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Added in proof: It has come to the attention of the author that theorem 2 in Section 3 already has been proved by Arhangel'skiĭ in [8]. Our treatment of the subject is, however, entirely different from Arhangel'skiĭ's and contains other viewpoints.

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On distributive n-lattices and n-quasilattices

by

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0. In this paper we give a representation theorem for a class of abstract algebras (which we shall call *distributive n-quasilattices*), having n binary fundamental operations $o_1, \ldots, o_n$, which are idempotent, commutative, associative and distributive with respect to each other. A distributive n-quasilattice will be called a *distributive n-lattice*, if it satisfies moreover formula (5) below, which generalizes the familiar absorption law for lattices.

We shall show that every distributive n-lattice can be treated as a subalgebra of an algebra defined in a natural way in a product of distributive lattices, and every distributive n-quasilattice can be represented as a sum of a direct system (see [2]) of distributive n-lattices.

1. We shall call a *distributive n-quasilattice* every abstract algebra $Q = (X; o_1, \ldots, o_n)$ where $n \geq 2$ and $o_1, \ldots, o_n$ are binary operations which satisfy the following four conditions:

(1) $$x\, o_i\, x = x\,,$$

(2) $$x\, o_i\, y = y\, o_i\, x\,,$$

(3) $$(x\, o_i\, y)\, o_i\, z = x\, o_i\, (y\, o_i\, z)\,,$$

(4) $$(x\, o_i\, y)\, o_j\, z = (x\, o_j\, z)\, o_i\, (y\, o_j\, z)$$

$(i, j = 1, 2, \ldots, n)$.

A distributive n-quasilattice we shall call a *distributive n-lattice* if it satisfies moreover the following equality:

(5) $$x\, o_1 \big(x\, o_2 \big(\ldots x\, o_{n-1} (x\, o_n\, y) \ldots\big)\big) = x\,.$$

It is easy to see that in the case $n = 2$ a distributive n-lattice is a distributive lattice, and equation (5) coincides with the law of absorption. Similarly, a distributive n-quasilattice in the case $n = 2$ is a distributive quasilattice, as defined in [1].

Examples. 1. Let $X = \{a_1, a_2, \ldots, a_n, 0\}$ and let us define for $i = 1, 2, \ldots, n$ the operations o_i as follows: $x\, o_i\, x = x$, $x\, o_i\, a_i = a_i\, o_i\, x = a_i$, and $x o_i y = 0$ in all remaining cases. It is trivial to check that the algebra $(X; o_1, \ldots, o_n)$ is a distributive n-lattice.

2. Let X be the real n-space, and define, for $i = 1, 2, \ldots, n$,

$$[x_1, \ldots, x_n]\, o_i\, [y_1, \ldots, y_n]$$
$$= [\max(x_1, y_1), \ldots, \max(x_{i-1}, y_{i-1}), \min(x_i, y_i), \max(x_{i+1}, y_{i+1}), \ldots, \max(x_n, y_n)] .$$

Again it is easy to check that $(X; o_1, \ldots, o_n)$ is a distributive n-lattice.

Now we shall develop some properties of distributive n-quasilattices and n-lattices, which will be needed in the proof of the representation theorem.

LEMMA 1. *In a distributive n-quasilattice the following equalities hold*:

(i) $$(x o_i y) o_i (x o_j y) = x o_i y ,$$

(ii) $$x o_i (x o_j y) o_i (y o_j z) = x o_i (y o_j z),$$

(iii) $$x o_i (x o_j y) o_i (x o_j y o_j z) = x o_i (x o_j y o_j z) .$$

Proof. By (1)-(4) we have

$$x o_i y = (x o_i y) o_j (x o_i y) = x o_i (x o_j y) o_i (y o_j x) o_i y = x o_i y o_i (x o_j y) ,$$

proving (i).

