is a continuous non-decreasing function on I with $\varphi_1(0) = \varphi_2(0) = 0$, then the set $\{\varphi_1(x) - \varphi_2(x): x \in E\}$, transformed affinely to lie in I, has property (5). By giving an example of a Hamel basis with property (1), the author has shown [1] that if E has property (1), then $\{x - y: x, y \in E\}$ need not have property (5).

In order to show that (3) does not imply (1), it suffices to observe that N. Lusin's argument [3] for the existence of an uncountable set E with property (3) works in a Cantor set C contained in I with the relative topology: there is an uncountable subset E of C such that if $\{y_i\}$ is a countable dense subset of C and O_i is an open set containing y_i, $i = 1, 2, \ldots$, then $\operatorname{card}(E - \bigcup_i O_i) \leqslant \aleph_0$.

The purpose of this note is to construct an example to show that (5) does not imply (3). In our construction we need to assume the continuum hypothesis.

Construction. Let $\{x^a\}$ and $\{\mu_a\}$ be well orderings of the sequences $x = \{x_i\}$ of elements of I and the (non-trivial) non-atomic Baire measures μ on I, where each a has countably many predecessors. Let F_a be a first category F-sigma which supports μ_a so that $M_a = \bigcup_{\beta \leqslant a} F_\beta$ is a first category F-sigma. Denote by V the set of non-negative non-atomic Baire measures ν on I such that $\nu(I) = 1$. Let J_0 be a Cantor set in $I - M_0$, let ν_0 be an element of V which lives on J_0, and let $N^0 = \{N_i^0\}$ be a sequence of segments such that $x_i^0 \in N_i^0$ and $\sum_i \nu_0(N_i^0) < 1$. There is a Cantor set K_0 in $J_0 - \bigcup_i N_i^0$. Let S_0 be an uncountable subset of K_0 which satisfies (1) with respect to the space K_0. Suppose that J_β, ν_β, N^β, K_β, and S_β have been obtained for $\beta < a$ such that J_β is a Cantor set in $I - M_\beta \cup (\bigcup_{\gamma < \beta} K_\gamma)$, ν_β is an element of V which lives on J_β, N^β is a sequence of segments N_i^β such that $x_i^\beta \in N_i^\beta$ and $\sum_i \nu_\beta(N_i^\beta) < 1$, K_β is a Cantor set in $J_\beta - \bigcup_i N_i^\beta$, and S_β is an uncountable subset of K_β which satisfies (1) with respect to the space K_β. Then it is clear how to obtain S_a. Let $E = \bigcup_a S_a$. If $x = x^a$, then $S_a \subset (I - \bigcup_i N_i^a)$. If $\mu = \mu_a$, then $\bigcup_{\beta \geqslant a} S_\beta$ is a subset of $I - F_\beta$, which is a set of μ_a measure zero, and $\mu_a(S_\beta) = 0$ for each β and, hence, $\mu(E) = 0$.

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Polynomial factors of light mappings on an arc

by

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Introduction. In this paper we characterize light mappings of an arc onto an arc by a factorization property. It is shown that a mapping of an arc onto an arc is light if and only if it is topologically equivalent to a real valued continuous function f of $[0, 1]$ onto $[0, 1]$ such that f can be factored $f = Pg = P(g)$ where P is a polynomial and g is arbitrarily near the identity. Only techniques of classical real variables are employed.

DEFINITION 1. If f is a mapping (continuous function), then f is *light* if and only if for each x in the range of f, $\overset{-1}{f}(x)$ is totally disconnected.

The class of light real-valued continuous functions on an interval includes nowhere differentiable functions and nowhere monotone functions. The latter type was treated, for example, by Garg in [1]. There are also continuous functions f such that each inverse set $\overset{-1}{f}(x)$ is a Cantor set. An interesting example of a function having all three of the properties just mentioned was described by Jolly in [2].

