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Note on metrization

by

P. R. Andenæs (Oslo)

1. Introduction. In [1] Alexandroff has proved the following theorem:

A T_1-space is metrizable if and only if it is paracompact and has a uniform base.

A base $\mathcal{B}$ for a topological space X is called *uniform* if for each $x \in X$ and each neighbourhood U of x at most finitely many members of $\mathcal{B}$ contain x and intersect $X \setminus U$. The theorem quoted above contrasts other metrization theorems in the fact that it requires neither a decomposition of the base into countably many subfamilies nor the existence of a sequence of open covers with "nice" properties; cf. the theorems of Bing, Nagata-Smirnov ([5], p. 127), Arhangel'skiĭ, Morita, Stone ([4], p. 196), and Alexandroff–Urysohn ([2]). On the other hand, it invokes the explicit requirement of paracompactness. In Section 3 of the present paper we shall prove that *a T_1-space is metrizable if and only if it has a base which is locally finite outside closed sets.* (The necessary definitions are given in Section 2). Bases that are locally finite outside closed sets generalize in a natural way the concept of a uniform base, and, as we shall see, no decomposition into countably many subfamilies is required in their definition.

Section 2 contains the necessary lemmas for the proof of the metrization theorem in Section 3. As corollaries we obtain new characterizations of metacompact and paracompact spaces. In Section 3 we also briefly discuss how the classical metrization theorems of Urysohn ([5], p. 125, [7], [8]) can be deduced from our theorem.

For notation not explained here the reader is referred to Kelley [5]. We recall that a topological space is called *metacompact* (or *pointwise paracompact*) if each open cover has a point-finite open refinement. Finally, if $\{\mathcal{A}_i\}_{i \in I}$ is a finite collection of covers of a space X, then $\bigwedge\{\mathcal{A}_i \mid i \in I\}$ is the cover consisting of all non-empty sets of the form $\bigcap\{A_i \mid i \in I\}$, $A_i \in \mathcal{A}_i$.

2. Local finiteness outside closed sets. Let X be a topological space. If $\mathcal{A}$ is a cover of X and B a subset of X, we put

$$\mathcal{A}_B = \{A \mid A \in \mathcal{A},\ A \cap B \neq \emptyset\}.$$

A cover $\mathcal{A}$ of X is called *point-finite outside closed sets* if for each closed subset F of X a point $x \in X \setminus F$ is contained in at most finitely many members of $\mathcal{A}_F$. Similarly, $\mathcal{A}$ is called *locally finite outside closed sets* if the following condition holds for each closed subset F of X:

For each $x \in X \setminus F$ there exists a neighbourhood V of x intersecting at most finitely many members of $\mathcal{A}_F$.

Clearly, if a cover $\mathcal{A}$ of X is locally finite outside closed sets, then it is also point-finite outside closed sets. Furthermore, we observe that *a base for a topological space is uniform if and only if it is point-finite outside closed sets.*

Our first result is a generalization of theorem II in [1]:

LEMMA 1. *Let $\mathcal{A}$ be an open cover of a topological space X. If $\mathcal{A}$ is point-finite outside closed sets, then $\mathcal{A}$ has a point-finite subcover.*

Proof. Let $\mathcal{A}$ be an open cover of X which is point-finite outside closed sets. Let $\mathcal{A}$ be well-ordered by $\leqslant$, let A_0 be the first member of $\mathcal{A}$ w.r.t. $\leqslant$ and put $B(A_0) = A_0$. By transfinite induction we construct a family $\{B(A) \mid A \in \mathcal{A}\}$ such that for each $A \in \mathcal{A}$:

(1) $$B(A) \in \mathcal{A} \cup \{\emptyset\},$$

(2) $$\bigcup_{A' \leqslant A} B(A') \supset \bigcup_{A' \leqslant A} A',$$

(3) $$\begin{cases} B(A) \setminus \bigcup_{A' < A} B(A') \neq \emptyset & \text{if } \bigcup_{A' < A} B(A') \neq X, \\ B(A) = \emptyset & \text{if } \bigcup_{A' < A} B(A') = X. \end{cases}$$

