On some numerical constants associated with abstract algebras II

by

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1. Introduction. In this paper we use the terminology and notation of [4]. E. Marczewski introduced in [2] the order of enlargeability and the arity or the order of reducibility of abstract algebras. We recall his definition of these concepts. Let $\mathcal{A} = (A; F)$ be an abstract algebra and let $G$ run over all families of operations in $A$. Put

$$\varepsilon(\mathcal{A}) = \min \{ n : \bigwedge_{G} \{ A^{n}(F) = A^{n}(G) \} = \{ A(F) \supset A(G) \} \}$$

and

$$\varrho(\mathcal{A}) = \min \{ n : \bigwedge_{G} \{ A^{n}(F) = A^{n}(G) \} = \{ A(F) \subset A(G) \} \}$$

where the minimum of an empty set is assumed to be $\infty$. The constant $\varepsilon(\mathcal{A})$ is called the order of enlargeability of the algebra $\mathcal{A}$. The constant $\varrho(\mathcal{A})$ (denoted in [2] by $\beta(\mathcal{A})$) is called the arity or the order of reducibility of the algebra $\mathcal{A}$. In the sequel we shall sometimes write $\varepsilon$ and $\varrho$ instead of $\varepsilon(\mathcal{A})$ and $\varrho(\mathcal{A})$ respectively when no confusion will arise.

In [4] a relationship between the order of enlargeability and a substitute of the minimal number of generators was discussed. The aim of the present paper is to give a description of all possible pairs $(\varepsilon, \varrho)$ for abstract algebras.

The $p$-enlargement $\mathcal{G}_{p}(\mathcal{A})$ of the algebra $\mathcal{A}$ was defined in [4], Chapter 2. A relationship between the concepts of the order of enlargeability and the $p$-enlargement is given by the following simple theorem ([4], Theorem 2.2):

(i) The inequality $\varepsilon(\mathcal{A}) \leq p$ holds if and only if $\mathcal{A} = \mathcal{G}_{p}(\mathcal{A})$.

If $\mathcal{A} = (A; F)$, then by $\mathcal{R}_{p}(\mathcal{A})$ ($p \geq 1$) we shall denote the $p$-reduct $(A; A^{p}(F))$ of $\mathcal{A}$. It is clear that

(ii) The inequality $\varrho(\mathcal{A}) \leq p$ holds if and only if $\mathcal{A} = \mathcal{R}_{p}(\mathcal{A})$.

Many algebras usually treated in mathematics have small arity. We say that an algebra $\mathcal{A}$ is rigid if the inequality $\varepsilon(\mathcal{A}) < \varrho(\mathcal{A})$ holds. As an example of rigid algebras we quote complete algebras over an at
least two-element set, i.e. algebras for which every operation is algebraic. In fact, for complete algebras we have the formulas $\varepsilon = 0$ and $\varepsilon = 2$ (see [3], [5]).

2. A class of rigid algebras with finite arity. In this section we assume that $p$ and $q$ are arbitrary positive integers satisfying the inequality

$$q \geq p + 2.$$  

Consider a $2(q+1)$-element set $A_q = \{a_1, a_2, a_3, a_4, b_1, b_2, \ldots, b_{2q+1}\}$. Put

$$B_k = \{a_1, a_2, a_3, a_4, b_1, b_2, \ldots, b_{2q+1}\} (k = 1, 2, \ldots, g+1).$$

Further, for every $n$-tuple $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ (where $n = 1, 2, \ldots$) of elements of the set $A_q$ we define a set $D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}$ as follows:

1. $D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}$ is a $k$-element set if one of the following cases holds:
   a. $\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \rangle < g$ and $\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \rangle \subseteq B_k$ for every index $k (k = 1, 2, \ldots, g+1)$,
   b. there exists two different indices $i$ and $j$ ($1 \leq i < j \leq n$) such that $\langle \varepsilon_i, \varepsilon_j \rangle \subseteq B_k$ and $\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \rangle \subseteq B_k$ for every index $k (k = 1, 2, \ldots, g+1)$.

2. $D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n} = B_k$ if $\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \rangle \geq g$, $\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \rangle \subseteq B_k$ and $\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \rangle \subseteq B_k$ for every index $k (k = 1, 2, \ldots, g+1)$.

3. $D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n} = A_q$ in the remaining cases, i.e. if $\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \rangle > g$ and $\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \rangle \subseteq B_k$ for every index $k (k = 1, 2, \ldots, g+1)$.

It is easy to verify the following inclusions

$$D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n} \subseteq D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}$$

if $\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \rangle \subseteq B_k,$

$$D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n} \subseteq D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}$$

Hence we get the inclusion

$$D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n} \subseteq D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}$$

if $\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \rangle \subseteq D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}.$

LEMMA 2.1. If $n > g$ and $u \in D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}$ then there exists an index $\varepsilon (1 \leq \varepsilon \leq n)$ such that $u \in D_{\varepsilon, \varepsilon, \ldots, \varepsilon}.$

Proof. First consider the case $D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n} = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$. Then $u = \varepsilon_i$ for some index $i$ ($1 \leq i \leq n$). Taking $i \neq i$ we have the formula

$$u \in \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_n\} \subseteq D_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_n}.$$
all trivial operations and, by (2.4), is closed under the composition. Put \( A_{p,q} = (A_2, F) \) for any pair of positive integers \( p,q \) satisfying condition (2.1). Of course, \( A(\mathbb{K}_{p,q}) = F_p \).

**Lemma 2.2.** For every \((q+1)\)-ary algebraic operation \( f \) in \( R_{q-1}(\mathbb{K}_{p,q}) \) the relation
\[
(2.8) \quad f(a_1, a_2, \ldots, a_{q+1}) \in \{a_1, a_2, \ldots, a_{q+1}\}
\]
holds.

**Proof.** The class of all \((q+1)\)-ary algebraic operations in \( R_{q-1}(\mathbb{K}_{p,q}) \) is the union \( \bigcup_{k=0}^q A_{k+1} \), where the classes \( A_{k+1} \) are defined recursively as follows
\[
A_{k+1} = A_{k} \cup \{g(f_1, f_2, \ldots, f_{q+1}) : g \in A_{q}(\mathbb{K}_{p,q}), f_j \in A_{k}(\mathbb{K}_{p,q}) , j = 1, 2, \ldots, q+1\}
\]
(see [1], p. 47). Let \( f \) be an operation from \( A_{k+1} \). We shall prove the lemma by induction with respect to \( k \). If \( k = 0 \), then the operation \( f \) is trivial and, consequently, relation (2.8) is obvious. Suppose that \( k \geq 1 \) and relation (2.8) holds for all operations from \( A_{k} \). Moreover, we may assume that the operation \( f \) does not belong to \( A_{k} \). Then it can be written in the form
\[
(2.9) \quad f(a_1, a_2, \ldots, a_{q+1}) = g(f_1(a_1, a_2, \ldots, a_{q+1}), f_2(a_1, a_2, \ldots, a_{q+1}), \ldots, f_{q+1}(a_1, a_2, \ldots, a_{q+1}))
\]
where \( g \in A_{q}(\mathbb{K}_{p,q}) \) and \( f_j \in A_{k}(\mathbb{K}_{p,q}) \). Consequently, by the inductive assumption,
\[
(2.10) \quad f(a_1, a_2, \ldots, a_{q+1}) \in \{a_1, a_2, \ldots, a_{q+1}\} \quad (j = 1, 2, \ldots, q+1)
\]
Hence and from (2.9) it follows that
\[
(2.11) \quad f(a_1, a_2, \ldots, a_{q+1}) = g(a_1, a_2, \ldots, a_{q+1})
\]
where \( (i_1, i_2, \ldots, i_{q+1}) \subseteq (1, 2, \ldots, q+1) \). Consequently, there exist two indices \( s \neq r \) \((1 \leq s, r \leq q+1)\) which do not belong to the set \( (i_1, i_2, \ldots, i_{q+1}) \). Thus we have the inclusion
\[
(a_1, a_2, \ldots, a_{q+1}) \subseteq R_{s} \cap R_{r}
\]
which implies the equation \( D_{p}(a_1, a_2, \ldots, a_{q+1}) = (a_1, a_2, \ldots, a_{q+1}) \). Hence and from (2.10) we get the relation
\[
(2.12) \quad f(a_1, a_2, \ldots, a_{q+1}) \in \{a_1, a_2, \ldots, a_{q+1}\}
\]
which completes the proof of the lemma.

