Unified spaces and singular sets for mappings of locally compact spaces

by

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1. Introduction. This paper may be considered as a continuation of the research started by G. T. Whyburn in [13], [14] and [15]. The results stated in this section are due to him and may be found in these papers.

Let X and Y be disjoint Hausdorff spaces and let $f\colon X\to Y$ be a continuous map of X into Y. In [13] and [14] Whyburn defined a unified space for the mapping f as follows:

Let W denote the set-theoretic union of X and Y and define a set Q in W to be open if is satisfies:

(i) $Q\cdot X$ and $Q\cdot Y$ are open in X and Y respectively; and

(ii) for any compact set K in $Q\cdot Y$, $f^{-1}(K)\cdot(X-Q\cdot X)$ is compact in X.

The set W together with the collection of open sets so defined is a T_1-topological space which is called the *unified space of* f and which we denote by Z. (Actually in [13] Whyburn did not require X and Y to be disjoint, but took disjoint copies of X and Y for his construction. He did, however, require that f be an onto mapping. In [14] he simplified the treatment by assuming that X and Y were disjoint and by allowing f to be an into mapping.)

The injections of X and Y into Z are open and closed respectively, thus X is embedded in Z as an open set and Y is embedded in Z as a closed set.

Associated with Z is a retraction $r\colon Z\to Y$ of Z onto Y defined by $r(z)=f(z)$ for $z\in X$ and $r(z)=z$ for $z\in Y$. This retraction is continuous and compact. A mapping g from a topological space W into a topological space V is said to be *compact* provided for every compact set K in V,

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$g^{-1}(K)$ is compact in W. Thus when Y is (locally) compact, Z is also (locally) compact. Furthermore, whenever f is an open mapping, so also is r.

In [13] it was shown that when X and Y are both locally compact Hausdorff spaces, so also is Z and that when X and Y are both locally compact separable metric spaces, Z is also a locally compact separable metric space. Henceforth we shall assume that X and Y are at least locally compact Hausdorff spaces and thus Z will be locally compact and Hausdorff.

We adopt Whyburn's notation and use $\tilde{E}$ to denote the closure in Z of any subset E of Z. For any subset A of X (or Y), the closure of A in X (or in Y respectively) will be denoted by $\bar{A}$, the interior of A by $\operatorname{int} A$ and the boundary of A by $\operatorname{Fr} A$. Thus, since Y is closed in Z, for any set A in Y, $\bar{A} = \tilde{A}$.

The restriction $r|X$ of r to X is topologically equivalent to f and thus $r|\tilde{X}$ is a compact mapping which extends f. Furthermore, since Y is locally compact, r and $r|\tilde{X}$ are closed mappings ([19], Section 4, p. 106).

Heinz Bauer in his study of conservative maps in [1] showed that for every mapping $f\colon X \to Y$, where X and Y are locally compact Hausdorff spaces, there exist a locally compact Hausdorff space X_0 and a mapping f_0 satisfying:

(i) *X is a dense subset of X_0*,

(ii) $f_0|X \equiv f$,

(iii) *f_0 is a compact mapping*,

(iv) *f_0 is 1-1 on $X_0 - X$*.

Furthermore, he showed that every space satisfying (i)-(iv) is homeomorphic to X_0 under a homeomorphism that leaves points of X fixed. Thus his X_0 is homeomorphic to Whyburn's $\tilde{X}$ and f_0 is topologically equivalent to $r|\tilde{X}$.

Momentarily leaving the unified space Z, let us consider the mapping $f\colon X \to Y$ and define Q to be the union of the interiors of sets in Y having a compact inverse under f. Then if $P = f^{-1}(Q)$, $f|P\colon P \to Q$ is a compact mapping. The complement S of Q in Y is closed and contains all of the points of Y at which f is not compact, i.e. S is the set of those points in Y which are not interior to any set with a compact inverse under f. The set S is called the *singular set of f in Y* and the set $T = f^{-1}(S)$ is called the *singular set of f in X*. We see immediately that f is compact if and only if S is empty.