DEFINITION 2. If f is a continuous function of $[a, b]$ onto $[c, d]$, then $f = f_1 f_2$ is a *factorization* of f, which means that there exists an interval $[a', b']$ such that f_2 is a continuous function of $[a, b]$ onto $[a', b']$ and f_1 is a continuous function of $[a', b']$ onto $[c, d]$ and for each $x \in [a, b]$,

$$f(x) = f_1(f_2(x)).$$

THEOREM 1. *If f is a continuous light function of $[a, b]$ onto $[c, d]$ and $\varepsilon > 0$, there exists a factorization $f = Pg$ such that P is a polynomial of $[a, b]$ onto $[c, d]$ and g is a continuous function of $[a, b]$ onto $[a, b]$ such that*

$$|g(x) - x| < \varepsilon \quad \text{for all } x \in [a, b].$$

We will first establish four lemmas.

DEFINITION 3. If f is a continuous function, $V(f) = \{t$: there exists an open interval Ω containing t such that $f(x) - f(t)$ does not change sign on $\Omega \cap [t, \infty) \cap$ domain of f or on $\Omega \cap (-\infty, t] \cap$ domain of $f\}$.

LEMMA 1. *If f is a continuous function on $[a, b]$, then $V(f)$ is dense in $[a, b]$.*

Proof. Suppose that Ω is an open interval lying in the open interval (a, b). If f is monotone on Ω, then $\Omega \subset V(f)$. Suppose that f is not monotone on Ω. There exist $c, d \in \Omega$ such that $c < d$ and $f(c) = f(d)$. f must have a relative maximum or a relative minimum between c and d and such a point belongs to $V(f)$.

LEMMA 2. *If f is a continuous light function on $[a, b]$ and $a, b \in V(f)$, then there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that*

$$f([x_{i-1}, x_i]) = [\min\{f(x_{i-1}), f(x_i)\}, \max\{f(x_{i-1}), f(x_i)\}], \quad i = 1, 2, \ldots, n.$$

Proof. Let $a_0 \in f^{-1}\big(\max_{a \leqslant x \leqslant b} f(x)\big)$. The sequence $\{a_i\}$ is defined by the following relations: If i is odd,

$$a_{i+1} = \text{least element of } f^{-1}\big(\min_{a \leqslant x \leqslant a_1} f(x)\big).$$

If i is even,

$$a_{i+1} = \text{least element of } f^{-1}\big(\max_{a \leqslant x \leqslant a_i} f(x)\big).$$

Now $a_0 \geqslant a_1 \geqslant a_2 \geqslant \ldots \to c \geqslant a$ for some number c and so $f(a_0), f(a_1), f(a_2), \ldots \to f(c)$. From the way $\{a_i\}$ is defined it follows that $f(x) = f(a)$ for all $a \leqslant x \leqslant c$. Since f is light, $c = a$.

Because $a \in V(f)$, there exists $\delta > 0$ such that $f(x) - f(a)$ does not change sign in $[a, a+\delta]$. There exists $n \geqslant 0$ such that $a \leqslant a_{n+1} \leqslant a_n < a + \delta$. From the way $\{a_i\}$ is defined it follows that either $f(a_n)$ or $f(a_{n+1}) = f(a)$ and therefore either $a_n = a$ or $a_{n+1} = a$. So finally we have a finite sequence $a_0 \geqslant a_1 \geqslant \ldots \geqslant a_j = a$ and in fact $a_0 > a_1 > \ldots > a_j = a$ since f is light.

We can similarly obtain a finite sequence $a_0 < b_1 < b_2 < \ldots < b_k = b$. The composite of the two sequences $a = a_j < \ldots < a_0 < b_1 < \ldots < b_k = b$ forms the required partition.