**Lemma 2.3.** Let \( v_1, v_2, \ldots, v_n \) be an arbitrary \( n \)-tuple of elements of \( A_4 \). Every operation \( f \), satisfying the conditions
\[
(2.13) \quad f(v_1, v_2, \ldots, v_{n}) \subseteq D_{p}(v_1, v_2, \ldots, v_{n}) \quad \text{and} \quad f(x_1, x_2, \ldots, x_{n}) = x_1
\]
for \( [x_1, x_2, \ldots, x_{n}] = [v_1, v_2, \ldots, v_{n}] \), is algebraic in the algebra \( R_{q}(\mathbb{K}_{p,q}) \).

**Proof.** We shall prove the lemma by induction with respect to \( n \). It is evident that each operation satisfying the condition of the lemma is algebraic in the algebra \( \mathbb{K}_{p,q} \). Consequently, our statement is true for \( n = 1 \). Suppose that \( n > 1 \) and that the lemma is true for \((n-1)\)-ary operations. Put \( u = f(v_1, v_2, \ldots, v_{n}) \). By Lemma 2.1 there exists an index \( i \) \((1 \leq i \leq n)\) such that \( u \in D_{p}(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n}) \). First consider the case \( i > 1 \). By the inductive assumption the operation \( g \), defined by the formula \( g(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n}) = u \) and \( g(v_1, v_2, \ldots, v_{i-1}, x_{i+1}, \ldots, x_{n}) = x_i \) otherwise, is algebraic in the algebra \( R_{q}(\mathbb{K}_{p,q}) \). Further, the ternary operation \( h \), defined as \( h(v_1, v_2, u) = u \) and \( h(v_1, v_2, x_i) = x_i \) otherwise, is algebraic in the algebra \( R_{q}(\mathbb{K}_{p,q}) \) because of the inequality \( q > p+2 \geq 3 \). Thus the composition \( f(x_1, x_2, \ldots, x_{n}) = h(v_1, v_2, g(v_1, v_2, \ldots, v_{i-1}, x_{i+1}, \ldots, x_{n})) \) is algebraic in \( R_{q}(\mathbb{K}_{p,q}) \). It is easy to verify that \( f(v_1, v_2, \ldots, v_{n}) = u \) and \( f(v_1, v_2, x_1, \ldots, x_{n}) = x_i \) otherwise. Thus \( f = f_1 \) and, consequently, the operation \( f \) is algebraic in \( R_{q}(\mathbb{K}_{p,q}) \).

Now consider the case \( i = 1 \). Of course, we may assume that all elements \( v_1, v_2, \ldots, v_{n} \) are different, because in the opposite case as the index \( r \) an integer greater than 1 can be taken. Since \( n > q + p+2 \geq 3 \), there exists an index \( r \) \((2 \leq r \leq n)\) such that \( v_r \neq u \). Setting \( g(v_1, v_2, \ldots, v_{n}) = u \) and \( g(v_1, v_2, \ldots, v_{r-1}, v_{r+1}, \ldots, v_{n}) = x_r \), \( h(v_1, v_2, v_{r+1}, \ldots, v_{n}) = x_r \) otherwise, we get, by the inductive assumption, algebraic operations in the algebra \( R_{q}(\mathbb{K}_{p,q}) \). Consequently, the composition \( f(x_1, x_2, \ldots, x_{n}) = h(v_1, v_2, g(v_1, v_2, \ldots, v_{n})) \) is also algebraic in \( R_{q}(\mathbb{K}_{p,q}) \). Since \( f(v_1, v_2, \ldots, v_{n}) = u \) and \( f(v_1, v_2, x_1, \ldots, x_{n}) = x_i \) otherwise, we have the equation \( f = f_1 \). Thus the operation \( f \) is algebraic in \( R_{q}(\mathbb{K}_{p,q}) \), which completes the proof.
and \( h \) are algebraic in the algebra \( \mathcal{H}(\mathcal{W}_d) \). Consequently, the composition

\[
f_d(x_1, x_2, \ldots, x_n) = h(g(x_1, x_2, \ldots, x_n))
\]

is algebraic in \( \mathcal{H}(\mathcal{W}_d) \) too. From the equations

\[
f_d(x_1, x_2, \ldots, x_n) = h(g(x_1, x_2, \ldots, x_n)) \quad \text{and} \quad g(x_1, x_2, \ldots, x_n) = h(g(x_1, x_2, \ldots, x_n))
\]

for all \( n \)-tuples \( x_1, x_2, \ldots, x_n \) different from the \( n \)-tuple \( x_1, x_2, \ldots, x_n \) we get the formula \( f = f_d \) which shows that the operation \( f \) is algebraic in the algebra \( \mathcal{H}(\mathcal{W}_d) \). The lemma is thus proved.

**Theorem 2.1.** For any pair \( p, q \) of positive integers satisfying the inequality \( q \geq p+2 \) the formula \( \varepsilon(\mathcal{W}_d) = q \) holds.

**Proof.** The algebra \( \mathcal{W}_d \) is finite. Consequently, by consecutive application of Lemma 2.2 we obtain that each algebraic operation in \( \mathcal{W}_d \) is also algebraic in \( \mathcal{W}(\mathcal{W}_d) \). In other words, \( \mathcal{W}_d = \mathcal{W}(\mathcal{W}_d) \) which, by proposition (ii) in Section 1, implies the inequality \( \varepsilon(\mathcal{W}_d) \geq q \).

Put \( h(a_1, a_2, \ldots, a_{n+1}) = h \) and \( g(a_1, a_2, \ldots, a_{n+1}) = x_1 \) otherwise. Since \( D_h(a_1, a_2, \ldots, a_{n+1}) = A_d \), the operation \( h \) is algebraic in \( \mathcal{W}_d \). On the other hand, by Lemma 2.2, it is not algebraic in \( \mathcal{W}(\mathcal{W}_d) \). Thus \( \mathcal{W}_d \neq \mathcal{W}(\mathcal{W}_d) \), whence, by proposition (ii) in Section 1 the inequality \( \varepsilon(\mathcal{W}_d) \geq q \) follows.