The singular set S of f in Y is the boundary $\tilde{X}\cdot Y$ of X in Z. In order to see this let $y \in Y$ and suppose that $y \notin \tilde{X}\cdot Y$. Since Z is locally compact and Hausdorff, there exist conditionally compact open sets U and V of Z containing y such that $\tilde{V} \subset U \subset Z - X$. Then y is an element of the interior relative to Y of the compact set $\bar{V}$ $(=\tilde{V})$ and since U is open in Z, $f^{-1}(\bar{V}) = f^{-1}(\bar{V})\cdot(X - U\cdot X)$ is compact in X. Thus $y \notin S$. Now suppose that $y \notin S$. Then there exists a compact set K of Y such that $y \in \operatorname{int} K$ (relative to Y) and $f^{-1}(K)$ is compact. But this implies that $\operatorname{int} K$ is open in Z and since $\operatorname{int} K$ misses X, $y \notin \tilde{X}\cdot Y$. Thus $\tilde{X}\cdot Y = S$. This equality provides a clue to the relationship of the mapping properties of f and the topological properties of Z. For example we have that when X and Y are connected, Z is connected if and only if f is not compact. Whyburn has shown in [14] that the mapping $f\colon X \to f(X)$ is compact if and only if $r(x) = x = r^{-1}(x)$ for every $x \in \tilde{X}\cdot Y$ or equivalently if and only if $\tilde{X}\cdot f(X)$ is empty.

A fundamental aim of this paper is to relate the topological properties of X and Y and the mapping properties of f to the topological properties of Z. In Section 2 we show that Z is paracompact if and only if Y is paracompact. In this same section we give necessary and sufficient conditions for the metrizability of Z and obtain a bound for the covering dimension of Z in terms of the strong inductive dimension of Y and the covering dimension of point inverses of f. We also show that the domain of any compact retraction is a unified space of a mapping.

In Section 3 the local connectedness of Z is investigated and we give a necessary and sufficient condition for Z to be locally connected in terms of a mapping property of f. Several interesting maps are shown to have a locally connected unified space. In Section 4 we give the definitions of two topological properties and show how they are related to unicoherence. Necessary and sufficient conditions for the unicoherence of a compact and locally connected unified space are then given in terms of the topological properties of X and Y. In the last section of this paper we investigate the compactness and the connectedness of the singular sets S and T in Y and X respectively.

Summary of notation. As stated above $f\colon X \to Y$ will always denote a continuous map of X into Y where X and Y are at least locally compact Hausdorff spaces. When f is an onto mapping it will be stated explicitly. The singular sets of f in X and Y will be denoted by T and S respectively. We shall always use Z to denote the unified space of f and use $r\colon Z \to Y$ to denote the retraction of Z onto Y induced by f.

A region is an open connected set, a continuum is a compact connected set and a generalized continuum is a locally compact, connected and separable metric space. By a *mapping* we will mean a *continuous transformation*.

2. Metrizability, dimension and characterizations of the unified spaces. In this section we show that Z is paracompact if and only if Y is paracompact and prove a theorem giving a necessary and sufficient condition for the metrizability of Z. The covering dimension of Z is shown to be related to the covering dimension of point inverses of f and the strong inductive dimension of Y. Finally we show that the domain of any compact retraction is homeomorphic to a unified space for a mapping. In this section we assume that X and Y are at least locally compact Hausdorff spaces.

(2.1) LEMMA. *Let g be a closed mapping of the topological space W onto the topological space V such that point inverses of g are compact. If V is paracompact, so also is W.*

