{1\}$ in $U \times I$. Let V be a neighborhood of B in U such that $V \times I \subset E$ and such that $J(V \times I) \subset U$. Then the restriction of J to $V \times I$ defines a homotopy $j_t\colon V \to U$ such that j_0 is the identity on V, $j_1 = \varphi\psi|V$ and $j_t(B) \subset B$ for all t. Define a map $k\colon q(V) \times I \to Y/B$ by

$$k_t(z) = \begin{cases} q j_{2t} q^{-1}(z) & \text{for all } z \in q(V),\ 0 \leqslant t \leqslant \tfrac{1}{2}, \\ q\varphi p^{-1} h_{2t-1} p\psi q^{-1}(z) & \text{for all } z \in q(V),\ \tfrac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

It is easily verified that k strongly deforms $q(V)$ to $q(B)$, and the proof is complete.

4.5. COROLLARY. *Among metric spaces the property of being an* ANR *divisor is an invariant of homotopy type, and every retract of an* ANR *divisor is an* ANR *divisor.*

5. Quotients and unions of ANR divisors and absolutely neighborhood contractible spaces. Suppose that A is a compact ANR subset of a metric space B. If B is an ANR, then B/A is an ANR; (7) however, if B/A is an ANR, it does not follow that B is an ANR. It follows from our next result, however, that B is at least an ANR divisor.

(7) Every ANR is an ANR divisor [4].

5.1. THEOREM. *Suppose that (B, A) is a metric pair such that A is a compact* ANR *divisor. Then B is an* ANR *divisor if and only if B/A is an* ANR *divisor.*

Proof. Choose some $Y \in \mathrm{ANR}(B)$. Since A is a compact ANR divisor, Y/A is an ANR. Therefore $Y/A \in \mathrm{ANR}(B/A)$. Notice that Y/B is homeomorphic to $(Y/A)/(B/A)$. If B is an ANR divisor, then Y/B is an ANE. Since Y/A is an ANR and since $Y/B \cong (Y/A)/(B/A)$, it follows from 1.2 that B/A is an ANR divisor. If B/A is an ANR divisor, then $(Y/A)/(B/A)$ is an ANE. Since $(Y/A)/(B/A) \cong Y/B$, it follows that B is an ANR divisor.

Suppose that A is a compact AR subset of a metric space B. If B is an AR, then B/A is an AR; however, if B/A is an AR, it does not follow that B is an AR. By 5.1, we can say that B is an ANR divisor. Our next result (5.3) enables us to say even more, namely, that B is absolutely neighborhood contractible.

5.2. LEMMA. *Suppose that (B, A) is a pair such that both A and B are absolutely neighborhood contractible. Let $Y \in \mathrm{ANR}(B)$ and let U be a neighborhood of A in Y. Then B is deformable into U under a deformation that leaves A pointwise fixed.*

Proof. By 3.5, there is a neighborhood V of B in Y such that any map from a closed subset of a metric space into $\overline{V}$ is extendable over Y; similarly there is a neighborhood W of A in $U \cap V$ such that any map from a closed subset of a metric space into $\overline{W}$ is extendable over $U \cap V$. Choose a point $a \in A$ and define a map $f\colon A \cup (B - W) \to W$ by

$$f(x) = \begin{cases} x & \text{if } x \in A, \\ a & \text{if } x \in B - W. \end{cases}$$

f has an extension $F\colon B \to U \cap V$. Define a map $g\colon B \times \{0\} \cup A \times I \cup B \times \{1\} \to V$ by

$$g(x, 0) = x \qquad \text{if } x \in B,$$
$$g(x, t) = x \qquad \text{if } x \in A,\ 0 \leqslant t \leqslant 1,$$
$$g(x, 1) = F(x) \qquad \text{if } x \in B.$$

g extends to a map $G\colon B \times I \to Y$. G is the desired deformation.

5.3. THEOREM. *Suppose that (B, A) is a metric pair such that A is compact and absolutely neighborhood contractible. Then B is absolutely neighborhood contractible if and only if B/A is absolutely neighborhood contractible.*

Proof. Let $Y \in \mathrm{ANR}(B)$ and let $p\colon Y \to Y/A$ be the natural projection. By 3.7, Y/A is an ANR; therefore $Y/A \in \mathrm{ANR}(B/A)$.

Assume first that B is absolutely neighborhood contractible. To show that B/A is absolutely neighborhood contractible it is sufficient, by 3.2(a), to show that B/A is contractible in an arbitrary neighborhood U of B/A in Y/A. Since Y/A is an ANR, the open set U is an ANR, and it follows that there is a neighborhood V of $p(A)$ contractible in U. By 5.2, there is a deformation k_t: $B \to p^{-1}(U)$ leaving A pointwise fixed and such that $k_1(B) \subset p^{-1}(V)$. The homotopy $pk_t(p|B)^{-1}$: $B/A \to U$ deforms B/A into V, which is contractible in U. Therefore B/A is contractible in U, and it follows that B/A is absolutely neighborhood contractible.

