Note on metrization

by

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I. Introduction. In [1] Alexandroff has proved the following theorem:

A T₁-space is metrizable if and only if it is paracompact and has a uniform base.

A base $\mathcal{B}$ for a topological space $X$ is called uniform if for each $x \in X$ and each neighbourhood $U$ of $x$ at most finitely many members of $\mathcal{B}$ contain $x$ and intersect $X \setminus U$. The theorem quoted above contrasts other metrization theorems in the fact that it requires neither a decomposition of the base into countably many subfamilies nor the existence of a sequence of open covers with "nice" properties; cf. the theorems of Bing, Nagata-Smirnov ([3], p. 127), Arhangel'skiï, Morita, Stone ([4], p. 190), and Alexandroff-Urysohn ([2]). On the other hand, it invokes the explicit requirement of paracompactness. In Section 3 of the present paper we shall prove that a T₁-space is metrizable if and only if it has a base which is locally finite outside closed sets. (The necessary definitions are given in Section 2). Bases that are locally finite outside closed sets generalize in a natural way the concept of a uniform base, and, as we shall see, no decomposition into countably many subfamilies is required in their definition.

Section 2 contains the necessary lemmas for the proof of the metrization theorem in Section 3. As corollaries we obtain new characterizations of metacompact and paracompact spaces. In Section 3 we also briefly discuss how the classical metrization theorems of Urysohn ([2]), p. 135, [7], [8]) can be deduced from our theorem.

For notation not explained here the reader is referred to Kelley [5].

We recall that a topological space is called metacompact (or pointwise paracompact) if each open cover has a point-finite open refinement. Finally, if $\{A_i\}_{i \in I}$ is a finite collection of covers of a space $X$, then $\bigwedge \{A_i\}_{i \in I}$ is the cover consisting of all non-empty sets of the form $\bigcap \{A_i\}_{i \in I}$.
2. Local finiteness outside closed sets. Let \( X \) be a topological space. If \( A \) is a cover of \( X \) and \( B \) a subset of \( X \), we put
\[
A_B = \{ A \in A : A \cap B = \emptyset \}.
\]
A cover \( A \) of \( X \) is called point-finite outside closed sets if for each closed subset \( F \) of \( X \) a point \( x \in X \setminus F \) is contained in at most finitely many members of \( A_B \). Similarly, \( A \) is called locally finite outside closed sets if the following condition holds for each closed subset \( F \) of \( X \):

For each \( x \in X \setminus F \) there exists a neighbourhood \( V \) of \( x \) intersecting at most finitely many members of \( A_B \).

Clearly, if a cover \( A \) of \( X \) is locally finite outside closed sets, then it is also point-finite outside closed sets. Furthermore, we observe that a base for a topological space is uniform if and only if it is point-finite outside closed sets.

Our first result is a generalization of theorem II in [1]:

**Lemma 1.** Let \( A \) be an open cover of a topological space \( X \). If \( A \) is point-finite outside closed sets, then \( A \) has a point-finite subcover.

**Proof.** Let \( A \) be an open cover of \( X \) which is point-finite outside closed sets. Let \( A \) be well-ordered by \( \subseteq \), let \( A_1 \) be the first member of \( A \) w.r.t. \( \subseteq \) and put \( B(A_1) = A_1 \). By transfinite induction we construct a family \( \{ B(A) : A \in A \} \) such that for each \( A \in A \):

\[
\begin{align*}
(1) & \quad B(A) = A \cup \{ \emptyset \}, \\
(2) & \quad \bigcup_{A' \subseteq A} B(A') \supset \bigcup_{A' \subseteq A} A', \\
(3) & \quad B(A) \neq \emptyset \quad \text{if} \quad \bigcup_{A' \subseteq A} B(A') = \emptyset, \\
& \quad B(A) = \emptyset \quad \text{if} \quad \bigcup_{A' \subseteq A} B(A') = X.
\end{align*}
\]

(1), (2) and (3) are evidently satisfied for \( A = A_1 \). Now, suppose that \( B(A) \) has been chosen for each \( A < A_1 \). If
\[
\bigcup_{A < A_1} B(A) = X,
\]
we put \( B(A_1) = \emptyset \), and (1), (2) and (3) are trivially satisfied. If
\[
\bigcup_{A < A_1} B(A) \neq X,
\]
let \( B(A_1) \) be the first member of \( A \) w.r.t. \( \subseteq \) such that
\[
B(A_1) \setminus \bigcup_{A < A_1} B(A) \neq \emptyset.
\]

We must verify that (2) is satisfied. ((1) and (3) are trivial.) If \( A_1 \subseteq \bigcup_{A < A_1} B(A) \), there is nothing to prove. On the other hand, if
\[
A_1 \setminus \bigcup_{A < A_1} B(A) \neq \emptyset,
\]
we necessarily have \( B(A_1) = A_1 \), and (2) follows. (The assumptions \( B(A_1) > A_1 \) and \( B(A_1) < A_1 \) both contradict the choice of \( B(A_1) \).) We now put \( A_2 = (B(A_1)) \setminus \bigcup_{A \in A} (B(A)) \). From (1) and (3) it follows that \( B \) is a subcover of \( A \). Let \( x \in X \) be arbitrary, and let \( A_2 \) be the first member of \( A \) w.r.t. \( \subseteq \) such that \( x \in B(A_2) \). If \( x \in B(A_1) \), let \( x \in A_2 \), then we have \( A_2 < A_1 \), and from (3) it follows that \( B(A_1) \cap B(A_2) = \emptyset \), i.e.
\[
B(A_1) \setminus B(A_2) \subseteq A_2 \setminus B(A_2).
\]

Since \( A \) is point-finite outside closed sets, it follows that there are at most finitely many \( B(A) \) such that \( x \in B(A) \), and the proof is complete.