Proof. Let $\{U_a\}$ $(a \in \Gamma)$ be an open covering of W. For each $y \in V$ select a finite union H_y of elements of $\{U_a\}$ $(a \in \Gamma)$ that contains the compact set $g^{-1}(y)$. Since g is closed, $G_y = V - g(W - H_y)$ is an open set containing y and $\{G_y\}$ $(y \in V)$ is an open covering of V. Since V is paracompact, there exists a locally finite open cover $\{K_\beta\}$ $(\beta \in \Delta)$ of V which refines $\{G_y\}$ $(y \in V)$. We note that for each $y \in V$, $g^{-1}(G_y)$ is contained in H_y. Thus for each $\beta \in \Delta$ we may and do choose a finite subset Γ_β of Γ such that $g^{-1}(K_\beta) \subset \sum U_a$ $(a \in \Gamma_\beta)$. We now assert that $\{g^{-1}(K_\beta)\cdot U_a\}$ $(\beta \in \Delta$ and $a \in \Gamma_\beta)$ is a locally finite covering of W. In order to see this let $x \in W$. Since $\{K_\beta\}$ $(\beta \in \Delta)$ is locally finite, there exists an open set P of V containing $g(x)$ such that P meets only finitely many K_β. Then $g^{-1}(P)$ is an open subset of W containing x that meets only finitely many of the sets $g^{-1}(K_\beta)$ and thus meets only finitely many of the sets $g^{-1}(K_\beta)\cdot U_a$, $a \in \Gamma_\beta$. Thus every open cover of W has a locally finite open refinement and W is paracompact.

(2.2) THEMOREM. *A necessary and sufficient condition that Z be paracompact is that Y be paracompact.*

Proof. Since Y is locally compact and Hausdorff, $r\colon Z \to Y$ is a closed mapping ([19], Section 4, p. 106). Thus the sufficiency is a consequence of Lemma (2.1). The necessity follows from a well-known result of E. Michael [9].

(2.3) **Remark.** Note that whenever Z is paracompact, the closed set $\tilde{X}$ is also paracompact. Thus in this case even when X is not normal, X is embedded as a dense subset of the paracompact, locally compact Hausdorff space $\tilde{X}$ on which f has a compact extension.

THEOREM. *Suppose that Y is a metric space. A necessary and sufficient condition for Z to be a metric space is that inverse image under f of any compact set in Y be a separable metric space.*

Proof of the necessity. Let K be a compact subset of Y; then $r^{-1}(K)$ is a compact subset of Z and as such is a separable metric space. Hence $f^{-1}(K)$ is also a separable metric space.

Proof of the sufficiency. By a result of A. H. Stone, Y is paracompact. Thus there exists a locally finite covering $\{K_a\}$ $(a \in \Gamma)$ of Y by compact sets. By our assumption each $H_a = f^{-1}(K_a)$, $a \in \Gamma$, is a separable metric space. Whyburn has shown that when the range and domain of a mapping are both separable metric spaces, so also is the unified space [13]. Hence the unified space Z_a of the mapping $f|H_a\colon H_a \to K_a$ is a separable metric space. One can easily show that for each $a \in \Gamma$, Z_a is homeomorphic to $r^{-1}(K_a)$. Thus $\{r^{-1}(K_a)\}$ $(a \in \Gamma)$ is a locally finite closed covering of Z by metric spaces and by a result of Nagata [10], Z is metrizable.

(2.5) **Remark.** As an immediate consequence of Theorem (2.4) we see that *when Z is a metric space, X is separable if and only if Y is separable.*

V. V. Proizvokov in [12], Cor. 1, p. 154, has shown that the domain of any mapping whose point inverses have weak-inductive dimension zero is a metric space whenever it is locally connected and locally compact and the range is a metric space. Thus if we consider the case where f is a light mapping (i.e. point inverses are totally disconnected) of X onto Y and X and Y are metric spaces, Z is also a metric space whenever it is locally connected. This is because the weak-inductive dimension of $f^{-1}(y)$ is not increased by the addition of the point y in Z, so that the weak-inductive dimension of $r^{-1}(y) = y + f^{-1}(y)$ is still zero. The local connectedness of Z is studied in Section 3.

(2.6) DEFINITIONS. The empty set and only the empty set has *strong-inductive (covering) dimension* -1. A space W has strong-inductive dimension $\leqslant n$ $(n \geqslant 0)$, written $\operatorname{Ind} W \leqslant n$, provided every pair of disjoint closed sets in W can be separated by a closed set of strong-inductive dimension $\leqslant n-1$. We say that the strong-inductive dimension of W is n written $\operatorname{Ind} W = n$, if $\operatorname{Ind} W \leqslant n$ is true and $\operatorname{Ind} W \leqslant n-1$ is false.

