$f = Pg$ as in Theorem 1 with $|g(x)-x| < 1/2$ length of i for all $x \in [a, b]$. $i \subset \overset{-1}{f}(z) = \overset{-1}{g}\,\overset{-1}{P}(z)$. Since $\overset{-1}{P}(z)$ is finite, then $i \subset \overset{-1}{g}(z_0)$ for some $z_0 \in [a, b]$. Therefore there exists an $x_0 \in i$ such that $|g(x_0)-x_0| \geqslant 1/2$ length of i. A contradiction has been reached and so the converse of Theorem 1 is proved.

We can now state Theorem 1 together with its converse and do so in slightly more general terms.

THEOREM 2. *If T is a mapping of an arc onto a non-degenerate arc, then T is light if and only if T is topologically equivalent to a mapping f of $[0,1]$ onto $[0,1]$ such that if $\varepsilon > 0$, there exists a factorization $f = Pg$ where P is a polynomial of $[0,1]$ onto $[0,1]$ and g is a mapping of $[0,1]$ onto $[0,1]$ such that*

$$|g(x)-x| < \varepsilon \quad \text{for all } x \in [0,1].$$

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Reçu par la Rédaction le 15. 2. 1967

Multiple complementation in the lattice of topologies*

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1. Introduction. Hartmanis [4] showed that in the lattice of all topologies on a finite set with at least three elements every proper (i.e., neither discrete nor trivial) topology has at least two complements. In the light of Steiner's result [7] that the lattice of topologies on an arbitrary set is complemented, the question of Berri [1] in this journal may be rephrased as follows. Does every proper topology on an infinite set have at least two complements? This paper answers the question affirmatively. Further evidence of the pathological nature of the lattice of topologies is the result that a non-discrete T_1 topology never possesses a maximal complement (or a maximal principal complement). The result of Hartmanis above is sharpened. It is shown that every proper topology on a finite set with $n \geqslant 2$ elements has at least $n-1$ complements. Finally, utilizing these results it is shown that every proper topology on an infinite set actually has infinitely many principal complements.

2. Basic facts. The paper of Steiner [7] provides an ideal reference on the background material for this paper. It is possible to quickly outline the basic facts needed here. If (X, t) is a topological space on the set X, then the topology t consists of the open sets. (Note: Hartmanis [4] considers the closed sets.) If t_1 and t_2 are topologies on X and t_1 is a subset of t_2, then $t_1 \leqslant t_2$ and under this partial order the set of all topologies, Σ, on a fixed set X is a complete lattice with greatest element 1, the discrete topology, and least element 0, the trivial topology. If $t, t' \in \Sigma$ and $t \vee t' = 1$ while $t \wedge t' = 0$, then t' is a *complement* for t.

A maximal proper topology is an *ultraspace*. Given a filter $\mathcal{F}$ on X and a fixed point $x \in X$ one can define a topology $S(x, \mathcal{F}) = \{A \subset X: x \in A \Rightarrow A \in \mathcal{F}\}$. A filter $\mathcal{U}$ on X with the property that $A \cup B \in \mathcal{U}$ implies $A \in \mathcal{U}$ or $B \in \mathcal{U}$ is an *ultrafilter*. An ultrafilter of the form $\mathcal{U} = \{U \subset X: p \in U\}$ is *principal* and denoted $\mathcal{U}(p)$. An ultrafilter on X is principal

* This paper is part of a doctoral dissertation written under the direction of Professor M. P. Berri of Tulane University of Louisiana.

** The author holds an NSF Science Faculty Fellowship.

iff it contains a finite subset of X as a member, otherwise, it is *non-principal*. Fröhlich [2] showed that the ultraspaces are precisely those spaces of the form $t = S(x, \mathfrak{U})$ where x and $\mathfrak{U}$ are uniquely determined and $\mathfrak{U}$ is an ultrafilter with $\mathfrak{U} \neq \mathfrak{U}(x)$. An ultraspace $S(x, \mathfrak{U})$ is principal or not accordingly as $\mathfrak{U}$ is principal or not. It can be shown that every proper topology is contained in an ultraspace. (In fact, the author [5] has shown, in the absence of the axiom of choice, that this is equivalent to the Boolean prime ideal theorem.) Every topology is the infimum of the collection of all ultraspaces which contain it. A topology is T_1 iff it is contained in no principal ultraspace. Steiner [7] calls a topology *principal* iff it is the infimum of a collection of principal ultraspaces. The set of principal topologies forms a sublattice, Π, of Σ which is complete and complemented [7]. If X is finite, $\Pi = \Sigma$. Steiner showed that every topology $t \in \Sigma$ has a principal complement $t' \in \Pi$. She also characterized a topology as principal iff each point has a minimum open neighborhood containing it.