Though it will not be needed in the sequel, we include the following

**Proposition.** A topological space \( X \) is metacompact if and only if each open cover has an open refinement which is point-finite outside closed sets.

**Proof.** To prove necessity it is sufficient to observe that a point-finite cover of \( X \) is trivially point-finite outside closed sets. Sufficiency follows from lemma 1.

**Lemma 2.** Let \( A \) be an open cover of a topological space \( X \). If \( A \) is locally finite outside closed sets, then \( A \) has a locally finite subcover.

**Proof.** Let \( A \) be an open cover of \( X \) which is locally finite outside closed sets. Then \( A \) is also point-finite outside closed sets, so, by lemma 1, \( A \) has a point-finite subcover \( B \). Then \( B \) is an irreducible subcover \( C \), i.e. no proper subfamily of \( C \) covers \( X \). Let \( x \in X \) be arbitrary and select \( C_x \in C \) such that \( x \in C_x \). \( C \) is a subcover of \( A \) and is therefore locally finite outside closed sets, i.e. there exists a neighbourhood \( V \) of \( x \) intersecting at most finitely many members of \( C \setminus C_x \). Since \( C \) is irreducible, no \( C \in C \) can be properly contained in \( C_x \), hence \( C = C_x \), and we conclude that \( V \) intersects only finitely many members of \( C \). This completes the proof.

**Theorem 1.** A regular space \( X \) is paracompact if and only if each open cover has an open refinement which is locally finite outside closed sets.

**Proof.** Since a locally finite cover is locally finite outside closed sets, necessity is obvious. Sufficiency follows from lemma 2.

3. Metrization. Now we prove

**Theorem 2.** A \( T_1 \)-space \( X \) is metrisable if and only if it has a base which is locally finite outside closed sets.
Proof. Let $X$ be metrizable with metric $d$. Let $3_0$ be a locally finite open refinement of the cover consisting of all open spheres with $d$-radius $1/n$. $\{3_n\}$ is a base for $X$; we claim that it is also locally finite outside closed sets. Let $F$ be a closed subset of $X$ ($\emptyset \neq F \neq X$), and let $x$ be a point in $X \setminus F$. For some $n_x$ we have $St(x, 3_{n_x}) \cap St(F, 3_{n_x}) = \emptyset$ for all $n > n_x$. On the other hand, there exists for each $n$ a neighbourhood $V_n$ of $x$ intersecting only finitely many members of $3_{n}$. Then

$$V = \bigcup_{n=1}^{\infty} V_n,$$

intersects only finitely many members of $3_1 \cup 3_2 \cup \ldots \cup 3_{n_x}$, hence $V = V_x \cap St(x, 3_{n_x})$ intersects at most finitely many members of $3_{n}$. To prove sufficiency, let $3$ be a base for $X$ which is locally finite outside closed sets. We first prove that $X$ is regular. Let $x \in X$ be arbitrary and let $U$ be an open neighbourhood of $x$. There exists a neighbourhood $V$ of $x$, $V \subseteq U$, intersecting at most finitely many $B \in 3 \cap U$. If $y \in V \setminus V$, then $y$ cannot be an isolated point in $X$, hence, since $X$ is $T_1$, there are infinitely many $B \in 3$ containing $y$. Thus the assumption $y \in V \setminus U$ immediately leads to a contradiction. Therefore $V \subseteq U$, and $X$ is regular. If $U$ is an open cover of $X$, we can refine $U$ by members of $3$, hence $U$ has an open refinement which is locally finite outside closed sets. From theorem 1 it follows that $X$ is paracompact. $3$, being locally finite outside closed sets, is evidently a uniform base. The metrizability of $X$ now follows from the theorem of Alexandroff quoted in the introduction. (A simplified proof of Alexandroff's theorem can be found in [3].)