The *order* of an open covering is the largest integer n such that there are $n+1$ members of the covering with a non-empty intersection. A space W has *covering dimension* $\leqslant n$ $(n \geqslant 0)$, written $\dim W \leqslant n$, provided every open covering of W has an open refinement of order $\leqslant n$. We say that the covering dimension of W is n, written $\dim W = n$, if $\dim W \leqslant n$ is true and $\dim W \leqslant n-1$ is false.

(2.7) THEOREM. *Suppose that Y is paracompact and that there is an integer $k \geqslant 0$ such that for every point $y \in Y$, $\dim f^{-1}(y) \leqslant k$. Then* $\dim Y \leqslant \dim Z \leqslant \operatorname{Ind} Y + k$.

Proof. By theorem (2.2), Z is paracompact and hence normal and by Theorem (2.3) of [3], $\dim f^{-1}(y) \geqslant \dim r^{-1}(y)$ for every $y \in Y$. Since Z is locally compact and Hausdorff, the compact map $r\colon Z \to Y$ is a closed map ([19], Section 4, p. 106). Thus applying Theorem VII. 7 of [11], p. 106 we have $\dim Z \leqslant \operatorname{Ind} Y + k$. Furthermore, since Y is closed in Z, $\dim Y \leqslant \dim Z$ and this completes the proof.

(2.8) **Remark.** If we knew that $\dim X \leqslant \dim Z$, Theorem (2.7) would extend Theorem VII. 7 of [11] which considers closed maps on normal spaces to continuous maps on locally compact Hausdorff spaces. However, there are not any theorems in dimension theory that will insure that $\dim X \leqslant \dim Z$. In fact there are examples of compact Hausdorff spaces R with subsets W where $\dim W > \dim R$; this is due to the lack of complete normality. The best we can say at present is that X is imbedded as a dense open subset of $\widetilde{X}$ where $\dim \widetilde{X} \leqslant \operatorname{Ind} Y + k$ and $r|\widetilde{X}$ is a compact extension of f with $\dim r^{-1}(y) \leqslant \dim f^{-1}(y)$ for all $y \in Y$.

(2.9) THEOREM. *Suppose that g: $W \to Y$ is a compact retraction where W is a locally compact Hausdorff space and Y is a proper subset of W. Then if X denotes the set $W - Y$ and f is the restriction of g to X, the identity transformation i: $Z \to W$ of the unified space Z of f: $X \to Y$ onto W is a homeomorphism. Thus the domain of any compact retraction onto a proper subset is a unified space for a mapping.*

Proof. I first wish to show that i is continuous. To this end let U be any open set in W. Clearly $U \cdot X$ and $U \cdot Y$ are open in X and Y respectively. Now if K is any compact set in $U \cdot Y$, $g^{-1}(K)$ is compact in W as is $g^{-1}(K) \cdot (W - U)$. But $g^{-1}(K) \cdot (W - U) = f^{-1}(K) \cdot (X - X \cdot U)$ and so U is open in Z and i is a continuous map.

Since W is a locally compact Hausdorff space, every compact mapping onto W is a closed mapping ([19], Section 4, p. 690). Thus in order to show that i is a homeomorphism it suffices to show that i is a compact mapping. To this end let K be a compact set in W. Then $g(K)$ is a compact set in Y and since $i|Y$ is a homeomorphism, $i^{-1}g(K)$ is a compact subset of Y considered as a subset of Z. The retraction r: $Z \to Y$ is a compact mapping, so $H = r^{-1}i^{-1}g(K)$ is a compact subset of Z. Furthermore, since i is continuous, $i^{-1}(K)$ is a closed subset of H and is compact. Thus i is a compact mapping and is a homeomorphism.