Similarly, we get

$$x o_i (y o_j z) = (x o_i y) o_j (x o_i z) = x o_i (x o_j y) o_i (x o_j z) o_i (y o_j z) ,$$

hence

$$x o_i (x o_j y) o_i (y o_j z)$$
$$= x o_i (y o_j z) o_i (x o_j y) = \big(x o_i (x o_j y) o_i (x o_j z) o_i (y o_j z)\big)\, o_i (x o_j y)$$
$$= x o_i (x o_j y) o_i (x o_j z)\, o_i (y o_j z) = x o_i (y o_j z) = x o_i (x o_j y) o_i (y o_j z) ,$$

proving (ii). Equality (iii) follows from (ii) by the substitution $y = x o_j y$.

For shortness let us write

$$f_{i_1, i_2, \ldots, i_k}(x, y) = x o_{i_1}\big(x o_{i_2}(\ldots x o_{i_{k-1}}(x o_{i_k} y) \ldots\big)$$

$(1 \leqslant i_1, \ldots, i_k \leqslant n$ and all i_j's are distinct).

Observe that for every permutation $p_1, \ldots, p_k$ of the letters $1, 2, \ldots, k$ we have

(iv) $$f_{i_1, \ldots, i_k}(x, y) = f_{i_{p_1}, \ldots, i_{p_k}}(x, y)$$

and, moreover,

(v) $$f_{i_1, \ldots, i_k}(x, y o_j z) = f_{i_1, \ldots, i_k}(x, y) o_j f_{i_1, \ldots, i_k}(x, z) .$$

LEMMA 2. *Every operation $f_{i_1, \ldots, i_k}(x, y)$ is associative.*

Proof. At first we prove the formula

(6) $$f_{i_1, \ldots, i_k}\big(f_{i_1, \ldots, i_k}(x, y), z\big) = x o_{i_1}\big(x o_{i_2}(\ldots (x o_{i_k} y o_{i_k} z) \ldots)\big) .$$

In the case $k = 2$ we have

$$f_{i_1, i_2}\big(f_{i_1, i_2}(x, y), z\big)$$
$$= \big(x o_{i_1}(x o_{i_2} y)\big)\, o_{i_1}\Big(\big(x o_{i_1} (x o_{i_2} y)\big)\, o^{i_2} z\Big)$$
$$= x o_{i_1} (x o_{i_2} y) o_{i_1} (x o_{i_2} z) o_{i_1}(x o_{i_2} z) o_{i_1} (x o_{i_2} y o_{i_2} z)$$
$$= x o_{i_1}(x o_{i_2} y o_{i_2} z)$$

(the last equality here follows from (iii)). Assume now that (6) holds for all sequences of indices with the length $\leqslant k-1$. Then, with the abbreviations $I = [i_1, \ldots, i_k]$, $J = [i_2, \ldots, i_k]$ and $K = [i_1, \ldots, i_{k-1}]$ we have