Gaifman [3] noted that by the axiom of choice it may be shown that every topological space (X, t) has a maximal (with respect to inclusion) T_1 subspace $(Y, t|Y)$ with $Y \subset X$ and $t|Y =$ the relativization of t to Y. This fact is used in the proof of Theorem 5.

3. Results. Taking advantage of the facts and definitions of the previous section one may easily prove:

THEOREM 1. *If* $|X| \geqslant 3$, *then* Σ *(resp.,* Π*) has an element with at least two principal complements.*

Proof. Let $x, y, z \in X$ be distinct and let $t = S(x, \mathfrak{U}(y))$. Then, $t' = \{\emptyset, \{x\}, X\}$ and $t'' = \{\emptyset, \{x\}, \{x, z\}, X\}$ are distinct principal complements for t.

Although Theorem 1 is known, this extremely simple proof establishes simultaneously results of Hartmanis (finite case) and Berri (infinite case) and, moreover, has as an immediate:

COROLLARY (Steiner). *If* $|X| \geqslant 3$, *then* Σ *(resp.* Π*) is not modular.*

The next, somewhat surprising theorem reduces Berri's question to the non-T_1 case and is of independent interest.

THEOREM 2. *A non-discrete* T_1 *topology has no maximal (principal) complement.*

Proof. It will be shown that, given a complement (resp., principal complement) t' for t, there exists another complement (resp., principal complement) t'' for t properly containing t'.

First consider the case in which X has no t-isolated points. Let $V_0 \in t'$ be a fixed proper subset of X. There exists a point $x_0 \in X \setminus V_0$ such that $V_0 \cup \{x_0\} \notin t'$. (Otherwise, pick any point $y_0 \in X \setminus V_0$ and note that $X \setminus \{y_0\} = \bigcup \{V_0 \cup \{y\}\colon\ y \in X \setminus V_0 \text{ and } y \neq y_0\}$ belongs to t'. This is impossible, since t is a T_1 topology, so that $X \setminus \{y_0\}$ is a member of t, and $t \wedge t' = t \cap t' = 0$.) Let $t'' = t' \vee \{\emptyset, \{x_0\}, X\}$ and note that t'' is principal, if t' is principal (being the join of two elements of Π). Furthermore, since $V_0 \cup \{x_0\} \in t'' \setminus t'$, then t'' properly contains t'. Clearly, $t \vee t'' = 1$. It remains to show that $t \wedge t'' = 0$. Now $\{x_0\} \in t'' \setminus t$ since t has no isolated points. Also, if $V \in t'$ with $V \neq \emptyset$ and $V \cup \{x_0\} \neq X$, then $V \cup \{x_0\} \notin t$. To see this note that t being a T_1 topology is the infimum of non-principal ultraspaces. Since $V \in t' \setminus t$, there exists a non-principal ultraspace $S(x, \mathfrak{U}) \geqslant t$ with $x \in V \notin \mathfrak{U}$. Since $\{x_0\} \notin \mathfrak{U}$, then $V \cup \{x_0\} \notin \mathfrak{U}$ and therefore, $V \cup \{x_0\} \notin t$. Thus, $t \wedge t'' = 0$.

Suppose then that the set of t-isolated points, denoted by I, is non-empty. Let $Y = X \setminus I$ and let (P) denote the following proposition.

P: *There exist elements* $V_i \in t'$, $i = 1, 2$ *such that* $V_1 \cup (V_2 \cap Y) \in t \setminus t'$.

If P is true, then $V_1 \cup \{y_0\} \notin t'$ for some $y_0 \in Y$ (actually, some $y_0 \in V_2 \cap Y$). Thus, $t'' = t' \vee \{\emptyset, \{y_0\}, X\}$ properly contains t' and is principal, if t is principal. $t \vee t'' = 1$. To show that $t \wedge t'' = 0$ one notes that $y_0 \in Y$, so that $\{y_0\} \notin t$ and proceeds as in the first case.