(2.10) Remark. Note that the unified space of any constant map on a non-compact, locally compact Hausdorff space W is homeomorphic to the one-point compactification of W. This fact together with the fact that the retraction of any unified space is a compact mapping yields: *A necessary and sufficient condition that a non-degenerate topological space be a compact Hausdorff space is that it be homeomorphic to the unified space of a mapping of a locally compact Hausdorff space into a compact Hausdorff space.*

3. Local connectedness of the unified space.

In this section necessary and sufficient conditions for the local connectedness of Z are given in terms of a mapping property of f and certain interesting maps are shown to have a locally connected unified space. Throughout this section we will assume that X and Y are locally connected and connected in addition to being locally compact and Hausdorff.

(3.1) THEOREM. *Suppose that X and Y are locally connected and connected and suppose also that Y satisfies the first axiom of countability. Then a necessary and sufficient condition for Z to be locally connected is that for every point $y \in Y$ and every open set U of Y containing y there is a region R of Y containing y such that $\bar{R} \subset U$ and $f^{-1}(\bar{R})$ has just finitely many compact components.*

Proof of the sufficiency. Suppose that Z is not locally connected. Then f is not compact. For if it were, Z would be the sum of two disjoint open and locally connected sets and would be locally connected contrary to our supposition. Hence Z is connected. By theorem (2.1) of [22], p. 102, there exist conditionally compact open sets U and V of Z where $\widetilde{V} \subset U$ and infinitely many components $\{C_\alpha\}$ $(\alpha \in \Gamma)$ of $\widetilde{U} - V$ such that for each $\alpha \in \Gamma$, $C_\alpha \cdot (\widetilde{U} - U) \neq \emptyset \neq C_\alpha \cdot (\widetilde{V} - V)$, $\limsup C_\alpha = L'$ meets both $\widetilde{U} - U$ and $\widetilde{V} - V$ and the component C of $\widetilde{U} - V$ that contains L' does not meet any C_α. If follows that Z is not locally connected at any point of $L' \cdot (U - \widetilde{V})$ and hence since X is locally connected and open in Z, $L' \cdot (U - \widetilde{V})$ must lie entirely in Y.

Let P and Q be open subsets of Z such that $\widetilde{V} \subset P \subset \widetilde{P} \subset Q \subset \widetilde{Q} \subset U$. For each $\alpha \in \Gamma$, let K_α be a component of $C_\alpha \cdot (\widetilde{Q} - P)$ such that $K_\alpha \cdot (\widetilde{P} - P) \neq \emptyset \neq K_\alpha \cdot (\widetilde{Q} - Q)$ and let $L = \limsup K_\alpha$. Then L meets both $\widetilde{P} - P$ and $\widetilde{Q} - Q$ and $L \subset L'$.

We assert that $K_\alpha \cdot Y = \emptyset$ for all but finitely many $\alpha \in \Gamma$. For suppose that $K_\beta \cdot Y \neq \emptyset$ for all $\beta \in \Delta$ where β is an infinite subset of Γ. For each $\beta \in \Delta$, let $y_\beta \in K_\beta \cdot Y$. Then $\sum y_\beta$ $(\beta \in \Delta)$ would have a limit point y in $Y \cdot (\widetilde{Q} - P)$. Let W be a region of Y containing y that is contained in $U - \widetilde{V}$. Then W must contain infinitely many y_β and hence intersect K_β for infinitely many $\beta \in \Delta$. But this is a contradiction for W must be a subset of C, the component of $\widetilde{U} - V$ that contains L', and hence W cannot meet any of the K_β, $\beta \in \Delta$. Therefore we may and do assume that $K_\alpha \cdot Y = \emptyset$ for all $\alpha \in \Gamma$.