Conversely, assume that B/A is absolutely neighborhood contractible. Since Y/A is an ANR, B/A is contractible in Y/A under a deformation k_t. By 4.1, p has a homotopy inverse q: $Y/A \to Y$. Let i: $B \to Y$ be the inclusion. Then we have $i \sim qp|B = qk_0p|B \sim qk_1p|B$. Since k_1 is constant, i is null-homotopic, and it follows from 3.2(b) that B is absolutely neighborhood contractible.

Given a finite collection of ANR's, it is possible to build up a new ANR by fitting them together in a sufficiently "smooth" way. For example, if a metric space X can be written as the union of two closed ANR subsets X_1 and X_2 such that $X_1 \cap X_2$ is an ANR, then X is an ANR. Similarly, we can build up compact ANR divisors from smaller ones by fitting them together properly.

5.4. THEOREM. *Let B be a compact metric space. Suppose that $B_1, \ldots, B_n$ are closed subsets of B such that $\bigcup_{i=1}^{n} B_i = B$ and such that for every subcollection $\{B_{i_1}, \ldots, B_{i_k}\}$ of $\{B_1, \ldots, B_n\}$, $\bigcap_{j=1}^{k} B_{i_j}$ is an ANR divisor (or empty). Then B is an ANR divisor.*

Proof of Theorem 5.4. We show first that the disjoint union $A_1 \cup A_2$ of compact ANR divisors A_1 and A_2 is an ANR divisor. Let $Y \in \mathrm{ANR}(A_1 \cup A_2)$. Then Y/A_1 is an ANR. Let p: $Y \to Y/A_1$ be the natural projection. Since $A_1 \cap A_2 = \emptyset$, $p|A_2$ is a homeomorphism. Therefore $Z = (Y/A_1)/p(A_2)$ is an ANR. Let q: $Y/A_1 \to Z$ be the natural projection. $qp(A_1)$ and $qp(A_2)$ are singletons, therefore $Z/(qp(A_1) \cup qp(A_2))$ is an ANR.(7) But $Z/(qp(A_1) \cup qp(A_2))$ is homeomorphic to $Y/(A_1 \cup A_2)$. Therefore $A_1 \cup A_2$ is an ANR divisor.

Returning to the theorem itself, we note that if $n = 1$, the result is trivial. Assume inductively that any metric space which can be written as the union of k compact subsets satisfying the conditions in the hypothesis is an ANR divisor, and let $n = k+1$. For $i = 1, \ldots, k$, let $C_i = B_i \cap B_{k+1}$. By hypothesis, each C_i is an ANR divisor (or empty); and by the induction hypothesis, $D = \bigcup_{i=1}^{k} B_i$ and $E = \bigcup_{i=1}^{k} C_i$ are ANR divisors (or empty). If $E \neq \emptyset$, B_{k+1}/E is an ANR divisor by 5.1. But B_{k+1}/E is homeomorphic to B/D. Therefore, by 5.1, B is an ANR divisor. If $E = \emptyset$, then B is the disjoint union of B_{k+1} and D. By the above paragraph, B is an ANR divisor.

An immediate consequence of 5.4 and 3.7 is

5.5. COROLLARY. *Let B be a compact metric space. Suppose that $B_1, \ldots, B_n$ are closed subsets of B such that $\bigcup_{i=1}^{n} B_i = B$ and such that for every subcollection $\{B_{i_1} \ldots, B_{i_k}\}$ of $\{B_1, \ldots, B_n\}$, $\bigcap_{j=1}^{k} B_{i_j}$ is absolutely neighborhood contractible (or empty). Then B is an ANR divisor.*

Most of the simple examples of ANR divisors can be written as the union of absolutely neighborhood contractible spaces satisfying the hypothesis of 5.5. For example, every compact polyhedron, when triangulated, is the union of cells, intersections of which are lower dimensional cells (or empty). Each cell, of course, is absolutely neighborhood contractible.

6. Absolute neighborhood contractibility and proximate retracts.

Recently, Yandl [5] extended the theory of retracts to the category of compact metric spaces and "approximately continuous" functions. The AR's of this category are called WPAR's. In this section we will show that the WPAR's are precisely the compact absolutely neighborhood contractible spaces.

6.1. DEFINITION. Let (X, d) and (Y, ϱ) be compact metric spaces. A function f: $X \to Y$ is called an *ε-map*, $\varepsilon > 0$, if there is a $\delta > 0$ such that $d(x_1, x_2) < \delta$ implies $\varrho(f(x_1), f(x_2)) < \varepsilon$.

6.2. DEFINITION. Let Y be a closed subset of a compact metric space X. An ε-map f: $X \to Y$ having the property that $f(y) = y$ for all $y \in Y$ is called an *ε-retraction*.(8) Y is called a *proximate retract of X* if for every $\varepsilon > 0$ there is an ε-retraction of X onto Y. A compact metric space is called a *weak proximate absolute retract* (WPAR) if it is a proximate retract of every compact metric space in which it is embedded.