**Theorem 2.2.** For any pair \( p, q \) of positive integers satisfying the inequality \( q \geq p+2 \) the formula \( \varepsilon(\mathcal{W}_d) = p \) holds.

**Proof.** Let \( m \) be an integer greater than \( p \) and \( f \) an arbitrary \( n \)-ary algebraic operation in the \( p \)-enlargement \( \mathcal{W}_d(\mathcal{W}_d) \). We shall prove that the operation \( f \) is algebraic in the algebra \( \mathcal{W}_d(\mathcal{W}_d) \), i.e. for every \( n \)-tuple \( x_1, x_2, \ldots, x_n \) of elements of \( A_d \) the relation

\[
f_t(x_1, x_2, \ldots, x_n) = D_f(x_1, x_2, \ldots, x_n)
\]

holds. If \( D_f(x_1, x_2, \ldots, x_n) = A_d \), then the above relation is obvious. Suppose that \( \varepsilon(\mathcal{W}_d) < q \). Then we can choose a system \( i_1, i_2, \ldots, i_q \) of indices for which the equation

\[
f_t(x_1, x_2, \ldots, x_n) = \{x_1, x_2, \ldots, x_n\}
\]

holds. For any \( k \) \( 1 \leq k \leq n \), let \( f_k \) denote the least index \( i \), for which \( x_k = x_i \). Put \( f_k(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) \). Of course, the operation \( f_k \) is algebraic in the algebra \( \mathcal{W}_d(\mathcal{W}_d) \) and

\[
f(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n).
\]

Consequently, \( f(x_1, x_2, \ldots, x_n) \in D_f(x_1, x_2, \ldots, x_n) \), which, by (2.2) and (2.12), implies relation (2.11).

It remains the case \( \{x_1, x_2, \ldots, x_n\} \subset B_d \) for an index \( k \) \( 1 \leq k \leq q+1 \) and card \( \{x_1, x_2, \ldots, x_n\} \geq p \). Let \( b_1, b_2, \ldots, b_p \) be the system of all indices \( b_k \) for which the inclusion \( \{x_1, x_2, \ldots, x_n\} \subset B_d \) holds. Of course,

\[
D_f(x_1, x_2, \ldots, x_n) = B_d \quad \text{if} \quad r = 1
\]

and

\[
D_f(x_1, x_2, \ldots, x_n) = \{x_1, x_2, \ldots, x_n\} = \bigcap_{k=1}^{p} B_{b_k} \quad \text{if} \quad r > 1.
\]

Since card \( \{x_1, x_2, \ldots, x_n\} \geq p \), we may assume without loss of generality that the elements \( x_1, x_2, \ldots, x_{p-1} \) are different and do not belong to the set \( \{b_1, b_2, \ldots, b_{p-1}\} \). Consequently,

\[
D_f(x_1, x_2, \ldots, x_n) = B_{b_p} \quad \text{if} \quad r = 1, 2, \ldots, r.
\]

Setting \( g_{p+1}(x_1, x_2, \ldots, x_{p-1}, b_{p+1}) = y_1 \) and \( g_{p+1}(x_1, x_2, \ldots, y_p) = x_1 \) otherwise \( (j = 1, 2, \ldots, p; s = 1, 2, \ldots, r) \), we get algebraic operations in \( \mathcal{W}_d \).

Thus the compositions

\[
f(x_1, x_2, \ldots, x_p) = f(g_{p+1}(x_1, x_2, \ldots, x_{p-1}, b_{p+1}), g_{p+1}(x_1, x_2, \ldots, x_p), \ldots, g_{p+1}(x_1, x_2, \ldots, x_p)) \quad (s = 1, 2, \ldots, r)
\]

are algebraic in \( \mathcal{W}_d \). Consequently, by (2.3) and (2.15), we have the relation \( f(x_1, x_2, \ldots, x_{p-1}, b_p) = B_{b_p} \quad (s = 1, 2, \ldots, r) \). Hence in view of the equations \( f(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_{p-1}, b_p) \quad (s = 1, 2, \ldots, r) \) we get the relation \( f(x_1, x_2, \ldots, x_n) = \bigcap_{k=1}^{p} B_{b_k} \), which, by (2.13) and (2.14), implies relation (2.11). Thus we have proved that each operation algebraic in \( \mathcal{W}_d(\mathcal{W}_d) \) is algebraic in \( \mathcal{W}_d \). Hence the equation \( \varepsilon(\mathcal{W}_d) = \varepsilon(\mathcal{W}_d(\mathcal{W}_d)) \) follows. Consequently, by proposition (i) in Section 1, we have the inequality \( \varepsilon(\mathcal{W}_d) \leq p \).

From the definition of the sets \( D_f(x_1, x_2, \ldots, x_n) \) and algebraic operations in \( \mathcal{W}_d \) it follows that each subalgebra of \( \mathcal{W}_d \) generated by \( p \) elements either is equal to \( B_k \) \( k = 1, 2, \ldots, q+1 \) or consists of \( p \) elements. Thus the minimal number of generators of \( \mathcal{W}_d \) is at least \( p+1 \). Consequently, \( \gamma(\mathcal{W}_d) \geq p+1 \) and, by Theorem 6.1 in [4], \( \varepsilon(\mathcal{W}_d) \geq \gamma(\mathcal{W}_d) - 1 \geq p \) which together with (2.16) implies the assertion of the theorem.

**3. A class of rigid algebras with infinite arity.** In the sequel we shall use the following analogue of Theorem 12.2 in [4].

**Theorem 3.1.** Suppose that the algebra \( \mathcal{W} \) contains an algebraic constant \( c \) such that...
Further, setting

\[ \gamma(t_1, t_2, \ldots, t_n) = \gamma(t_1, t_2, \ldots, t_{n-1}, t_{n+1}, \ldots, t_n), \]

\[ h(t_1, t_2, \ldots, t_n) = h(t_1, t_2, \ldots, t_{n-1}, t_{n+1}, \ldots, t_n) \]

for each pair of indices \( r \leq s \) \((r, s = 1, 2, \ldots, n)\) we get algebraic operations satisfying the conditions

\[ g(t_1, t_2, \ldots, t_n) = g(t_1, t_2, \ldots, t_{n-1}, t_{n+1}, \ldots, t_n), \]

\[ h(t_1, t_2, \ldots, t_n) = h(t_1, t_2, \ldots, t_{n-1}, t_{n+1}, \ldots, t_n) \]

and for \( j \neq r, s \) the conditions

\[ g(t_1, t_2, \ldots, t_n) = g(t_1, t_2, \ldots, t_{n-1}, t_{n+1}, \ldots, t_n), \]

\[ h(t_1, t_2, \ldots, t_n) = h(t_1, t_2, \ldots, t_{n-1}, t_{n+1}, \ldots, t_n) \]

where the \( n \)-tuples \( t_1, t_2, \ldots, t_n \) are defined by (3.1). Hence and from (3.3), (3.4) and (3.5) we get the equations

\[ g(t_1, t_2, \ldots, t_n) = h(t_1, t_2, \ldots, t_n) \]

\[ \gamma(t_1, t_2, \ldots, t_n) = \gamma(t_1, t_2, \ldots, t_{n-1}, t_{n+1}, \ldots, t_n) \]

where both \( g \) and \( h \) are algebraic and, consequently, uniquely determined by their values on the \( n \)-tuples \( t_1, t_2, \ldots, t_n \) \((j = 1, 2, \ldots, n)\); \( \gamma \) is defined in [4], Chapter 7. Since both exceptional algebras \( X \) and \( Y \) defined in [4], Chapter 8 do not satisfy the conditions of the theorem, we infer, by Theorem 9.2 in [4], that

\[ \gamma(t_1, t_2, \ldots, t_n) \]