$$f_I\big(f_I(x, y), z\big) = f_I(x, y) o_{i_1}\Big(f_I(x, y) o_{i_2}\big(\ldots(f_I(x, y) o_{i_k} z) \ldots\big)$$
$$= f_I(x, y) o_{i_k} \Big(f_I(x, y) o_{i_2}\big(\ldots(f_I(x, y) o_{i_1} z) \cdots\big)$$
$$= \big(x o_{i_1} f_J(x, y)\big) o_{i_k} \big(x o_{i_1} f_J(x, y)\big) o_{i_2}\big(\ldots(x o_{i_1} f_J(x, y)\, o_{i_1} z) \ldots\big)$$
$$= x o_{i_1} \Big(f_J(x, y) o_{i_k}\Big(f_J(x, y) o_{i_2} \big(\ldots(f_J(x, y) o_{i_1} z) \ldots\big)$$
$$= x o_{i_1}\Big(f_J(x, y) o_{i_1}\Big(f_J(x, y) o_{i_2} \big(\ldots(f_J (x, y) o_{i_k} z) \ldots\big)$$
$$= x o_{i_1} f_J(x, y) o_{i_1} f_J(x, y o_{i_k} z) = x o_{i_1} f_J(x, y) o_{i_1} x o_{i_1} f_J(x, y o_{i_k} z)$$
$$= f_I(x, y) o_{i_1} f_I(x, y o_{i_k} z) = f_K(x, x o_{i_k} y) o_{i_1} f_K(x, x o_{i_k} y o_{i_k} z)$$
$$= f_K\big(x, (x o_{i_k} y) o_{i_1} (x o_{i_k} y o_{i_k} z)\big)$$
$$= f_{i_{k-1}, i_2, \ldots, i_1}\big(x, (x o_{i_k} y) o_{i_1} (x o_{i_k} y o_{i_k} z)\big)$$
$$= f_{i_{k-1}, i_2 \ldots, i_{k-2}}\big(x, x o_{i_1} (x o_{i_k} y) o_{i_1} (x o_{i_k} y o_{i_k} z)\big)$$
$$= f_{i_{k-1}, i_2 \ldots i_{k-2}}\big(x, x o_{i_1} (x o_{i_k} y o_{i_k} z)\big)$$
$$= f_{i_{k-1}, i_2, \ldots, i_{k-2}, i_1}(x, x o_{i_k} y o_{i_k} z)$$
$$= f_{i_{k-1}, i_2, i_{k-2}, i_1, i_k}(x, y o_{i_k} z) = f_{i_1, \ldots i_k}(x, y o_{i_k} z) ,$$

thus proving (6).

Now we shall prove the formula

(7) $$f_I\big(x, f_I(y, z)\big) = x o_{i_1}\big(x o_{i_2} (\ldots (x o_{i_k} y o_{i_k} z) \ldots\big)$$

(where we preserve the abbreviations used above), which together with (6) gives the desired associativity. In the case $k = 2$ we have

$$f_{i_1, i_2}\big(x, f_{i_1, i_2}(y, z)\big) = x o_{i_1}\Big(x o_{i_2}\big(y o_{i_1} (y o_{i_2} z)\big)\Big)$$
$$= x o_{i_1} (x o_{i_2} y) o_{i_1} (x o_{i_2} y o_{i_2} z) = x o_{i_1} (x o_{i_2} y o_{i_2} z) ,$$

as needed. Assume now that (7) holds for all sequences of indices with the length not exceeding $k-1$. Then, using (v) and the inductional assumption, we have

$$f_I(x, f_I(y, z)) = x\, o_{i_1} \big(f_J(x, y)\, o_{i_1} f_J(x, f_J(y, z))\big) = x\, o_{i_1} f_J(x, y)\, o_{i_1} f_J(x, y\, o_{i_k} z)$$

and this equals $f_I(x, y\, o_{i_k} z)$ as we have just seen in the proof of (6).

Observe that (6) and (7) imply

$$\text{(vi)} \qquad f_I(f_I(x, y), z) = f_I(f_I(x, z), y)$$

for every sequence I of indices.

Lemma 3. *For every sequence I of indices, and every $i = 1, \ldots, n$ we have*

$$f_I(x\, o_i\, y, z) = f_I(x, z)\, o_i\, f_I(y, z)\,.$$

Proof. We use induction with respect to the length of I. If it is equal to 2, then we have (with $I = (i_1, i_2)$)