(1), (2) and (3) are evidently satisfied for $A = A_0$. Now, suppose that $B(A)$ has been chosen for each $A < A_1$. If

$$\bigcup_{A < A_1} B(A) = X,$$

we put $B(A_1) = \emptyset$, and (1), (2) and (3) are trivially satisfied. If

$$\bigcup_{A < A_1} B(A) \neq X,$$

let $B(A_1)$ be the first member of $\mathcal{A}$ w.r.t. $\leqslant$ such that

$$B(A_1) \setminus \bigcup_{A < A_1} B(A) \neq \emptyset.$$

We must verify that (2) is satisfied. ((1) and (3) are trivial.) If

$$A_1 \subset \bigcup_{A < A_1} B(A),$$

there is nothing to prove. On the other hand, if

$$A_1 \setminus \bigcup_{A < A_1} B(A) \neq \emptyset,$$

we necessarily have $B(A_1) = A_1$, and (2) follows. (The assumptions $B(A_1) > A_1$ and $B(A_1) < A_1$ both contradict the choice of $B(A_1)$.) We now put $\mathcal{B} = \{B(A) \mid A \in \mathcal{A}\} \setminus \{\emptyset\}$. From (1) and (2) it follows that $\mathcal{B}$ is a subcover of $\mathcal{A}$. Let $x \in X$ be arbitrary, and let A_x be the first member of $\mathcal{A}$ w.r.t. $\leqslant$ such that $x \in B(A_x)$. If $x \in B(A)$, $A \neq A_x$, then we have $A_x < A$, and from (3) it follows that $B(A) \setminus B(A_x) \neq \emptyset$, i.e.

$$B(A) \in \mathcal{B}_{X \setminus B(A_x)} \subset \mathcal{A}_{X \setminus B(A_x)}.$$

Since $\mathcal{A}$ is point-finite outside closed sets, it follows that there are at most finitely many $B(A) \in \mathcal{B}$ such that $x \in B(A)$, and the proof is complete.

Though it will not be needed in the sequel, we include the following

PROPOSITION. *A topological space X is metacompact if and only if each open cover has an open refinement which is point-finite outside closed sets.*

Proof. To prove necessity it is sufficient to observe that a point-finite cover of X is trivially point-finite outside closed sets. Sufficiency follows from lemma 1.

LEMMA 2. *Let $\mathcal{A}$ be an open cover of a topological space X. If $\mathcal{A}$ is locally finite outside closed sets, then $\mathcal{A}$ has a locally finite subcover.*

Proof. Let $\mathcal{A}$ be an open cover of X which is locally finite outside closed sets. Then $\mathcal{A}$ is also point-finite outside closed sets, so, by lemma 1, $\mathcal{A}$ has a point-finite subcover $\mathcal{B}$. Then $\mathcal{B}$ has an irreducible subcover $\mathcal{C}$, i.e. no proper subfamily of $\mathcal{C}$ covers X (cf. [4], p. 160). Let $x \in X$ be arbitrary and select $C_x \in \mathcal{C}$ such that $x \in C_x$. $\mathcal{C}$ is a subcover of $\mathcal{A}$ and is therefore locally finite outside closed sets, i.e. there exists a neighbourhood V of x intersecting at most finitely many members of $\mathcal{C}_{X \setminus C_x}$. Since $\mathcal{C}$ is irreducible, no $C \in \mathcal{C}$ can be properly contained in C_x, hence $\mathcal{C} = \mathcal{C}_{X \setminus C_x} \cup \{C_x\}$, and we conclude that V intersects only finitely many members of $\mathcal{C}$. This completes the proof.

THEOREM 1. *A regular space X is paracompact if and only if each open cover has an open refinement which is locally finite outside closed sets.*

Proof. Since a locally finite cover is locally finite outside closed sets, necessity is obvious. Sufficiency follows from lemma 2.