Now we shall prove that the algebra \( X \) has property (H) defined in [4], Chapter 12. Let \( g \) be an \( n \)-ary operation \((n > 3)\) such that for all operations \( g_1, g_2, \ldots, g_n \) \((j = 1, 2, \ldots, n)\) the composition \( g_1, g_2, \ldots, g_n \) belongs to \( \mathfrak{A} \). First, we suppose that \( g(x, y, z, \ldots, e) = x \). Then, by property of the algebra \( X \), we have the equation \( g(x_1, x_2, x_3, \ldots, x_n) = x_1 \) for all indices \( i \), \( j \) satisfying the inequality \( 2 < i \leq j \leq n \). Hence and from the inequality \( n > 3 \) it follows that \( g(x_1, x_2, x_3, \ldots, x_n) = x_1 \) whenever the \( n \)-tuple \( x_1, x_2, \ldots, x_n \) contains at most two different elements. Thus the algebra \( X \) has property (H).

By Theorem 12.1 in [4], we have the equation \( \gamma(t_1, t_2, \ldots, t_n) > 0 \) and the inequality \( \epsilon(t_1, t_2, \ldots, t_n) < 0 \) \( \gamma(t_1, t_2, \ldots, t_n) \leq 0 \). To prove this inequality it suffices, by the inequality \( \epsilon(t_1, t_2, \ldots, t_n) \leq 0 \), to prove that each algebraic ternary operation \( g \) in \( \mathfrak{A} \) is algebraic in \( X \). Put

\[ f_1(a) = g(x, y, z) \]

\[ f_2(a) = g(x, y, z) \]

\[ f_3(a) = g(x, y, z) \]

\[ f_4(a) = g(x, y, z) \]

and

\[ f_5(a) = g(x, y, z) \]

Of course, the operations \( f_1 \) and \( f_2 \) are algebraic in \( X \). Moreover,

\[ d_1(x, y, z) = f_1(x, y, z) \]

\[ d_2(x, y, z) = f_2(x, y, z) \]

\[ d_3(x, y, z) = f_3(x, y, z) \]

Now we shall prove that there exists a ternary algebraic operation \( \gamma \) in \( X \) such that

\[ g(x, y, z) = \gamma(x, y, z) \]

\[ \varepsilon(x, y, z) = \gamma(x, y, z) \]

\[ \delta(x, y, z) = \gamma(x, y, z) \]
First, let us suppose that at least one operation \( f_1 \) is trivial. Without loss of generality we may assume that \( f_1(x) = x \). Then, by (3.8) and condition (i), we have the equations \( d_1(x, y, z) = d_1(x, y, z) = x \) and \( d_1(x, y, z) = c \). Consequently, according to (3.7), the operation \( h(x, y, z) = c(x, y, z) \) satisfies (3.9).

Suppose now that all the operations \( f_1, f_2, \) and \( f_3 \) are non-trivial. Since \( f_1(c) = f_2(c) = f_3(c) \), there exists, by condition (ii), a ternary algebraic operation \( h \) such that

\[
f_1(x) = h(c, x, c), \quad f_2(x) = h(c, x, c), \quad f_3(x) = h(c, x, c).
\]

Moreover, by (3.8) and condition (ii), we have the equations

\[
d_1(x, y) = h(c, x, y), \quad d_1(x, y) = h(c, x, y), \quad d_1(x, y) = h(x, y, c)
\]

which together with (3.7) imply (3.9).

Let \( b \) be an arbitrary algebraic constant in \( \mathbb{F} \). Put \( v_b(x, y) = g(x, y, b) \) and \( v_b(x, y) = h(x, y, b) \). The operations \( v_b \) and \( v_b \) are algebraic in \( \mathbb{F} \) and, by (3.9), \( v_b(c, y) = w_b(c, y) \), \( v_b(c, x) = w_b(c, x) \). Consequently, by condition (iii), \( v_b = w_b \), i.e.

\[
g(x, y, b) = h(x, y, b) \quad \text{if} \quad b \in \mathbb{A}^0(\mathbb{F}).
\]

For any pair \( q_1, q_2 \) of unary algebraic operations in \( \mathbb{F} \) we put \( v_b(x, y) = g(x, y, b), v_b(x, y) = h(x, y, b) \), \( x \), \( y \), \( z \), and \( w_b(x, y) = h(x, y, b) \). Both operations \( v_b \) and \( v_b \) are algebraic in \( \mathbb{F} \) and, by (3.9) and (3.10), \( v_b(c, y) = w_b(c, y) \), \( v_b(c, x) = w_b(c, x) \). Consequently, by condition (ii), \( v_b = w_b \), i.e.

\[
g(x, y, b) = h(x, y, b) \quad \text{if} \quad b \in \mathbb{A}^0(\mathbb{F}).
\]

Thus, for any pair \( q_1, q_2 \) of binary algebraic operations in \( \mathbb{F} \) we put

\[
v_b(x, y) = g(x, y, b), v_b(x, y) = h(x, y, b)
\]

Both operations \( v_b \) and \( v_b \) are algebraic in \( \mathbb{F} \) and, by (3.11), \( v_b(c, y) = w_b(c, y) \), \( v_b(c, x) = w_b(c, x) \) which, by condition (i), implies the identity \( v_b = w_b \). Thus

\[
g(x, y, b) = h(x, y, b)
\]

for all \( b \in \mathbb{A}^0(\mathbb{F}) \).

Let \( a_1, a_2, a_3 \) be an arbitrary triple of elements of \( \mathbb{F} \). Since \( \gamma(\mathbb{F}) \leq 2 \), there exist elements \( b_1, b_2 \in \mathbb{F} \) and operations \( g_1, g_2 \) such that \( a_3 = g_1(b_1, b_2) \). Taking into account (3.12) we obtain the equation \( g_1(a_1, a_2, a_3) = h(a_1, a_2, a_3) \) which implies the identity \( g = h \). Thus the operation \( g \) is algebraic in \( \mathbb{F} \) which completes the proof.

Let \( A_1 \) be the set of all rationals and \( F_1 \) the family of all operations \( f \) of the form

\[
f(x_1, x_2, ..., x_n) = \sum_{k=1}^{n} a_k x_k + a_0 \quad (n = 1, 2, ...),
\]

where \( a_1, a_2, ..., a_n \) are non-negative even integers and \( a \in A_1 \). We denote the algebra \((A_1, F_1)\) by \( \mathbb{F}_{\infty} \).

For any \( p \) satisfying the inequality \( 2 \leq p \leq \infty \) we put \( A_p = \{0, 1, ..., p+1\} \). Let \( F_p \) be the family of all operations \( f_p (n \geq 1) \) defined as \( f_p(x_1, x_2, ..., x_n) = \sum_{k=1}^{n} a_k x_k \) \( \mod \) \( 4 \) if \( \{x_1, x_2, ..., x_n\} \subset \{0, 1, 2\} \) and \( f_p(x_1, x_2, ..., x_n) = 0 \) otherwise. The algebra \((A_p, F_p)\) will be denoted by \( \mathbb{F}_{p, \infty} \).