Let $s \in L \cdot (Q - \widetilde{P})$. By hypothesis there exists a region R of Y containing s such that $\bar{R} \subset Y \cdot (Q - \widetilde{P})$ and $f^{-1}(\bar{R})$ has only finitely many compact components. We note that only one component of $r^{-1}(\bar{R})$ meets Y since $r^{-1}(\bar{R}) \cdot Y = \bar{R}$ is a continuum. Furthermore, if B is a component of $r^{-1}(\bar{R})$ that misses Y, B is a compact component of $f^{-1}(\bar{R})$. Thus since $f^{-1}(\bar{R})$ has only finitely many compact components, $r^{-1}(\bar{R})$ has only finitely many components, say $A_1, A_2, \ldots, A_n$. Since r is continuous and since $s \in \limsup K_\alpha$, $r(s) \in \limsup r(K_\alpha)$. Also since $s \in Y$ and each $K_\alpha \subset X$, $r(s) = s$ and $r(K_\alpha) = f(K_\alpha)$ and so $s \in \limsup f(K_\alpha)$. Therefore R must intersect infinitely many of the sets $f(K_\alpha)$ and some A_j, say A_1, must meet infinitely many of the sets K_α, $\alpha \in \Gamma$. Let $\varkappa$ be an infinite subset of Γ such that $K_\sigma \cdot A_1 \neq \emptyset$ for each $\sigma \in \varkappa$. If follows that A_1 is not a subset of $\widetilde{Q} - P$. For if it were, A_1 would not be connected.

Let F denote the set $(\tilde{Q}-Q)+(\tilde{P}-P)$. For each $\sigma \in \varkappa$, $A_1 \cdot K_\sigma \neq \emptyset \neq A_1 \cdot (Z-K_\sigma)$ and A_1 must meet the boundary of K_σ. Since each K_σ, $\sigma \in \varkappa$, is a subset of X and X is locally connected, the boundary of each K_σ is the set $K_\sigma \cdot F$. Thus for each $\sigma \in \varkappa$, $A_1 \cdot K_\sigma \cdot F \neq \emptyset$.

For each $\sigma \in \varkappa$ let $x_\sigma \in A_1 \cdot K_\sigma \cdot F$. Then $\sum x_\sigma$ $(\sigma \in \varkappa)$ is a subset of $f^{-1}(\bar{R}) \cdot \big(X - X \cdot (Q-\tilde{P})\big)$ and the latter set is compact in X by the definition of the topology in Z. Thus $\sum x_\sigma$ $(\sigma \in \varkappa)$ has a limit point x in X. But this is a contradiction since x would then be in $X \cdot L' \cdot (U-\tilde{V})$ and this set is empty. Hence Z is locally connected.

Proof of the necessity. Suppose that Z is locally connected and suppose that there exists a sequence $\{U_i\}$ $(i = 1, 2, \ldots)$ of conditionally compact regions of Y closing down on a point $y \in Y$ such that for each i, $\bar{U}_{i+1} \subset U_i$ and $f^{-1}(\bar{U}_i)$ has infinitely many compact components. Since every compact component of $f^{-1}(\bar{U}_i)$ is a component of $r^{-1}(\bar{U}_i)$, each $r^{-1}(\bar{U}_i)$ must have infinitely many components.

Since $r^{-1}(\bar{U}_2)$ is compact and every component of $r^{-1}(U_1)$ is open in Z, $r^{-1}(\bar{U}_2)$ is covered by a finite collection of components of $r^{-1}(\bar{U}_1)$. Let C_1 be a component of $r^{-1}(\bar{U}_1)$ that misses $r^{-1}(\bar{U}_2)$. Note that C_1 must lie entirely in X since $\bar{U}_1 = r^{-1}(\bar{U}_1) \cdot Y$ is a continuum that intersects $r^{-1}(\bar{U}_2)$. Let K_2 be a component of $r^{-1}(\bar{U}_2)$ that misses $r^{-1}(\bar{U}_3)$ and let C_2 be the component of $r^{-1}(\bar{U}_1)$ that contains K_2. Since $C_1 \cdot r^{-1}(\bar{U}_2) = \emptyset$, $C_1 \neq C_2$. Continue in this manner and select a sequence $\{C_i\}$ $(i = 1, 2, \ldots)$ of distinct components of $r^{-1}(\bar{U}_1)$ such that for each $i = 1, 2, \ldots$, $r^{-1}(\bar{U}_i) \cdot C_i \neq \emptyset$.