LEMMA 3. *If f is a continuous light function on $[a, b]$ and $\varepsilon > 0$, there exists a partition $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ of mesh $< \varepsilon$ such that*

$$f([x_{i-1}, x_i]) = [\min\{f(x_{i-1}), f(x_i)\}, \max\{f(x_{i-1}), f(x_i)\}], \quad i = 2, 3, \ldots, n-1,$$

$$f(x_1) = \textit{either} \max_{a \leqslant x \leqslant x_1} f(x) \textit{ or } \min_{a \leqslant x \leqslant x_1} f(x)$$

and

$$f(x_{n-1}) = \textit{either} \max_{x_{n-1} \leqslant x \leqslant b} f(x) \textit{ or } \min_{x_{n-1} \leqslant x \leqslant b} f(x).$$

Proof. By Lemma 1, there exists a partition $a = y_0 < y_1 < \ldots < y_k = b$ such that $y_i - y_{i-1} < \varepsilon$ $(i = 1, 2, \ldots, k)$ and $y_i \in V(f)$ $(i = 1, 2, \ldots, k-1)$. Let x_1 be a number such that $a \leqslant x_1 \leqslant y_1$ and

$$|f(x_1) - f(a)| = \max_{0 \leqslant x \leqslant y_1} |f(x) - f(a)|.$$

Because of the lightness of f, $x_1 > a$. If $x_1 = y_1$, then no further partition of $[a, y_1]$ is necessary. If $x_1 < y_1$, then $x_1 \in V(f)$ and

$$f(x_1) = \text{ either } \max_{a \leqslant x \leqslant x_1} f(x) \text{ or } \min_{a \leqslant x \leqslant x_1} f(x).$$

We now similarly choose a number z from $[y_{n-1}, b]$ so that $z \in V(f)$ and

$$f(z) = \text{ either } \max_{z \leqslant x \leqslant b} f(x) \text{ or } \min_{z \leqslant x \leqslant b} f(x).$$

Now we partition each of the intervals $[x_1, y_1]$, $[y_{n-1}, z]$ and $[y_{i-1}, y_i]$ $i = 2, 3, \ldots, n-1$ according to Lemma 2. The refinement thus obtained is a partition with the required properties.

LEMMA 4. *Suppose that n is a positive integer, $x_{i-1} < x_i$ and $y_{i-1} \neq y_i$ $(i = 1, 2, \ldots, n)$. Then there exists a polynomial P such that $P(x_i) = y_i$ $(i = 0, 1, \ldots, n)$ and P is monotone in each of the intervals $[x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$.*

This was proved in [4].

Proof of Theorem 1. Let $a = x_0 < x_1 < \ldots < x_n = b$ be a partition as in Lemma 3. $y_0 =$ opposite end of the interval $f([a, x_1])$ from $f(x_1)$. $y_n =$ opposite end of the interval $f([x_{n-1}, b])$ from $f(x_{n-1})$. $y_i = f(x_i)$ $(i = 1, 2, \ldots, n-1)$. Now let P be a polynomial as in Lemma 4. Let P_i denote the contraction of P to the interval $[x_{i-1}, x_i|$ $(i = 1, 2, \ldots, n)$. The function g is defined to be $g(x) = P_i^{-1} f(x)$ for all $x_{i-1} \leqslant x \leqslant x_i$ $(i = 1, 2, \ldots, n)$. g is consistently defined and continuous because $g(x_i) = P_i^{-1} f(x_i) = P_i^{-1}(y_i) = x_i$ and $g(x_i) = P_{i+1}^{-1} f(x_i) = P_{i+1}^{-1}(y_i) = x_i$ $(i = 1, 2, \ldots, n-1)$. P is onto $[c, d]$ since one of the numbers $\{y_i\}$ must be equal to c and one of them must be equal to d. g is onto $[a, b]$ since $g([x_{i-1}, x_i]) = P_i^{-1} f([x_{i-1}, x_i]) = P_i^{-1}([y_{i-1}, y_i]) = [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$. Also if $x_{i-1} \leqslant x \leqslant x_i$, then $x_{i-1} \leqslant g(x) \leqslant x_i$ and so $|g(x) - x| < \varepsilon$. Finally, if $x_{i-1} \leqslant x \leqslant x_i$, then

$$P\big(g(x)\big) = P\Big(P_i^{-1}\big(f(x)\big)\Big) = P_i\Big(P_i^{-1}\big(f(x)\big)\Big) = f(x)$$

and the theorem is proved.