**Theorem 3.2.** For any \( p \) satisfying the inequality \( 1 \leq p \leq \infty \) the formulas \( \epsilon(\mathbb{F}_{p, \infty}) = p \) and \( g(\mathbb{F}_{p, \infty}) = \infty \) hold. 

**Proof.** First consider the algebra \( \mathbb{F}_{p, \infty} \). Since all its elements are algebraic constants, we have the formula \( \gamma(\mathbb{F}_{p, \infty}) = 0 \). Moreover, the algebra \( \mathbb{F}_{p, \infty} \) satisfies conditions of Theorem 3.1. Consequently, \( \epsilon(\mathbb{F}_{p, \infty}) \leq 2 \).

Obviously, \( \mathbb{F}_{p, \infty} \) is not the complete algebra over the set \( A_1 \). Therefore, we have the inequality \( \epsilon(\mathbb{F}_{p, \infty}) \gg 1 \). Consequently, to prove the formula \( \epsilon(\mathbb{F}_{p, \infty}) = 1 \) it suffices to prove that each binary algebraic operation \( f \) in \( \mathbb{F}_{p, \infty} \) is algebraic in \( \mathbb{F}_{p, \infty} \). For such operation \( f \) and for arbitrary elements \( a \) and \( b \) from \( A_1 \) we have the equation

\[
f(a, y) = a_0 y + a_0, \quad f(x, b) = a_0 x + a_0,
\]

where either \( a_0 = 0 \) or \( a_0 = 0 \)

Consequently,

\[
d_0 = a_0 = 0, \quad a_0 = 0 = 0, \quad a_0 + d_0 = a_0 + d_0, \quad a_0 + d_0 = a_0 + d_0, \quad a_0 + d_0 = a_0 + d_0
\]

Thus \( a_0 = a_0 + d_0 \) and \( a_0 = a_0 + d_0 \) for all \( a \in A_1 \) and the equation \( f(a, y) = a_0 x + a_0 \) holds for all \( x \in \mathbb{A} \). Moreover, the equation \( a_0 = 1 \) implies the equation \( a_0 + d_0 = 0 \) for all \( b \in A_1 \) and, consequently, the equation \( a_0 = d_0 = 0 \). In the same way we prove that the equation \( a_0 = d_0 = 0 \) is algebraic in \( \mathbb{F}_{p, \infty} \).
It is clear that the operation $\sum_{i=1}^n 2x_i$ is algebraic in $\mathbb{R}_{\infty}$ and is not algebraic in $\mathbb{R}_{\infty}(\mathbb{R}_{\infty})$ ($n = 2, 3, \ldots$). Hence and from proposition (ii) in Section 1 we get the formula $\varphi(\mathbb{R}_{\infty}) = \infty$.

Now consider the algebra $\mathbb{R}_{\infty,p}$ for $p > 2$. It is evident that each non-trivial $n$-ary algebraic operation $f$ in $\mathbb{R}_{\infty}$ is either identically equal to 0 or of the form

$$f(x_1, x_2, \ldots, x_n) = f(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$$

where $1 \leq k \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. Hence it follows that the set $A_n(\mathbb{R}_{\infty}) = \{0, 2\}$ is the only set of generators of the algebra $\mathbb{R}_{\infty}$. Thus $\varphi(\mathbb{R}_{\infty}) = p$. Moreover, the algebra $\mathbb{R}_{\infty,b}$ satisfies conditions of Theorem 3.1 which implies the equation $\varphi(\mathbb{R}_{\infty}) = p$. Further, it is easy to verify that for all non-trivial algebraic operations $f$ and $g$ in $\mathbb{R}_{\infty}$ the equation

$$f(y_1, y_2, \ldots, y_n) = f(0, y_2, \ldots, y_n)$$

is true. Hence it follows that the fundamental operation $f_0 (n > 2)$ is not algebraic in $\mathbb{R}_{\infty}(\mathbb{R}_{\infty})$. Consequently, by proposition (ii) in Section 1, $\varphi(\mathbb{R}_{\infty}) = \infty$ which completes the proof of the Theorem.

4. Algebras whose order of enlargeability is 0. The purpose of this section is to compute the arity of algebras whose order of enlargeability is 0. First we shall prove some lemmas.

Lemma 4.1. If $\varphi(\mathbb{R}) = 0$, then $A(\mathbb{R}) = 0$.

Proof. Contrary to this suppose that $\varphi(\mathbb{R}) = 0$ and $A(\mathbb{R}) = \frac{1}{2}$. Hence, in particular, it follows that the algebra in question contains at least two elements. Let $a, b$ be a pair of elements of $\mathbb{R}$. We define two operations $f$ and $g$ as follows: $f(a) = b, f(b) = a, g(a, b) = b$ and $g(b, a) = a$. Denoting the carrier of $\mathbb{R}$ by $A$ and the set $\mathbb{R} = \{A; \{f, g\}\}$ and $\mathbb{A} = \{A; \{f, g\}\}$. Since $A(\mathbb{R}) = A(\mathbb{A}) = 0$, we infer that both algebras $\mathbb{R}$ and $\mathbb{A}$ are reducts of the algebra $\mathbb{R}$. Consequently, both operations $f$ and $g$ are algebraic. But the composition $g(a, f(a))$ is a constant operation identically equal to 0 which contradicts the equation $A(\mathbb{R}) = \frac{1}{2}$. The lemma is thus proved.

Lemma 4.2. Let $B_1, B_2, \ldots$ be disjoint subsets of the carrier of an algebra $\mathbb{R}$ with $\varphi(\mathbb{R}) = 0$. Then the $(k+1)$-ary operation $h_{B_0, B_1, \ldots, B_k}$ defined by the condition $h_{B_0, B_1, \ldots, B_k}(x_1, x_2, \ldots, x_{k+1}) = x_{k+1}$ if $x_{k+1} \in B_j$ ($j = 1, 2, \ldots, k$) and $h_{B_0, B_1, \ldots, B_k}(x_1, x_2, \ldots, x_{k+1}) = x_{k+1}$ if $x_{k+1} \notin \bigcup_{j=1}^k B_j$ is algebraic in the reduct $\mathbb{R}(\mathbb{R})$.

Proof. We shall prove the lemma by induction with respect to $k$.

First consider the case $k = 1$. By Lemma 4.1 there exists an algebraic constant, say $c$, in $\mathbb{R}$. Put $g(x, y) = c$ if $x \in B_1$, $g(x, y) = c$ if $y \in B_1$,

$$g(x, y) = c \text{ if } y \in B_1, g(x, y) = c \text{ if } y \notin B_1, \quad g(x, c) = g(c, x) = x \text{ and}$$

$$g(x, y) = c \text{ otherwise.}$$

The operations $g_1, g_2$, and $g_3$ preserve the algebraic constants in $\mathbb{R}$ and, consequently, are algebraic in $\mathbb{R}$ because of the formula $\varphi(\mathbb{R}) = 0$. Moreover, being binary operations they are also algebraic in $\mathbb{R}(\mathbb{R})$. Further, it is easy to verify the equation

$$h_{B_0}(x_1, x_2, x_3) = g_3(g_2(x_1, x_2), g_2(x_3, x_4))$$

which shows that the operation $h_{B_0}$ is algebraic in $\mathbb{R}(\mathbb{R})$.