For each $i = 1, 2, \ldots$, let $x_i \in r^{-1}(\bar{U}_1) \cdot C_i$. Since $r^{-1}(\bar{U}_1)$ is compact, $\sum x_i$ $(i = 1, 2, \ldots)$ has a limit point p and since r is continuous, $p \in r^{-1}(y)$. But this is a contradiction since Z is locally connected and p is interior to some component of $r^{-1}(\bar{U}_1)$. This component must intersect infinitely many C_i $(i = 1, 2, \ldots)$ and this is absurd. This completes the proof.

(3.2) **Remark.** We note that in the proof of the necessity of Theorem (3.1) we only used the fact that for some sequence $\{U_i\}$ $(i = 1, 2, \ldots)$ of regions closing down on y, $f^{-1}(\bar{U}_{i+1})$ has a compact component that missed $f^{-1}(\bar{U}_i)$, $i = 1, 2, \ldots$, in order to obtain a contradiction. Thus if Z is locally connected, about every point $y \in Y$ there is a conditionally compact region R such that if U and V are any regions in Y containing y where $\bar{U} \subset \bar{V} \subset \bar{V} \subset R$, then every compact component of $f^{-1}(V)$ contains a component of $f^{-1}(\bar{U})$ and thus every compact component of $f^{-1}(\bar{V})$ meets $f^{-1}(y)$.

EXAMPLE. In this example Z is locally connected and Y contains a point y such that for every positive integer n there exists an open set U of Y such that the inverse under f of every region R of Y containing y

with $\bar{R} \neq U$ has at least n compact components. Thus there is no upper bound on the number of compact components of $f^{-1}(\bar{R})$ in Theorem (3.1).

Let L_0 be the segment in the plane joining the origin and the point $(1, 1)$ and for each positive integer i, let L_i be the segment from $(0, 1/i)$ to $(1/i, 1/i)$. Let X be set $\sum L_i$ $(i = 0, 1, 2, \ldots)$ minus the origin and let Y be the interval $[0, 1]$ on the x-axis. Define $f\colon X \to Y$ by $f(x, y) = x$. Then the unified space Z of f is merely the set $X+Y$ with the usual topology induced from the plane. The inverse image of the set $R = [0, 1/(n+1))$ in Y containing $y = 0$ has exactly n compact components.

EXAMPLE. We cannot weaken the hypothesis of the necessity of Theorem (3.1) to that of Z being locally connected at a point $y \in Z$. In this example every sufficiently small region R of Y about y has infinitely many compact components in $f^{-1}(\bar{R})$ and Z is locally connected at y. We construct this example in the plane.

Let L_0 and M_0 denote the line segments from the origin to the points $(1, 1)$ and $(1, -1)$ respectively. For every positive integer i, let L_i denote the segment from $(2^{1-i}, 2^{1-i})$ to $(2^{1-i}, 0)$ minus the latter endpoint and let M_i denote the segment from $(3^{1-i}, -3^{1-i})$ to $(3^{1-i}, 0)$ minus the latter endpoint. For every pair of positive integers (i, j) where $j > i$, let $P(i, j)$ denote the segment from $(2^{-i}, 2^{-j})$ to $(2^{-i}-2^{-i-1}(j-1)/j, 2^{-j})$ and let $Q(i, j)$ denote the segment from $(3^{-i}, -3^{-j})$ to $(3^{-i}-3^{-i-1}(j-1)/j, -3^{-j})$. Let A denote the right half of the circle centered at $(1, 0)$ with radius 1. Finally let X denote the set

$$A + \sum_{i \geqslant 0} L_i + \sum_{i \geqslant 0} M_i + \sum_{j>i>0} P(i, j) + \sum_{j>i>0} Q(i, j) \text{ minus the origin}.$$

Then if Y is the interval $[0, 1]$ on the x-axis, X and Y are locally connected generalized continua. The unified space of the mapping $f\colon X \to Y$ defined by $f(x, y) = \min\{x, 1\}$ is homeomorphic to the set $X+Y$ with the induced topology from the plane and this set is locally connected at the origin. However every region $R = [0, t)$ $(t < 1/2)$ has infinitely compact components in $f^{-1}(\bar{R})$.