$$f_I(x\, o_i\, y, z) = (x\, o_i\, y)\, o_{i_1} \big((x\, o_i\, y)\, o_{i_2} z\big)$$
$$= (x\, o_i\, y)\, o_{i_1} \big((x\, o_{i_2} z)\, o_i\, (y\, o_{i_2} z)\big)$$
$$= \big(x\, o_{i_1} (x\, o_{i_2} z)\big)\, o_i \big(x\, o_{i_1} (y\, o_{i_2} z)\big)\, o_i \big(y\, o_{i_1} (x\, o_{i_2} z)\, o_{i_1} \big(y\, o_{i_1} (y\, o_{i_2} z)\big)\big)$$
$$= \big(x\, o_{i_1} (x\, o_{i_2} z)\big)\, o_i \big(x\, o_{i_1} (x\, o_{i_2} z)\, o_{i_1} (y\, o_{i_2} z)\big)\, o_i \big(y\, o_{i_1} (y\, o_{i_2} z)\, o_{i_1} (x\, o_{i_2} z)\big)\, o_i$$
$$o_i \big(y\, o_{i_1} (y\, o_{i_2} z)\big)\, o_i \big(x\, o_{i_1} y\, o_{i_1} (x\, o_{i_2} z)\, o_{i_1} (y\, o_{i_2} z)\big)$$
$$= \big(x\, o_{i_1} (x\, o_{i_2} z)\big)\, o_i \big(y\, o_{i_1} (y\, o_{i_2} z)\big)\, o_i \big(x\, o_{i_1} y\, o_{i_1} (x\, o_{i_2} z)\, o_{i_1} (y\, o_{i_2} z)\big)$$
$$= \big(x\, o_{i_1} (x\, o_{i_2} z)\big)\, o_i \big(y\, o_{i_1} (y\, o_{i_2} z)\big) = f_I(x, z)\, o_i\, f_I(y, z)\,,$$

as needed.

Now assume that the lemma is true for all sequences of indices with the length not exceeding $k-1$, and let $I = [i_1, \ldots, i_k]$. Then we have with $J = [i_2, \ldots, i_k]$ and $K = [i_2, \ldots, i_{k-1}]$)

$$\text{(8)} \quad f_I(x\, o_i\, y, z) = (x\, o_i\, y)\, o_{i_1} f_J(x\, o_i\, y, z)$$
$$= (x\, o_i\, y)\, o_{i_1} \big(f_J(x, z)\, o_i\, f_J(y, z)\big)$$
$$= \big(x\, o_{i_1} f_J(x, z)\big)\, o_i \big(x\, o_{i_1} f_J(y, z)\big)\, o_i \big(y\, o_{i_1} f_J(x, z)\big)\, o_i \big(y\, o_{i_1} f_J(y, z)\big)\,.$$

The inductional assumption implies

$$\text{(9)} \qquad x\, o_{i_1} f_J(y, z) = x\, o_{i_1} f_J(x, z)\, o_{i_1} f_J(y, z)\,,$$

because

$$x\, o_{i_1} f_J(x, z)\, o_{i_1} f_J(y, z) = x\, o_{i_1} f_J(x\, o_{i_1} y, z)$$
$$= x\, o_{i_1} f_K\big(x\, o_{i_1} y, (x\, o_{i_k} z)\, o_{i_1} (y\, o_{i_k} z)\big)$$
$$= f_K\big(x\, o_{i_1} y, x\, o_{i_1} (x\, o_{i_k} z)\, o_{i_1} (y\, o_{i_k} z)\big)$$
$$= f_K\big(x\, o_{i_1} y, x\, o_{i_1} (y\, o_{i_k} z)\big)$$
$$= x\, o_{i_1} f_J(y, z)\,.$$

From (8) and (9) we get

$$f_I(x\, o_i\, y, z) =$$
$$= \big(x\, o_{i_1} f_J(x, z)\big)\, o_i \big(x\, o_{i_1} f_J(y, z)\big)\, o_i \big(y\, o_{i_1} f_J(x, z)\big)\, o_i \big(y\, o_{i_1} f_J(y, z)\big)$$
$$= \big(x\, o_{i_1} f_J(x, z)\big)\, o_i \big(x\, o_{i_1} f_J(x, z)\, o_{i_1} f_J(y, z)\big)\, o_i \big(y\, o_{i_1} f_J(x, z)\, o_{i_1} f_J(y, z)\big)\, o_i$$
$$o_i \big(y\, o_{i_1} f_J(y, z)\big)\, o_i \big(x\, o_{i_1} y\, o_{i_1} f_J(x, z)\, o_{i_1} f_J(y, z)\big)$$
$$= \big(x\, o_{i_1} f_J(x, z)\big)\, o_i \big(y\, o_{i_1} f_J(y, z)\big)\, o_i \big(x\, o_{i_1} y\, o_{i_1} f_J(x, z)\, o_{i_1} f_J(y, z)\big)$$
$$= \big(x\, o_{i_1} f_J(x, z)\big)\, o_i \big(y\, o_{i_1} f_J(y, z)\big) = f_I(x, z)\, o_i\, f_I(y, z)\,,$$