It is clear that if X, Y and Z are compact metric spaces, if f: $X \to Y$ is a map and if g: $Y \to Z$ is an ε-map, then the composite gf is an ε-map. Also, a compact metric space is a WPAR if and only if it is a proximate retract of the Hilbert cube [5].

6.3. THEOREM. *A compact metric space B is absolutely neighborhood contractible if and only if it is a WPAR.*

Proof. Consider B to be embedded in the Hilbert cube H, and let d be a metric on H.

(8) This definition differs slightly from that in [5]; however, it is easily seen that the class of WPAR's considered here is the same as that in [5].

Assume first that B is absolutely neighborhood contractible. To show that B is a WPAR, it is sufficient, by the above remarks, to ε-retract H onto B, where $\varepsilon > 0$ is given. Let $U = \{x \in H \mid d(x, B) < \varepsilon/3\}$. Since U is open in H, U is an ANR; and since the inclusion of B in U is nullhomotopic, there is by 3.4 a map $f\colon H \to U$ such that $f(b) = b$ for all $b \in B$. Define a function $g\colon \overline{U} \to B$ by assigning to each $u \in \overline{U} - B$ a point $g(u) \in B$ such that $d(g(u), u) \leqslant \varepsilon/3$, and by setting $g(u) = u$ for each $u \in B$. If $d(u_1, u_2) < \varepsilon/3$, then $d(g(u_1), g(u_2)) < \varepsilon$; therefore g is an ε-map. Consequently $gf\colon H \to B$ is an ε-map, and clearly $gf(b) = b$ for all $b \in B$. This proves that B is a WPAR.

Conversely, assume that B is a WPAR. Let W be a neighborhood of B in H. To show that B is absolutely neighborhood contractible, it is sufficient, by 3.2(a), to contract B to a point over W. Let X be a compact ANR such that $B \subset X \subset W$. (9) Since X is closed in H, there exists a retraction $r\colon U \to X$, where U is a neighborhood of X in H. Since U is open in H, U is locally convex; and since B is compact, there exists an $\varepsilon > 0$ such that

(*) *any set of diameter* $< \varepsilon$ *meeting* B *lies in a convex subset of* U.

Let $s\colon H \to B$ be an $\varepsilon/3$-retraction; there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(s(x), s(y)) < \varepsilon/3$. We may and do assume that $\delta < \varepsilon/3$. Let $\{V_a\}_{a \in A}$ be a canonical cover [3] of $H - B$ such that $\operatorname{diam}\{\operatorname{star}(V_a)\} < \delta/2$ for all a. Choose, for each a, a point $p_a \in V_a$, and let $P = \{p_a \mid a \in A\}$. Define a retraction $t\colon P \cup B \to B$ by assigning to each $p_a \in P$ a point $t(p_a) \in B$ such that $d(p_a, t(p_a)) < 2d(p_a, B)$, and by setting $t(b) = b$ for all $b \in B$. Define a function $\lambda\colon P \cup B \to B$ by

$$\lambda(x) = \begin{cases} t(x) & \text{if} \quad d(x, B) < \delta/2, \\ s(x) & \text{if} \quad d(x, B) \geqslant \delta/2. \end{cases}$$

Since λ agrees with t on a neighborhood of B and since P is discrete, it follows that λ is continuous. Moreover, if $V_a \cap V_\beta \neq \emptyset$, then $d(\lambda(p_a), \lambda(p_\beta)) < \varepsilon$: For if $d(p_a, B) < \delta/2$ and $d(p_\beta, B) \geqslant \delta/2$, choose a $b \in B$ such that $d(p_a, b) < \delta/2$. Then

$$d(\lambda(p_a), b) \leqslant d(\lambda(p_a), p_a) + d(p_a, b) < \delta + \delta/2,$$

and

$$d(p_\beta, b) \leqslant d(p_\beta, p_a) + d(p_a, b) < \delta/2 + \delta/2 = \delta,$$

which implies that

$$d(\lambda(p_\beta), b) = d(s(p_\beta), s(b)) < \varepsilon/3.$$

(9) The existence of such a set X follows from the methods in [1].

Therefore

$$d(\lambda(p_a), \lambda(p_\beta)) \leqslant d(\lambda(p_a), b) + d(b, \lambda(p_\beta)) < \delta + \delta/2 + \varepsilon/3 < \varepsilon.$$

The other cases are easier to verify and omitted.

It now follows from (*) that there is a map $g\colon H \to U$ defined by

$$g(b) = b \quad \text{if} \quad b \in B,$$
$$g(x) = \sum_a \varphi_a(x) \cdot \lambda(p_a) \quad \text{if} \quad x \in H - B,$$

where $\{\varphi_a\}$ is a partition of unity subordinated to $\{V_a\}$. (10) Let $k_t\colon B \to H$ be a contraction of B to a point in H. Then the homotopy $rgk_t\colon B \to X$ contracts B in $X \subset W$. This completes the proof.

(10) The continuity of g follows as in [3], 4, 3.

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