If P is false, then $t' \vee \{\emptyset, Y, X\}$ is a complement for t and is principal if t' is principal. (The falsity of P implies that $t \wedge (t' \vee \{\emptyset, Y, X\}) = 0$.) Hence, if $Y \notin t'$, then t' is properly contained in $t'' = t' \vee \{\emptyset, Y, X\}$. If $Y \in t'$, then $Y \cup \{x_0\} \notin t'$ for some $x_0 \in I$. (Otherwise, t' would have to contain the complement of an isolated point.) Thus, $t'' = t' \vee \{\emptyset, Y \cup \{x_0\}, X\}$ properly contains t' and is principal, if t' is principal. $t \vee t'' = 1$. If $U \in t'' \setminus t'$, then $U = V \cup \{x_0\}$, where $\emptyset \neq V \neq X$ and $V \in t'$. This is easily verified using the facts that the join of two topologies has their union for a subbase and every non-empty element of t' intersects Y. As in the earlier cases, it follows that $U \notin t$. Hence, $t \wedge t'' = 0$.

One has an immediate:

COROLLARY. *A non-discrete* T_1 *topology has infinitely many principal complements. In particular, a non-principal ultraspace has infinitely many principal complements.*

It is now possible to give a new characterization of principal ultraspaces.

THEOREM 3. *Let* t *be an ultraspace. Then these are equivalent statements.*

(i) t *is principal;*

(ii) t *has a greatest complement;*

(iii) t *has a greatest principal complement;*

(iv) t *has a maximal complement;*

(v) t *has a maximal principal complement.*

Proof. If t is non-principal, then by Theorem 2, t has no maximal (principal) complement. To complete the proof it suffices to assume that t is principal and to exhibit the greatest complement for t which is in addition, principal. If $t = S(x, \mathfrak{U}(y))$ with $x \neq y$, then $t' = \{V \subset X: x \in V \subset X \setminus \{y\}\} \cup \{\emptyset, X\}$ is the desired topology.

The next theorem improves the result of Hartmanis [4] cited. The inductive proof shows that the construction of at least $n-1$ complements for a proper topology on a set with n elements is amenable to being programmed for a digital computer.

THEOREM 4. *If $|X| = n \geqslant 2$ and t is a proper topology on X, then t has at least $n-1$ complements.*

Proof. For $n = 2$ the result is trivial and for $n = 3$ it is already known. Suppose inductively that $n \geqslant 4$ and that the theorem holds for all j with $2 \leqslant j < n = |X|$. Through the proof suppose that t has a non-empty element W with k points, but no non-empty element with fewer points. Let $V = X \setminus W$ and $m = |V|$.

Case 1. $1 \leqslant k \leqslant n-2$. There are three subcases to consider.

(i) $t|V$ is proper. Since $|V| = m \geqslant 2$, then $t|V$ has at least $m-1$ complements, say s^j, $j = 1, \ldots, m-1$. Since $t|V$ is proper, there exist distinct points $v_i \in V$, $i = 1, 2$ with $v_1 \in \mathrm{Cl}(\{v_2\})$ (where Cl = closure in (X, t)). For $w \in W$, $i = 1, 2$ and $j = 1, \ldots, m-1$ define complements for t as follows. If $V \notin t$, let $t(w, i, j) = S(w, \mathfrak{U}(v_i)) \wedge \{C \subset X: C \cap V \in s^j\}$. If $V \in t$, let $t'(w, i, j) = S(v_i, \mathfrak{U}(w)) \wedge t(w, i, j)$. Clearly, these are topologies. It will be shown that in either case they provide us with $2k(m-1) \geqslant n-1$ complements for t.

Suppose that $V \notin t$. Fix $w \in W$, i and j. For each $w' \in W$, $\{w'\} = W \cap (\{w'\} \cup V) \in t \vee t(w, i, j)$. For each $v \in V$, $\{v\} = (U \cap V) \cap B$, where $U \in t$ and $B \in s^j$. But, $B \subset V$ and $B \in t(w, i, j)$; thus $\{v\} = U \cap B \in t \vee t(w, i, j)$. Consequently, $t \vee t(w, i, j) = 1$. Let $C \in t \wedge t(w, i, j)$. If $C \cap W = \emptyset$, then $C \in (t|V) \wedge s^j$. Since $V \notin t$, $C = \emptyset$. If $C \cap W \neq \emptyset$, then $W \subset C$. Thus, $v_i \in C \cap V \in (t|V) \wedge s^j$ and $C \cap V = V$, i.e. $C = X$. Hence, $t \wedge t(w, i, j) = 0$. If $w \neq w'$, then $\{w'\} \in t(w, i, j) \setminus t(w', i', j')$. Suppose $w = w'$. For any j, $\{v_1\} = (U \cap V) \cap B$ with $B \in s^j$ and $U \in t$. Since $v_1 \in \mathrm{Cl}(\{v_2\})$, then $v_2 \in U \cap V$ and, therefore, $v_2 \notin B$. Thus, for any j' one has $\{w\} \cup B \in t(w, 1, j) \setminus t(w, 2, j')$. Suppose $w = w'$, $i = i'$ and $j \neq j'$. Then there exists $B \in s^j \setminus s^{j'}$, say. But, then $B \in t(w, i, j) \setminus t(w, i, j')$.