thus proving the lemma.

Lemma 4. $f_{1,2,\ldots,n}(x\, o_i\, y, x) = x\, o_i\, y$ *and* $f_{1,2,\ldots n}(x\, o_i\, y, y) = x\, o_i\, y$ $(i = 1, 2, \ldots, n)$.

Proof. We have

$$f_{1,\ldots,n}(x\, o_i\, y, x) = f_{1,\ldots,i-1,i+1,\ldots,n,i}(x\, o_i\, y, x)$$
$$= f_{1,\ldots,i-1,i+1,\ldots,n}\big(x\, o_i\, y, (x\, o_i\, y)\, o_i\, x\big)$$
$$= f_{1,\ldots,i-1,i+1,\ldots,n}(x\, o_i\, y, x\, o_i\, y) = x\, o_i\, y\,.$$

The proof of the second formula is similar.

Lemma 5. *If for some elements x, y, u, v of a distributive n-quasi-lattice Q and some $i, j \leqslant n$ we have $f_{i,j}(x, y) = x$ and $f_{i,j}(u, v) = u$, then*

$$f_{i,j}(x\, o_k\, u, y\, o_k\, v) = x\, o_k\, u \quad \textit{for} \quad k = 1, 2, \ldots, n\,.$$

Proof. By lemma 3 we have

$$f_{i,j}(x\, o_k\, u, y\, o_k\, v) = (x\, o_k\, u)\, o_i \big((x\, o_k\, u)\, o_j\, (y\, o_k\, v)\big)$$
$$= \big(x\, o_i\, (x\, o_j\, y)\big)\, o_k \big(x\, o_i\, (x\, o_j\, v)\big)\, o_k \big(u\, o_i\, (u\, o_j\, y)\big)\, o_k \big(u\, o_i\, (u\, o_j\, v)\big)$$
$$= \big(x\, o_i (x\, o_j\, y)\big)\, o_k \big(x\, o_i (x\, o_j\, v)\big)\, o_k \big(u\, o_i\, (u\, o_j\, y)\, o_k \big(u\, o_j\, (u\, o_j\, v)\big)\, o_k$$
$$o_k \big(x\, o_i\, (x\, o_j\, v)\big)\, o_i\, (u\, o_j\, v)\, o_i\, u\, o_i\, (u\, o_j\, y)\, o_i\, (x\, o_j\, y)$$
$$= x\, o_k \big(x\, o_i (x\, o_j v)\big)\, o_k \big(u\, o_i (u\, o_j y)\big)\, o_k\, u\, o_k \big(x\, o_i (x\, o_j v)\, o_i (u\, o_j v)\, o_i\, u\, o_i (u\, o_j y)\, o_i (x\, o_j y)\big)$$
$$= x\, o_k\, u\, o_k \big(x\, o_i\, (x\, o_j\, v)\, o_i\, (u\, o_j\, v)\, o_i\, u\, o_i\, (u\, o_j\, y)\, o_i\, (x\, o_j\, y)\big)$$
$$= x\, o_k\, u\, o_k \big(x\, o_i\, (x\, o_i\, y\, o_i\, u\, o_i\, (u\, o_j\, v)\big) = x\, o_k\, u\, o_k\, (x\, o_i\, u) = x\, o_k\, u\,.$$