Remark. We shall give another proof of the sufficiency part in the preceding theorem; this proof is based on a technique used by Alexandroff in [1]. Let $3$ be a base for $X$ which is locally finite outside closed sets. We put

$$3 = \{x \in X \mid x \text{ is an isolated point in } X\}$$

and $3_1 \cup 3_2 \cup \ldots \cup 3_{n_x}$. Then $3_1 \cup 3_2 \cup \ldots \cup 3_{n_x}$ is also locally finite outside closed sets. Using lemma 2 of section 2 we can find a locally finite subcover $3_{n_x}$ of $3_1 \cup 3_2 \cup \ldots \cup 3_{n_x}$. Proceeding by induction we obtain sequences $(3_{n_x})$ and $(3_{n_x})$ such that $3_{n_x}$ is a locally finite subcover of $3_1 \cup 3_2 \cup \ldots \cup 3_{n_x}$ for each $n$. Let $x \in X$ be arbitrary and select, for each $n$, $B_n \in 3_{n_x}$ such that $x \in B_n$. If, for some $n_x$, $B_n \in 3_{n_x}$ then $B_n$ is a sequence of distinct members of $3_{n_x}$, and, since $3$ is a uniform base, $B_n$ must still be a neigh-
bourhood base at $x$. Thus, $\bigcup_{n=1}^{\infty} 3_{n_x}$ is a $\sigma$-locally finite base. We have already seen that $X$ is regular, and metrizability follows from the Nagata-Smirnov theorem.

$\Box$

One of the merits of the Nagata-Smirnov theorem is that the following metrization theorems of Urysohn are easily deducible as corollaries:

1. A regular $T_1$-space with a countable base is metrizable ([5], p. 126, and [7]).

2. A compact Hausdorff space is metrizable if and only if it has a countable base ([6]).

It is also easy to deduce these theorems from theorem 2 of the present paper. Let $X$ be a metrizable compact Hausdorff space and let $3$ be a base for $X$ which is locally finite outside closed sets. Instead of using lemma 2 of section 2 we now use compactness to select a finite subcover $3_{n_x}$ of $3_{n_x}$ for each $x$. In the remark following theorem 2, $\bigcup_{n=1}^{\infty} 3_{n_x}$ is then a countable base for $X$.

It remains to prove that if $X$ is regular and has a countable base, then $X$ also has a base which is locally finite outside closed sets. We first note that the family of covers

$$A(U, F) = \{V, X \setminus \overline{U}\}, U, V \in A, U \subseteq V,$$

is countable. Let $n(U, V)$ be the number of $A(U, F)$ in an enumeration of $(A(U, F))_n$ and put

$$\bigcup_{n=1}^{\infty} 3_{n_x}$$

Then $\bigcup_{n=1}^{\infty} 3_{n_x}$ is a base for $X$ which is locally finite outside closed sets. To see this, let $F$ be a closed subset of $X$ and let $x$ be an arbitrary point in $X \setminus F$. Using regularity we can choose $U, V, W \in A$ such that $x \in U \subseteq V \subseteq F \subseteq W \subseteq X \setminus F$. For $n > \max(n(U, V), n(V, W))$ we then have $St(U, 3_{n_x}) \cap St(F, 3_{n_x}) = \emptyset$, and therefore (since $3_{n_x}$ refines $3_{n_x}$) $3_{n_x}$ is a finite cover for each $n$) at most finitely many $B \in 3$ can intersect both $U$ and $F$.

References


Fundamenta Mathematicae LXII
On distributive $n$-lattices and $n$-quasilattices

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0. In this paper we give a representation theorem for a class of abstract algebras (which we shall call distributive $n$-quasilattices), having $n$ binary fundamental operations $o_1, \ldots, o_n$, which are idempotent, commutative, associative and distributive with respect to each other. A distributive $n$-quasilattice will be called a distributive $n$-lattice, if it satisfies moreover formula (5) below, which generalizes the familiar absorption law for lattices.

We shall show that every distributive $n$-lattice can be treated as a subalgebra of an algebra defined in a natural way in a product of distributive lattices, and every distributive $n$-quasilattice can be represented as a sum of a direct system (see [2]) of distributive $n$-lattices.

1. We shall call a distributive $n$-quasilattice every abstract algebra $Q = (X; o_1, \ldots, o_n)$ where $n \geq 2$ and $o_1, \ldots, o_n$ are binary operations which satisfy the following four conditions:

\begin{align*}
(1) & \quad x o_1 x = x, \\
(2) & \quad x o_i y = y o_i x, \\
(3) & \quad (x o_i y) o_i z = x o_i (y o_i z), \\
(4) & \quad (x o_i y) o_j z = (x o_j z) o_i (y o_j z) \\
& \quad (i, j = 1, 2, \ldots, n).
\end{align*}

A distributive $n$-quasilattice we shall call a distributive $n$-lattice if it satisfies moreover the following equality:

\begin{equation}
(5) \quad x_{o_1 \ldots o_{n-1} x_{o_n y}} = x.
\end{equation}

It is easy to see that in the case $n = 2$ a distributive $n$-lattice is a distributive lattice, and equation (5) coincides with the law of absorption. Similarly, a distributive $n$-quasilattice in the case $n = 2$ is a distributive quasilattice, as defined in [1].

**Examples.** Let $X = (a_1, a_2, \ldots, a_n, 0)$ and let us define for $i = 1, 2, \ldots, n$ the operations $o_i$ as follows: $x o_i x = x$, $x o_i a_i = a_i o_i x = a_i$, $x o_i a_j = a_j o_i x = a_i$, $x o_i 0 = 0 = 0 o_i x$. Then $X$ is an $n$-quasilattice.