Suppose that $V \in t$. Fix $w \in W$, i and j. For each $w' \in W \setminus \{w\}$, $\{w'\} = W \cap \{w'\} \subset t \vee t'(w, i, j)$ and $\{w\} = W \cap (\{w\} \cup V) \in t \vee t'(w, i, j)$. If $v \in V$, then $\{v\} = U \cap B$, where $U \subset V$, $U \in t$ and $B \in s^j$. If $v_i \notin B$, then $\{v\} = U \cap B \in t \vee t'(w, i, j)$; if $v_i \in B$, then $\{v\} = U \cap (\{w\} \cup B) \notin t \vee t'(w, i, j)$. Thus, $t \vee t'(w, i, j) = 1$. Let $C \in t \wedge t'(w, i, j)$. If $C \cap W = \emptyset$, then $C \subset V \setminus \{v_i\}$. Since $C \in (t|V) \wedge s^j$, then $C = \emptyset$. If $C \cap W \neq \emptyset$, then $W \cup \{v_i\} \subset C$. Since $C \cap V \in (t|V) \wedge s^j$, then $C \cap V = V$. Hence, $C = X$. Thus, $t \wedge t'(w, i, j) = 0$. If $w \neq w'$, then $\{w\} \cup V \in t'(w, i, j) \setminus t'(w', i', j')$. Suppose $w = w'$. For any j, there exists $B \in s^j$ with $v_1 \in B \subset V \setminus \{v_2\}$. Hence, $\{w\} \cup B \in t'(w, 1, j) \setminus t'(w, 2, j')$ for any j'. Suppose $w = w'$, $i = i'$, and $j \neq j'$. Then there exists $B \in s^j \setminus s^{j'}$, say. But, then $B \in t'(w, i, j) \setminus t'(w, i, j')$ if $v_i \notin B$; and $\{w\} \cup B \in t'(w, i, j) \setminus t'(w, i, j')$ if $v_i \in B$.

(ii) $t|V$ *is trivial*. For each $w \in W$ and $v \in V$ there is a complement for t defined by

$$t(w, v) = \begin{cases} S(w, \mathfrak{U}(v)), & \text{if } V \text{ is not open;} \\ S(w, \mathfrak{U}(v)) \wedge S(v, \mathfrak{U}(w)), & \text{if } V \text{ is open.} \end{cases}$$

In all, this yields $km \geqslant n-1$ complements as is easily verified.

(iii) $t|V$ *is discrete*. Note that V is not open in this case. (Otherwise t is discrete.) One obtains k complements for t of the form:

$$t(w) = \{A \cup B: A \subset W,\ B = \emptyset \text{ or } B = V \text{ and } (w \in A \Rightarrow B = V)\}.$$

Since V is not open, there exists $v_0 \in V$ with $\{v_0\} \notin t$. Thus, there exist $m-1$ v's in V with $\{v_0, v\} \notin t$ where the possibility that $v = v_0$ is allowed. Each such v yields k more complements for t, viz. $t(w) \vee \{\emptyset, \{v_0\}, \{v_0, v\}, X\}$. Thus, there are at least $km \geqslant n-1$ complements for t in all as is easily verified.

Case 2. $k = n-1$. Then $t = \{\emptyset, W, X\}$.

All of the following $n-1$ ultraspaces are complements for t: $S(w, \mathfrak{U}(v))$, where $w \in W$ and $v \in V$.

Thus, in either case t has at least $n-1$ complements and the theorem is eastablished.

Remark. This estimate is the best possible. If $|X| = n \geqslant 3$, then the proper topology $\{\emptyset, \{w\}, X \setminus \{w\}, X\}$, where $w \in X$, has exactly $n-1$ complements.

To establish the last theorem and an affirmative answer to Berri's question, the following lemma is extremely useful. The case $n = 1$ is essentially Steiner's Theorem 7.1 [7] (pp. 392-393).