Now suppose that $k > 2$ and the $(k+1)$-ary operation $h_{B_0, B_1, \ldots, B_k}$ is algebraic in $\mathbb{R}(\mathbb{R})$. Since

$$h_{B_0, B_1, \ldots, B_k}(x_1, x_2, \ldots, x_{k+1}) = h_{B_0, B_1, \ldots, B_k}(x_1, x_2, \ldots, x_{k+1}, x_{k+2}, x_{k+3}),$$

where $B = \bigcup_{j=1}^k B_j$, we infer that the operation $h_{B_0, B_1, \ldots, B_k}$ is algebraic in $\mathbb{R}(\mathbb{R})$ which completes the proof.

It is evident that $c = 0$ is the only algebraic operation. Moreover, it is well known that the arity of the complete algebra over an at least two-element set is 2 (see [3] and [5]). Now we shall prove a sharpening of this theorem. We note that for infinite algebras the proof is based on the axiom of choice.

Theorem 4.1. If $\mathbb{A}$ is an algebra over an at least two-element set and $\varphi(\mathbb{A}) = 0$, then $\varphi(\mathbb{A}) = 2$.

Proof. Let $A$ be the carrier of the algebra $\mathbb{A}$ and $\varphi(\mathbb{A}) = 2$. Then the ternary operation $h_2$ defined in Lemma 4.2 is algebraic in $\mathbb{A}(\mathbb{A})$ and, of course, depends on every variable. Thus $\mathbb{A}$ cannot be a unary algebra and consequently, $\varphi(\mathbb{A}) = 2$. To prove the converse inequality it suffices, by proposition (ii) in Section 1, to prove that each $n$-ary algebraic operation in $\mathbb{A}$ is algebraic in $\mathbb{R}(\mathbb{R})$. We shall prove this statement by induction with respect to $n$. For $n = 2$ it is obvious. Suppose that $n > 2$ and that the statement is true for $(n-1)$-ary operations. Let $f$ be an arbitrary $n$-ary algebraic operation in $\mathbb{A}$. We shall consider three cases.

Case 1: card $A < n$. The set $A(\mathbb{A})$ of algebraic constants in $\mathbb{A}$ will be briefly denoted by $A(\mathbb{A})$. Further, we introduce the notation $A(\mathbb{A}) = \{a_1, a_2, \ldots, a_n\}$ and $A(\mathbb{A}) = \{a_1, a_2, \ldots, a_n\}$. Put $B_0 = \{a_j \mid j = 1, 2, \ldots, n\}$ and

$$g_1(x_1, x_2, \ldots, x_n) = h_{B_0, B_1, \ldots, B_k}(f(x_1, x_2, \ldots, x_{k-1}, a_1, a_{k+1}, \ldots, a_n),$$

$$f(a_1, x_2, \ldots, x_{k-1}, a_1, a_{k+1}, \ldots, a_n), \ldots, f(x_1, x_2, \ldots, x_{k-1}, a_1, a_{k+1}, \ldots, a_n))$$

which says, in particular, that the operations $g_1, g_2, \ldots, g_n$ are algebraic in $\mathbb{R}(\mathbb{R})$. Moreover, $g_1(x_1, x_2, \ldots, x_n)$ if
Now we define operations $h_1, h_2, ..., h_n$ recursively as follows

$$h_1(x_1, x_2, ..., x_n) = g(x_1, x_2, ..., x_n),$$
$$h_{b+1}(x_1, x_2, ..., x_n) = h_{b}(g_{b+1}(x_1, x_2, ..., x_n), p_{b}(x_1, x_2, ..., x_n), x_n)$$

($b = 1, 2, ..., n-1$),

where $B = A^0$ and the operation $h_B$ is defined in Lemma 4.2. Obviously, the operations $h_1, h_2, ..., h_n$ are algebraic in $B$. Moreover, it is easy to verify that $h_B(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n)$ if $x_n \in A^0$ for some index $i$ satisfying the inequality $1 \leq i \leq k$. In particular, we have the equation

$$h_B(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) \quad \text{if} \quad (x_1, x_2, ..., x_n) \cap A^0 \neq \emptyset.$$

From these equations it follows that the operation $f$ is algebraic in $B$ if $A = A^0$. Suppose now that $A \neq A^0$ and put $p_B(x_1, x_2, ..., x_{n-1}) = f(x_1, x_2, ..., x_{n-1})$ if all $x_1, x_2, ..., x_{n-1} \in A^0$ and $p_B(x_1, x_2, ..., x_{n-1}) = c_1$ otherwise ($j = 1, 2, ..., s$). The operations $p_j$ preserve all algebraic constants in $B$ and, consequently, are algebraic in $B$. By the inductive assumption they are also algebraic in $B$. Thus the composition

$$p_B(x_1, x_2, ..., x_n) = h_{n-1}(p_B(x_1, x_2, ..., x_{n-1}), ..., p_1(x_1, x_2, ..., x_{n-1}), x_{n-1}, x_n),$$

where $D_j = \{d_j\}$ ($j = 1, 2, ..., s$) and the operation $h_{n-1}, h_{n-2}, ..., h_1, \in B$ is algebraic in $\mathcal{R}(B)$. Moreover,

$$f(x_1, x_2, ..., x_n) = h_B(x_1, x_2, ..., x_n) \quad \text{if} \quad (x_1, x_2, ..., x_n) \cap A^0 \neq \emptyset.$$

Now we define auxilary operations $g_1, g_2, ..., g_{n-1}$ recursively as follows

$$g_1(x_1, x_2, x_3) = h_B(x_1, x_2, x_3),$$
$$g_{j+1}(x_1, x_2, ..., x_{j+2}) = g_j(g_{j+1}(x_1, x_2, ..., x_{j+1}), x_1, x_2, ..., x_{j+1}) \quad (j = 1, 2, ..., n-1),$$

where $B = A^0$ and the operation $h_B$ is defined in Lemma 4.2. Of course, all these operations are algebraic in $B$. Moreover, $g_{j+1}(x_1, x_2, ..., x_{j+3}) = x_1$ for all $x_1 \in A^0$ for some index $i$ satisfying the inequality $3 \leq i \leq j+2$ and $g_{n-1}(x_1, x_2, ..., x_{n+1}) = c_1$ otherwise. The composition

$$f_B(x_1, x_2, ..., x_n) = g_B(h_B(x_1, x_2, ..., x_n), p_B(x_1, x_2, ..., x_n), x_1, x_2, ..., x_n)$$

is algebraic in $B$. If $(x_1, x_2, ..., x_n) \cap A^0 \neq \emptyset$, then, by (4.1),

$$f_B(x_1, x_2, ..., x_n) = h_B(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n).$$

Further, if $(x_1, x_2, ..., x_n) \cap A^0 = \emptyset$, then, by (4.3),

$$f_B(x_1, x_2, ..., x_n) = p(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n).$$

Thus $f = f_B$ and, consequently, the operation $f$ is algebraic in $B$. This completes the proof of Case 1.