LEMMA 6. *If for some elements x, y of an n-quasilattice Q and some i_1, i_2, i_3 we have*

$$f_{i_1,i_2}(x, y) = f_{i_1,i_3}(x, y) = f_{i_2,i_3}(x, y) = x\,,$$

then

$$x = (x\,o_{i_2}\,y)\,o_{i_1}\,(x\,o_{i_3}\,y)\,.$$

Proof. We have

$$\begin{aligned} x = x\,o_{i_1}(x\,o_{i_2}\,y) &= \big(x\,o_{i_1}(x\,o_{i_3}\,y)\big)\,o_{i_1}(x\,o_{i_2}\,y) \\ &= x\,o_{i_1}(x\,o_{i_3}\,y)\,o_{i_1}(x\,o_{i_2}\,y) \\ &= \big(x\,o_{i_2}(x\,o_{i_3}\,y)\big)\,o_{i_1}(x\,o_{i_3}\,y)\,o_{i_1}(x\,o_{i_2}\,y) \\ &= \big(x\,o_{i_2}(x\,o_{i_3}\,y)\big)\,o_{i_1}(x\,o_{i_3}\,y)\,o_{i_1}(x\,o_{i_2}\,y)\,o_{i_1}\big((x\,o_{i_2}\,y)\,o_{i_2}(x\,o_{i_3}\,y)\big) \\ &= (x\,o_{i_3}\,y)\,o_{i_1}(x\,o_{i_2}\,y)\,o_{i_1}\big((x\,o_{i_2}\,y)\,o_{i_2}(x\,o_{i_3}\,y)\big) \\ &= (x\,o_{i_2}\,y)\,o_{i_1}(x\,o_{i_3}\,y)\,. \end{aligned}$$

LEMMA 7. *The following equalities hold:*

$$f_{i,j}(x\,o_i\,y, x\,o_j\,y) = x\,o_i\,y\,, \qquad f_{i,j}(x\,o_j\,y, x\,o_i\,y) = x\,o_j\,y\,.$$

Proof. In view of (iv) it is enough to prove one of these equalities. We have

$$f_{i,j}(x\,o_i\,y, x\,o_j\,y) = (x\,o_i\,y)\,o_i\big((x\,o_i\,y)\,o_j\,(x\,o_j\,y)\big) = (x\,o_i\,y)\,o_i\,(x\,o_j\,y) = x\,o_i\,y.$$

2. In this section we prove the representation theorem for distributive n-quasilattices and distributive n-lattices. Let us denote by $\mathcal{N}$ the set $\{1, 2, \ldots, n\}$ and by $\mathfrak{J}$ the system of all subsets of $\mathcal{N}$ containing 1 and different from $\mathcal{N}$.

Clearly, $\mathfrak{J}$ has $m = 2^{n-1}-1$ elements, and we may number them in an arbitrary but fixed manner with numbers $1, 2, \ldots, m$. We define now an algebra $\mathfrak{P}$ as follows: the carrier of $\mathfrak{P}$ is equal to $\mathop{\mathrm{P}}_{I\in\mathfrak{J}} L^{(I)}$, where for each $I \in \mathfrak{J}$, $L^{(I)}$ is a distributive lattice, with the fundamental operations $\cup$ and $\cap$. The fundamental operations $o_1, \ldots, o_n$ of $\mathfrak{P}$ are all binary, and are defined by

$$[x_1, \ldots, x_m]\,o_i\,[y_1, \ldots, y_m] = [x_1\,\epsilon_1\,y_1, \ldots, x_m\,\epsilon_m\,y_m]\,,$$

where $x_k\,\epsilon_k\,y_k = x \cup y$ if $i \in I_k$ and $x_k\,\epsilon_k\,y_k = x \cap y$ otherwise. (Here I_k is the set with number k in $\mathfrak{J}$.)