We will consider now the converse of Theorem 1. Suppose that f is a continuous function of $[a, b]$ onto $[c, d]$, $c < d$, which is not light. Then there exists a $z \in [c, d]$ such that $\overset{-1}{f}(z)$ contains a nondegenerate closed interval i. Suppose furthermore that there exists a factorization

$f = Pg$ as in Theorem 1 with $|g(x)-x| < 1/2$ length of i for all $x \in [a, b]$. $i \subset \overset{-1}{f}(z) = \overset{-1}{g}\,\overset{-1}{P}(z)$. Since $\overset{-1}{P}(z)$ is finite, then $i \subset \overset{-1}{g}(z_0)$ for some $z_0 \in [a, b]$. Therefore there exists an $x_0 \in i$ such that $|g(x_0)-x_0| \geqslant 1/2$ length of i. A contradiction has been reached and so the converse of Theorem 1 is proved.

We can now state Theorem 1 together with its converse and do so in slightly more general terms.

THEOREM 2. *If T is a mapping of an arc onto a non-degenerate arc, then T is light if and only if T is topologically equivalent to a mapping f of $[0, 1]$ onto $[0, 1]$ such that if $\varepsilon > 0$, there exists a factorization $f = Pg$ where P is a polynomial of $[0, 1]$ onto $[0, 1]$ and g is a mapping of $[0, 1]$ onto $[0, 1]$ such that*

$$|g(x)-x| < \varepsilon \quad \textit{for all } x \in [0, 1].$$

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Multiple complementation in the lattice of topologies*

by

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1. Introduction. Hartmanis [4] showed that in the lattice of all topologies on a finite set with at least three elements every proper (i.e., neither discrete nor trivial) topology has at least two complements. In the light of Steiner's result [7] that the lattice of topologies on an arbitrary set is complemented, the question of Berri [1] in this journal may be rephrased as follows. Does every proper topology on an infinite set have at least two complements? This paper answers the question affirmatively. Further evidence of the pathological nature of the lattice of topologies is the result that a non-discrete T_1 topology never possesses a maximal complement (or a maximal principal complement). The result of Hartmanis above is sharpened. It is shown that every proper topology on a finite set with $n \geqslant 2$ elements has at least $n-1$ complements. Finally, utilizing these results it is shown that every proper topology on an infinite set actually has infinitely many principal complements.

2. Basic facts. The paper of Steiner [7] provides an ideal reference on the background material for this paper. It is possible to quickly outline the basic facts needed here. If (X, t) is a topological space on the set X, then the topology t consists of the open sets. (Note: Hartmanis [4] considers the closed sets.) If t_1 and t_2 are topologies on X and t_1 is a subset of t_2, then $t_1 \leqslant t_2$ and under this partial order the set of all topologies, Σ, on a fixed set X is a complete lattice with greatest element 1, the discrete topology, and least element 0, the trivial topology. If $t, t' \in \Sigma$ and $t \vee t' = 1$ while $t \wedge t' = 0$, then t' is a *complement* for t.

A maximal proper topology is an *ultraspace*. Given a filter $\mathcal{F}$ on X and a fixed point $x \in X$ one can define a topology $S(x, \mathcal{F}) = \{A \subset X\colon x \in A \Rightarrow A \in \mathcal{F}\}$. A filter $\mathfrak{U}$ on X with the property that $A \cup B \in \mathfrak{U}$ implies $A \in \mathfrak{U}$ or $B \in \mathfrak{U}$ is an *ultrafilter*. An ultrafilter of the form $\mathfrak{U} = \{U \subset X\colon p \in U\}$ is *principal* and denoted $\mathfrak{U}(p)$. An ultrafilter on X is principal

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