3. Metrization. Now we prove

THEOREM 2. *A T_1-space X is metrizable if and only if it has a base which is locally finite outside closed sets.*

Proof. Let X be metrizable with metric d. Let $\mathcal{B}_n$ be a locally finite open refinement of the cover consisting of all open spheres with d-radius $1/n$. $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a base for X; we claim that it is also locally finite outside closed sets. Let F be a closed subset of X ($\emptyset \neq F \neq X$), and let x be a point in $X \setminus F$. For some n_0 we have $\mathrm{St}(x, \mathcal{B}_n) \cap \mathrm{St}(F, \mathcal{B}_n) = \emptyset$ for all $n > n_0$. On the other hand, there exists for each n a neighbourhood V_n of x intersecting only finitely many members of $\mathcal{B}_n$. Then

$$V_0 = \bigcap_{i=1}^{n_0} V_i$$

intersects only finitely many members of $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_{n_0}$, hence $V = V_0 \cap \mathrm{St}(x, \mathcal{B}_{n_0+1})$ intersects at most finitely many members of $\mathcal{B}_F$.

To prove sufficiency, let $\mathcal{B}$ be a base for X which is locally finite outside closed sets. We first prove that X is regular. Let $x \in X$ be arbitrary and let U be an open neighbourhood of x. There exists a neighbourhood V of x, $V \subset U$, intersecting at most finitely many $B \in \mathcal{B}_{X \setminus U}$. If $y \in \overline{V} \setminus V$, then y cannot be an isolated point in X, hence, since X is T_1, there are infinitely many $B \in \mathcal{B}$ containing y. Thus the assumption $y \in X \setminus U$ immediately leads to a contradiction. Therefore $\overline{V} \subset U$, and X is regular. If $\mathcal{U}$ is an open cover of X, we can refine $\mathcal{U}$ by members of $\mathcal{B}$, hence $\mathcal{U}$ has an open refinement which is locally finite outside closed sets. From theorem 1 it follows that X is paracompact. $\mathcal{B}$, being locally finite outside closed sets, is evidently a uniform base. The metrizability of X now follows from the theorem of Alexandroff quoted in the introduction. (A simplified proof of Alexandroff's theorem can be found in [3].)

Remark. We shall give another proof of the sufficiency part in the preceding theorem; this proof is based on a technique used by Alexandroff in [1]. Let $\mathcal{B}$ be a base for X which is locally finite outside closed sets. We put

$$\mathcal{J} = \{\{x\} | x \text{ is an isolated point in } X\}$$

and $\mathcal{A}_1 = \mathcal{B} \setminus \mathcal{J}$. Then $\mathcal{A}_1 \cup \mathcal{J}$ is also locally finite outside closed sets. Using lemma 2 of Section 2 we can find a locally finite subcover $\mathcal{B}_1$ of $\mathcal{A}_1 \cup \mathcal{J}$. Let $\mathcal{A}_2 = \mathcal{A}_1 \setminus \mathcal{B}_1$; then $\mathcal{A}_2 \cup \mathcal{J}$ is a cover of X which is locally finite outside closed sets. (It is easy to see that $\mathcal{A}_2 \cup \mathcal{J}$ covers X: if $\{x\} \notin \mathcal{J}$, infinitely many members of $\mathcal{A}_1$ must contain x since X is T_1, on the other hand $\mathcal{B}_1$ is point-finite.) Proceeding by induction we obtain sequences $\{\mathcal{A}_n\}$ and $\{\mathcal{B}_n\}$ such that $\mathcal{B}_n$ is a locally finite subcover of $\mathcal{A}_n \cup \mathcal{J}$ and $\mathcal{A}_{n+1} = \mathcal{A}_n \setminus \mathcal{B}_n$ for each n. Let $x \in X$ be arbitrary and select, for each n, $B_n \in \mathcal{B}_n$ such that $x \in B_n$. If, for some n_0, $B_{n_0} \in \mathcal{J}$, then $\{B_n\}$ is evidently a neighbourhood base at x; if each $B_n \in \mathcal{A}_n$, then $\{B_n\}$ is a sequence of distinct members of $\mathcal{B}$, and, since $\mathcal{B}$ is a uniform base, $\{B_n\}$ must still be a neighbourhood base at x. Thus, $\bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a σ-locally finite base. We have already seen that X is regular, and metrizability follows from the Nagata-Smirnov theorem.

*

One of the merits of the Nagata-Smirnov theorem is that the following metrization theorems of Urysohn are easily deducible as corollaries:

(1) *A regular T_1-space with a countable base is metrizable* ([5], p. 125, and [7]).

(2) *A compact Hausdorff space is metrizable if and only if it has a countable base* ([6]).