We prove now the following

THEOREM I. *An algebra $\mathfrak{A} = (X; o_1, \ldots, o_n)$, $n \geqslant 2$, is a distributive n-lattice if and only if $\mathfrak{A}$ is isomorphic with a subalgebra of some algebra $\mathfrak{P}$, as defined above.*

Proof. The sufficiency is trivial. The necessity in the case $n = 2$ is trivial too. To prove the necessity for the case $n \geqslant 3$ we introduce $n(n-1)/2$ relations R_{ij} in the set X, defining: $xR_{ij}y$ if and only if $f_{i,j}(x, y) = x$ and $f_{i,j}(y, x) = y$ (for $i, j \in N$, $i \neq j$). Lemma 2 implies that every relation R_{ij} is an equivalence, and from lemma 5 it follows that the relation R_{ij} is an equivalence, and from lemma 5 it follows that the relations R_{ij} are congruences in A. Let $[x]_{ij}$ be the class (mod R_{ij}) determined by $x \in X$ and consider the mapping $x \to ([x]_{12}, [x]_{13}, \ldots, [x]_{n-1,n})$. This mapping is an imbedding, which follows from lemma 6, and it results that the algebra $\mathfrak{A}$ can be isomorphically imbedded in the product $\mathop{\mathrm{P}}_{i,j}(\mathfrak{A}/R_{ij})$. (Note that by lemma 7 the operations o_i and o_j coincide in $\mathfrak{A}/R_{ij}$, so we may under circumstances treat $\mathfrak{A}/R_{ij}$ as an $(n-1)$-lattice.) Now we may consider in every algebra $\mathfrak{A}/R_{ij}$ congruences R_{st} with $s, t \neq i$, and similarly as before, we may imbed $\mathfrak{A}/R_{ij}$ in the product $\mathop{\mathrm{P}}_{s,t}(\mathfrak{A}/R_{ij})/R_{st}$. Collecting these imbeddings together we obtain an imbedding of A into the product $\mathop{\mathrm{P}}_{i,j}\mathop{\mathrm{P}}_{s,t}(\mathfrak{A}/R_{ij})/R_{st}$. (Note that in the factor $(\mathfrak{A}/R_{ij})/R_{st}$ of this product $o_i = o_j$ and $o_s = o_t$.)

Proceeding in this way we finally obtain an imbedding of $\mathfrak{A}$ into a product L of distriburive n-lattices, in which the fundamental operations $o_i, \ldots, o_n$ can be partitioned in two classes, say $(o_{i_1}, \ldots, o_{i_k})$ and $(o_{i_{k+1}}, \ldots, o_{i_n})$ such that every two operations of the same class coincide.

Note that if $I = (i_1, \ldots, i_k) \subset N$, and $N \setminus I = (i_{k+1}, \ldots, i_n)$, then there exists a factor of L in which $o_{i_1} = \ldots = o_{i_k}$ and $o_{i_{k+1}} = \ldots = o_{i_n}$. In fact, the algebra

$$\big(\ldots(\mathfrak{A}/R_{i_1i_2})/R_{i_2i_3}\big)\ldots/R_{i_{k-1}i_k}/R_{i_{k+1}i_{k+2}}/\ldots/R_{i_{n-1},n}\big)$$

is clearly such a factor.

If $I = (i_1, \ldots, i_k)$ is a subset of N containing 1, then by L_I we shall denote the set of all factors of L such that $o_{i_j} = o_{i_1}$ $(j = 1, 2, \ldots, k)$ and $o_{i_j} = o_{i_n}$ $(j = k+1, k+2, \ldots, n)$. If now for $\mathfrak{B} \in L_I$ we define $x \cup y = x o_1 y$ and $x \cap y = x\,o_{i_n}\,y$, then we obtain an algbera with two fundamental binary operations. Observe that this algebra is a lattice. In fact, in view of (1)-(4) it is enough to show that $x = x \cup (x \cap y)$, but this results from (1), (5) and Lemma 1, (iii) as follows:

$$\begin{aligned} x &= x\,o_{i_1}\big(x\,o_{i_2}(\ldots(x\,o_{i_k}\,y)\ldots)\big) \\ &= x\,\underbrace{o_{i_1}\big(x\,o_{i_1}\big(\ldots(x\,o_{i_1}}_{k\text{ times}}\,(x\,o_{i_{k+1}}(\ldots(x\,o_{i_{k+1}}\,y)\ldots)\big) \\ &= x\,o_{i_1}(x\,o_{i_{k+1}}y) = x \cup (x \cap y)\,. \end{aligned}$$