Case 2: card $A^0 \leq \kappa$ and card $A \geq \kappa$. Denoting by $B^0$ the Cartesian product of $k$ copies of the set $B$ we have the equation card $(A^{k-1}\setminus(A^0)^{k-1}) = card(A\setminus A^0)$. Let $g_B$ be a one-to-one mapping from $A^{k-1}\setminus(A^0)^{k-1}$ onto $A\setminus A^0$. We note that, by Lemma 4.1, the set $A^0$ is non-void. Let $A^{00} = (x_1, x_2, ..., x_\kappa)$.

We define an auxiliary operation $g$ as follows:

$$g(x_1, x_2, ..., x_{n-1}) = g_B(x_1, x_2, ..., x_{n-1}) \quad \text{if} \quad (x_1, x_2, ..., x_{n-1}) \in A^{k-1}\setminus(A^0)^{k-1}$$
$$g(x_1, x_2, ..., x_{n-1}) = c_1 \quad \text{otherwise}.$$
Further, $d(x, y) = \epsilon$ in the opposite case. The operations $d_1, d_2, \ldots, d_n$ are algebraic in $\mathcal{A}(\mathbb{R})$, because they preserve algebraic constants in $\mathbb{R}$. Moreover, for the compositions

$$s_j(x_1, x_2, \ldots, x_n) = d_j(s(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n), x_j) \quad (j = 1, 2, \ldots, n)$$

we have the formula

$$s_j(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)$$

if either $x_1, x_2, \ldots, x_n \in \mathbb{A}^{\omega}$ or $x_j \in \mathbb{A}^{\omega}$ \hspace{1cm} (j = 1, 2, \ldots, n).

Setting $B = \mathbb{A}^{\omega}$ we define operations $f_1, f_2, \ldots, f_n$ recursively as follows

$$f_j(x_1, x_2, \ldots, x_n) = s_j(x_1, x_2, \ldots, x_n)$$

and

$$f_{j+1}(x_1, x_2, \ldots, x_n) = h_k(f_1(x_1, x_2, \ldots, x_n), s_1(x_2, x_3, \ldots, x_n), x_{j+1}) \quad (j = 1, 2, \ldots, n - 1)$$

where the operation $h_k$ is defined in Lemma 4.2. Of course, all operations $f_1, f_2, \ldots, f_n$ are algebraic in $\mathcal{A}(\mathbb{R})$. Moreover, taking into account (4.3), we can easily prove by induction with respect to $j$ that the formula $f_j(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)$ if either $x_1, x_2, \ldots, x_n \in \mathbb{A}^{\omega}$ or $x_j \in \mathbb{A}^{\omega}$ for some index $i$ satisfying the inequality $1 \leq i \leq j$. Thus $f = f_n$ and, consequently, the operation $f$ is algebraic in $\mathcal{A}(\mathbb{R})$ which completes the proof of the theorem.

**5. Some reducts of Boolean algebras.** Let $\mathbb{B}$ be a Boolean algebra with a denumerable set of generators, $\mathbb{B}$ as the neutral element and $1 = 0^*$. Let us introduce the notation

$$t(x) = x^* \quad s_i(x, y) = x \lor y \quad x_i(x, y) = x \land y$$

$$\mathbb{B}_n = (\mathbb{A}; \{0, 1\}) \quad \mathbb{B}_i = (\mathbb{A}; \{0, 1, t, s_i\})$$

Moreover, by 0 and 1 we shall denote the constant operations equal to 0 and 1 respectively. Denoting the carrier of $\mathbb{B}$ by $\mathcal{A}$ we put

$$\mathbb{B}_n = (\mathcal{A}; \{0, 1\}) \quad \mathbb{B}_i = (\mathcal{A}; \{0, 1, t, s_i\})$$

Furthermore, we put $\mathbb{B}_n = (\mathcal{A}; \{0, 1\}; \{0, 1, t, s_i\})$.

**Theorem 5.1.** For any $p$ and $q$ satisfying the condition $q = 0, 1, \ldots, \text{and } \max(2, q-1) \leq p \leq \infty$ by $\mathcal{A}$ we shall denote a subalgebra of $\mathcal{A}$ with $\mathcal{A}(\mathbb{B}_n) = p$ containing both elements 0 and 1. Moreover, we put $\mathbb{B}_n = (\{0, 1\}; \{0, 1, t, s_i\})$.

**Proof.** It is clear that the elements 0 and 1 form a two-element subalgebra of $\mathcal{A}$ which will be denoted by $\mathbb{B}$. The algebra $\mathbb{B}$ is a subalgebra of $\mathbb{B}_n$ for all indices $p$ satisfying the inequality $\max(2, q-1) \leq p \leq \infty$. Moreover,

$$\mathbb{B}_n = \mathbb{B}.$$

Each algebraic operation in $\mathbb{B}_n$ is uniquely determined by its restriction to $\mathbb{B}$.

Moreover,

$$\mathcal{E}(\mathbb{B}_n) = \mathcal{E}(\mathbb{B})$$

and, by Theorem 4.1 in [1],

$$\mathcal{E}(\mathbb{B}_n) = \mathcal{E}(\mathbb{B})$$

we get the formulas

$$\mathbb{B}_n = (\mathcal{A}; \{0, 1\}) \quad \mathbb{B}_i = (\mathcal{A}; \{0, 1, t, s_i\}) \quad \mathbb{B}_n = (\mathcal{A}; \{0, 1, t, s_i\}) \quad \mathbb{B}_n = (\mathcal{A}; \{0, 1, t, s_i\})$$

with $(n = 3, 4, \ldots)$. Hence from the table in [1], p. 274 we get the following formulas

$$\mathcal{E}(\mathbb{B}_n) = \mathcal{E}(\mathbb{B}_n) = \mathbb{B}$$

We note that in the definition of $\mathbb{B}_n$ in [1], p. 273 $\nu_{n+1}$ instead of $\nu_n$ should be written. Hence and from the table in [1], p. 274 we get the following formulas

$$\mathcal{E}(\mathbb{B}_n) = \mathcal{E}(\mathbb{B}_n) = \mathbb{B}$$

Now the assertion of the theorem is a direct consequence of formulas (5.1), (5.2), (5.3) and (5.4).

**6. A class of unary algebras.** An algebra is said to be unary if all its algebraic operations essentially depend on at most one variable. It was proved in [1] (Theorem 13.1) that for unary algebras with $\mathcal{E}(\mathbb{B}) \geq 3$ the equation $\mathcal{E}(\mathbb{B}) = \mathcal{E}(\mathbb{B})$ holds. In this section we shall study unary algebras whose all elements are algebraic constants.

**Theorem 6.1.** For unary algebras with $\mathcal{E}(\mathbb{B}) = 0$ the inequality $\mathcal{E}(\mathbb{B}) \leq 0$ holds.

**Proof.** By Theorems 3.1 and 13.1 in [1], we have the inequality $\mathcal{E}(\mathbb{B}) \leq 0$. Consequently, by proposition (i) in Section 1, to prove the theorem it suffices to prove that each ternary algebraic operation $f$ in $\mathbb{B}$ is algebraic in $\mathbb{B}$. Of course, without loss of generality we may assume that the operation $f$ depends on the first variable. Consequently, there
exist elements $a_0$ and $b_0$ and a non-constant unary operation $g$ algebraic in $\mathbb{A}$ such that

\begin{equation}
(6.1)
    f(x, a_0, b_0) = g(x).
\end{equation}

Further, the binary operation $f(x, a_0, x)$ is algebraic in $\mathbb{A}$ and, consequently, depends on at most one variable. By (6.1) it depends on the variable $x$ which implies the formula

\begin{equation}
(6.2)
    f(x, a_0, x) = g(x).
\end{equation}

Let $c$ be an arbitrary element of the algebra $\mathbb{A}$. The binary operation $f(x, y, c)$ is also algebraic in $\mathbb{A}$ and, consequently, depends on at most one variable. Taking into account (6.2) we infer that it depends on the variable $x$ and, consequently, $f(x, y, c) = g(x)$ for all elements $c$. Thus the operation $f$ is algebraic in $\mathbb{A}$ which completes the proof.