LEMMA. *Let $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \emptyset$. Suppose that t is given such that (with $t_i = t|X_i$, $i = 1, 2$) t_1 has $n \geqslant 1$ principal complements. Then t has at least n principal complements.*

Proof. Steiner [7] has proved the lemma for the case $n = 1$. Suppose that $n \geqslant 2$ and assume without loss of generality that $X_2 \neq \emptyset$. Let $\{t_1^j: j \in J \text{ and } |J| = n\}$ be a set of n distinct principal complements for t_1; let t_2' be a principal complement for t_2. Such exists by Steiner's Theorem ([7], p. 397). Define n distinct principal complements for t: t^j, $j \in J$ as follows. First let $t_1^j * t_2' = \{U \cup V: U \in t_1^j \text{ and } V \in t_2'\}$. Next fix $x_i \in X_i$, $i = 1, 2$.

Case 1. $X_i \notin t$ *for* $i = 1, 2$. Let $t^j = t_1^j * t_2^j$.
Case 2. $X_1 \in t$ *and* $X_2 \notin t$.
Let $t^j = (t_1^j * t_2^j) \wedge S(x_1, \mathfrak{U}(x_2))$.
Case 3. $X_1 \notin t$ *and* $X_2 \in t$.
Let $t^j = (t_1^j * t_2^j) \wedge S(x_2, \mathfrak{U}(x_1))$.
Case 4. $X_i \in t$ *for* $i = 1, 2$.
Let $t^j = (t_1^j * t_2^j) \wedge S(x_1, \mathfrak{U}(x_2)) \wedge S(x_2, \mathfrak{U}(x_1))$.

Clearly t^j is a principal topology for each $j \in J$. The straightforward verification that t^j is a complement for t is omitted. (The interested reader will find a proof of this fact in Steiner [7], pp. 392-393. Note that she only considered the case where $n = 1$. The formulae above somewhat simplify the description of the topologies involved in the various cases.)

The map which assigns to each $j \in J$ the principal complement t^j is one-to-one. For suppose that $i, j \in J$ with $i \neq j$ so that $t_1^i \neq t_2^j$. Then there exists a set U with, say, $U \in t_1^i \setminus t_1^j$. In Cases 1 and 3, $U \in t^i \setminus t^j$. In Cases 2 and 4, if $x_1 \notin U$, then $U \in t^i \setminus t^j$; if $x_1 \in U$, then $U \cup X_2 \in t^i \setminus t^j$. Hence, $t^i \neq t^j$.

Remark. It is clear from the proof that the lemma remains valid even if n is an infinite cardinal.

Theorems 2 and 4 enable one to concentrate on non-T_1 topologies on infinite sets. The idea of the proof of Theorem 5 is to show that in a proper non-T_1 infinite space there always exists an arbitrarily large finite proper subspace X_1 (proper with the relative topology) and then to apply the previous lemma.

THEOREM 5. *If X is infinite, then every proper topology on X has infinitely many principal complements.*

Proof. By the corollary to Theorem 2 it may be assumed that the given topology t is not T_1.

Suppose that X has a maximal T_1 subspace M with at least two points. Let $n \in X \setminus M$. Since $M \cup \{n\}$ is not T_1, there exists $m \in M$ such that either (i) $m \in \mathrm{Cl}(\{n\})$ or (ii) $n \in \mathrm{Cl}(\{m\})$. In either case, let $X_1 = \{m, m', n\}$, where $m' \in M \setminus \{m\}$. Then, $t_1 = t | X_1$ is a proper topology on X_1. (In case (i) $\{m, n\}$ is contained in $\mathrm{Cl}(\{n\}) \cap X_1$, so t_1 is not discrete; and since $m' \notin \mathrm{Cl}(\{m\}) \cap X_1$, then t_1 is not trivial. In case (ii) $\mathrm{Cl}(\{m\}) \cap X_1 = \{m, n\}$, so t_1 is neither trivial nor discrete.)

On the other hand, if every maximal T_1 subspace of X is a singleton, let $W \in t$ with $\emptyset \neq W \neq X$. Let X_1 consist of three distinct points such that $X_1 \cap W \neq \emptyset \neq X_1 \setminus W$. Then $t_1 = t | X_1$ is proper. ($X_1 \cap W$ is a proper t_1-open subset of X_1, so t_1 is not trivial; t_1 is not T_1, hence not discrete.)

Now given $k \geq 2$ any subset of X consisting of $k+1$ points and containing X_1 as a subset forms a proper subspace in the relative topology.

Designating this set again by X_1 and applying the previous lemma it follows that t has at least k complements. Since k is arbitrary, the theorem is proved.

Added in proof: The author has solved the problem of determining the cardinality of the set of complements (resp., principal complements) for a proper topology on an infinite set X. The solution will appear in a forthcoming paper.

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Reçu par la Rédaction le 17. 2. 1967