If we now define, for $I \subset N$ containing 1, $L^{(I)}$ as the product of all lattices in L_I, each taken as many times as it appears as a factor of L, then we get clearly

$$L = \mathop{\mathrm{P}}_{I} L^{(I)}$$

and this proves theorem I.

Remark. If in this proof one replaces the word "lattice" by "quasilattice", and do the same in the statement of the theorem, then we get a representation theorem for distributive n-quasilattices. However, the following characterization of distributive n-quasilattices seems to be simpler:

THEOREM II. *An algebra $\mathfrak{A} = (X; o_1, \ldots, o_n)$, $n \geqslant 2$, is a distributive n-quasilattice if and only if it is the sum of a direct system of distributive n-lattices.*

(For the definition of the sum of direct systems of algebras, see [2].)

Proof. The sufficiency is nearly trivial (cf. theorem 3 of [1]). The necessity follows from theorem 3 of [2], as the operation $f_{1,2,\ldots,n}(x, y)$ satisfies the conditions characterizing the partition functions, which follows from (1), lemma 2, (vi), lemmas 3 and 4 and (v).

References

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Some remarks on sums of direct systems of algebras

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0. Introduction. In this paper we give some additional remarks concerning the notion of a sum of direct system (with the least upper bound property) of abstract algebras defined in [1]. At first we recall the following definition:

Let $\mathcal{A}$ be a direct system of abstract algebras of a fixed similarity type without nullary fundamental operations, indexed by elements of a partially ordered set I, the ordering relation of which has the least upper bound property. Moreover, we assume (which is not an essential restriction) that the carriers of the algebras $\mathfrak{A}_i$ $(i \in I)$ of this system are mutually disjoint. The sum $S(\mathcal{A})$ of the system $\mathcal{A}$ is an abstract algebra of the same similarity type as the algebras $\mathfrak{A}_i$, the carrier of which is the *sum* of the carriers A_i of all algebras of the system $\mathcal{A}$ and whose fundamental operations are defined by

$$F_t(x_1, \ldots, x_n) = F_t\big(\varphi_{i_1,i_0}(x_1), \ldots, \varphi_{i_n,i_0}(x_n)\big),$$

where $x_1 \in A_{i_1}, \ldots, x_n \in A_{i_n}$, $i_0 = \text{l.u.b.}\,(i_1, \ldots, i_n)$, $\{F_t\}$ is the set of fundamental operations of the algebras in the system $\mathcal{A}$, and φ_{ij} are the canonical homomorphisms of $\mathcal{A}$.

Let us also recall the definition of a P-function (partition function) of a given abstract algebra $\mathfrak{A} = (A, F)$ without nullary fundamental operations.

A mapping $f\colon A^2 \to A$ is called a *P-function* if it satisfies the following conditions:

(1) $$f(x, x) = x,$$

(2) $$f\big(x, f(y, z)\big) = f\big(x, f(z, y)\big),$$

(3) $$f\big(f(x, y), z\big) = f\big(x, f(y, z)\big),$$

(4) $$f\big(F(x_1, \ldots, x_n), y\big) = F\big(f(x_1, y), \ldots, f(x_n, y)\big),$$

(5) $$f\big(F(x_1, \ldots, x_n), x_k\big) = F(x_1, \ldots, x_n) \qquad (1 \leqslant k \leqslant n),$$