We note that if all unary operations are algebraic in an at least two-element algebra $\mathbb{A}$, then $\gamma_0(\mathbb{A}) = 0$ and $\epsilon(\mathbb{A}) = 3$.

**Theorem 6.2.** Suppose that the unary algebra $\mathbb{A}$ with $\gamma_0(\mathbb{A}) = 0$ satisfies the condition

\begin{equation}
(6.3)
    \min\{\operatorname{card}(\mathbb{A}): f(x, y, z) \in \mathbb{A}\} > \operatorname{card}(\mathbb{A})^2 + 1,
\end{equation}

where $\mathbb{A}$ is the carrier of $\mathbb{A}$. Then $\epsilon(\mathbb{A}) = 1$.

**Proof.** Since $\mathbb{A}$ is not the complete algebra over the set $\mathbb{A}$, we have the inequality $\epsilon(\mathbb{A}) > 1$. Moreover, by Theorem 6.1, $\epsilon(\mathbb{A}) \leq 2$. Thus, to prove the Theorem it suffices to prove that each algebraic operation $f$ in $\mathbb{A}$ is algebraic in $\mathbb{A}$. Without loss of generality we may assume that $f$ depends on the first variable. Consequently, there exist an element $a_0$ in $\mathbb{A}$ and a non-constant operation $h \in \mathbb{A}(\mathbb{A})$ such that

\begin{equation}
(6.4)
    f(x, a_0) = h(x).
\end{equation}

Put

\begin{equation}
B = \{b: h(b) \neq g(a_0) \text{ for all } g \in \mathbb{A}(\mathbb{A})\}.
\end{equation}

From inequality (6.3) it follows that

\begin{equation}
(6.5)
    \operatorname{card} B > 2.
\end{equation}

For every $b \in B$ the unary operation $f(b, x)$ is algebraic in $\mathbb{A}$ and does not depend on the variable $x$. Indeed, the equation $f(b, x) = g(a_0)$, where $g \in \mathbb{A}(\mathbb{A})$, would imply, by (6.4), the equation $h(b) = g(a_0)$ and, consequently, the relation $b \in B$. Thus for every $b \in B$ the equation $f(b, x)$ is constant. Hence and from (6.4) we get the formula

\begin{equation}
(6.6)
    f(b, x) = h(b) \quad (b \in B).
\end{equation}

Let $c$ be an arbitrary element of $\mathcal{A}$. Suppose that the operation $f(x, c)$ is constant, say $f(x, c) = c_0$. From (6.5) it follows that there exists an element $b_0 \in B$ such that $h(b_0) = c_0$. Further, by (6.6), $h(b_0) = f(b_0, c) = c_0$ which gives the contradiction. Thus for every element $c \in \mathcal{A}$ the operation $f(x, c)$ depends on the variable $x$. In other words, for every $c \in \mathcal{A}$ there exists an operation $g_c : \mathbb{A}(\mathbb{A}) \to \mathbb{A}(\mathbb{A})$ such that

\begin{equation}
(6.7)
    f(x, c) = g_c(x) \quad (x, c \in \mathcal{A}).
\end{equation}

Suppose now that the operation $f$ depends on both variables. Then there exist an element $c_0 \in \mathcal{A}$ and an operation $h_0 : \mathbb{A}(\mathbb{A}) \to \mathbb{A}(\mathbb{A})$ such that

\begin{equation}
(6.8)
    f(x, y) = g_{c_0}(y) \quad (y \in \mathcal{A}).
\end{equation}

Hence and from (6.7) we get the inclusion

\begin{equation}
(6.9)
    h_0(\mathcal{A}) = \{g_{c_0}(x): x \in \mathcal{A}\} \subseteq \mathbb{A}(\mathbb{A})
\end{equation}

which contradicts condition (6.3). Thus the operation $f$ depends on one variable and, consequently, is algebraic in $\mathcal{A}$ which completes the proof of the theorem.

Let $\mathcal{A}_m = (\mathbb{A}, \ldots, \mathbb{A})$, where $m > 3$. By $\mathcal{F}_m$ we shall denote the family of all constant operations on $\mathcal{A}_m$ and by $\mathcal{H}$ the unary operation defined by the conditions $h(0) = 1$ and $h(x) = x$ for $x \neq 1$. Put $\mathcal{H}_{m+1} = (\mathcal{A}_m, \mathcal{F}_m)$ and $\mathcal{H}_{m+1} = (\mathcal{A}_m, \mathcal{H}, \mathcal{F}_m)$. The formulas

\begin{equation}
(6.10)
    \varphi(\mathcal{H}_{m+1}) = 0 \quad \text{and} \quad \varphi(\mathcal{H}_{m+1}) = 1
\end{equation}

are obvious. Moreover, $\gamma_0(\mathcal{H}_{m+1}) = \gamma(\mathcal{H}_{m+1}) = 0$. Further, non-constant algebraic operations in $\mathcal{H}_{m+1}$ are trivial and the operation $h$ is the only non-constant and non-trivial unary operation in $\mathcal{H}_{m+1}$. Thus for the algebra $\mathcal{H}_{m+1}$ the left-hand side and the right-hand side of (6.3) are equal to $3$ and $2$ respectively. The same quantities for the algebra $\mathcal{H}_{m+1}$ are equal to $m + 1$ and $2$ respectively. Thus, both algebras satisfy the condition of theorem 6.2 and, consequently,

\begin{equation}
(6.11)
    \epsilon(\mathcal{H}_{m+1}) = \epsilon(\mathcal{H}_{m+1}) = 1.
\end{equation}

7. **Description of all pairs $((\epsilon, \varphi), \mathcal{H}_{m+1})$.** The algebras $\mathcal{H}_{m+1}$ ($p = 1, 2, \ldots, \infty; q = 0, 1, \ldots, \infty$) defined in the preceding sections satisfy the conditions $\varphi(\mathcal{H}_{m+1}) = p$ and $\varphi(\mathcal{H}_{m+1}) = q$ (see Theorems 2.1, 2.2, 3.2 and 5.1 and formulas (6.8) and (6.9)). Moreover, for a one-element algebra we have the formula $\epsilon = \varphi = 0$ and for complete algebras over an at least two-element set the formulas $\epsilon = 0$ and $\varphi = 2$. On the other hand, by Theorem 4.1, for algebras with $\epsilon = 0$ we have either $\varphi = 0$ or $\varphi = 2$. Thus we have proved the following theorem.
Theorem 7.1. The set of all possible pairs $(e, q)$ for abstract algebras is the set of all pairs $(p, q)$, where either $p = 1, 2, \ldots, \infty$, $q = 0, 1, \ldots, \infty$ or $p = 0$ and $q = 0, 1$.

References