It is also easy to deduce these theorems from theorem 2 of the present paper. Let X be a metrizable compact Hausdorff space and let $\mathcal{B}$ be a base for X which is locally finite outside closed sets. Instead of using lemma 2 of Section 2 we now use compactness to select a finite subcover $\mathcal{B}_n$ of $\mathcal{A}_n \cup \mathcal{J}$ for each n in the remark following theorem 2. $\bigcup_{n=1}^{\infty} \mathcal{B}_n$ is then a countable base for X.

It remains to prove that if X is regular and has a countable base $\mathcal{A}$, then X also has a base $\mathcal{B}$ which is locally finite outside closed sets. We first note that the family of covers

$$\mathcal{A}_{(U,V)} = \{V, X \setminus \overline{U}\}, \quad U, V \in \mathcal{A}, \quad \overline{U} \subset V,$$

is countable. Let $n(U, V)$ be the number of $\mathcal{A}_{(U,V)}$ in an enumeration of $\{\mathcal{A}_{(U,V)}\}$ and put

$$\mathcal{B}_n = \bigwedge \{\mathcal{A}_{(U,V)} | n(U, V) \leqslant n\}.$$

Then $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a base for X which is locally finite outside closed sets. To see this, let F be a closed subset of X and let x be an arbitrary point in $X \setminus F$. Using regularity we can choose $U, V, W \in \mathcal{A}$ such that $x \in U \subset \overline{U} \subset V \subset \overline{V} \subset W \subset X \setminus F$. For $n \geqslant \max(n(U, V), n(V, W))$ we then have $\mathrm{St}(U, \mathcal{B}_n) \cap \mathrm{St}(F, \mathcal{B}_n) = \emptyset$, and therefore (since $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$ and $\mathcal{B}_n$ is a finite cover for each n) at most finitely many $B \in \mathcal{B}$ can intersect both U and F.

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Added in proof: It has come to the attention of the author that theorem 2 in Section 3 already has been proved by Arhangel'skiĭ in [8]. Our treatment of the subject is, however, entirely different from Arhangel'skiĭ's and contains other viewpoints.

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On distributive n-lattices and n-quasilattices

by

J. Płonka (Wrocław)

0. In this paper we give a representation theorem for a class of abstract algebras (which we shall call *distributive n-quasilattices*), having n binary fundamental operations $o_1, \ldots, o_n$, which are idempotent, commutative, associative and distributive with respect to each other. A distributive n-quasilattice will be called a *distributive n-lattice*, if it satisfies moreover formula (5) below, which generalizes the familiar absorption law for lattices.

We shall show that every distributive n-lattice can be treated as a subalgebra of an algebra defined in a natural way in a product of distributive lattices, and every distributive n-quasilattice can be represented as a sum of a direct system (see [2]) of distributive n-lattices.

1. We shall call a *distributive n-quasilattice* every abstract algebra $Q = (X; o_1, \ldots, o_n)$ where $n \geqslant 2$ and $o_1, \ldots, o_n$ are binary operations which satisfy the following four conditions:

(1) $$x \, o_i \, x = x\,,$$

(2) $$x \, o_i \, y = y \, o_i \, x\,,$$

(3) $$(x \, o_i \, y) \, o_i \, z = x \, o_i \, (y \, o_i \, z)\,,$$

(4) $$(x \, o_i \, y) \, o_j \, z = (x \, o_j \, z) \, o_i \, (y \, o_j \, z)$$

$(i, j = 1, 2, \ldots, n)$.

A distributive n-quasilattice we shall call a *distributive n-lattice* if it satisfies moreover the following equality:

(5) $$x \, o_1 \Big(x \, o_2 \big(\ldots x \, o_{n-1} (x \, o_n \, y) \ldots\big)\Big) = x\,.$$

It is easy to see that in the case $n = 2$ a distributive n-lattice is a distributive lattice, and equation (5) coincides with the law of absorption. Similarly, a distributive n-quasilattice in the case $n = 2$ is a distributive quasilattice, as defined in [1].

EXAMPLES. 1. Let $X = \{a_1, a_2, \ldots, a_n, 0\}$ and let us define for $i = 1, 2, \ldots, n$ the operations o_i as follows: $x \, o_i \, x = x$, $x \, o_i \, a_i = a_i \, o_i \, x = a_i$,