DEFINITIONS. A locally connected and connected space W is said to be *regular* provided for every point $p \in W$ and every open set U of W containing p there is an open set V containing p such that $\bar{V} \subset U$ and $\mathrm{Fr}\, V$ is a finite set. A locally connected generalized continuum is said to be a *dendrite* provided it contains no simple closed curves.

(3.4) THEOREM. *Suppose that X is locally connected and connected and that Y is a regular space that satisfies the first axiom of countability. If point inverses of f have compact boundaries, Z is locally connected.*

Proof. We wish to show that f satisfies the condition of Theorem (3.1). Suppose to the contrary that there exists a point $y \in Y$ and an open set U

of Y containing y such that every region R of Y containing y with $\bar{R}\subset U$ has infinitely many compact components in $f^{-1}(\bar{R})$. Then since Y is regular and satisfies the first axiom of countability, we may select a sequence $\{U_i\}$ $(i=1,2,\ldots)$ of conditionally compact regions of Y closing down on y such that for each i, $\bar{U}_{i+1}\subset U_i$, $\mathrm{Fr}\,U_i$ is a non-empty finite set and $f^{-1}(\bar{U}_i)$ has infinitely many compact components. First of all we note that for each i, $H_i=\mathrm{Fr}\,f^{-1}(\bar{U}_i)$ is a subset of $f^{-1}(\mathrm{Fr}\,\bar{U}_i)$ and since $\mathrm{Fr}\,\bar{U}_i$ is a finite set and point inverses of f have compact boundaries, H_i is a compact subset of X.

Since H_2 is compact and every component of $f^{-1}(U_1)$ is open in X, H_2 is covered by a finite collection of components of $f^{-1}(\bar{U}_1)$. Let C_1 be a component of $f^{-1}(\bar{U}_1)$ that misses H_2. Let K_2 be a component of $f^{-1}(\bar{U}_2)$ that misses H_3 and let C_2 be the component of $f^{-1}(\bar{U}_1)$ that contains K_2. Since K_2 meets H_2, $C_1\neq C_2$. Continue in this manner and select a sequence $\{C_i\}$ $(i=1,2,\ldots)$ of distinct components of $f^{-1}(\bar{U}_1)$ such that for each $i=1,2,\ldots$, C_i contains a component of $f^{-1}(\bar{U}_i)$. Then for each $i\geqslant 2$, C_i contains a component of $f^{-1}(\bar{U}_2)$ and thus meets H_2. For each $i\geqslant 2$, let $x_i\in C_i\cdot H_2$. Then $\sum x_i$ $(i=2,3,\ldots)$ has a limit point p in the compact set H_2. But this is a contradiction; X is locally connected and $H_2\subset f^{-1}(U_1)$ and thus p is interior to some component of $f^{-1}(\bar{U}_1)$. This component must then intersect infinitely many C_i $(i=2,3,\ldots)$ which is impossible. Thus f satisfies the condition of Theorem (3.1) and Z is locally connected.

COROLLARY. *Suppose that Y is a dendrite and X is a locally connected generalized continuum and further suppose that f is a mapping of X onto Y such that for any compact set K in X, $f^{-1}f(K)$ is also compact in X. Then f is a compact mapping.*

Proof. Suppose that f is not compact. It is well known that Y is a regular space and so by Theorem (3.4), Z is a connected and locally connected space. Whyburn has shown that Z is also a separable metric space and thus Z is arcwise connected. Let $x\in X$ and let P be an arc in Z from x to a point $p\in Y$ such that $P\cdot Y=p$. If $f^{-1}(p)\cdot P\neq\emptyset$ let u be the last point of P in $f^{-1}(p)$ in the order from x to p; if $f^{-1}(p)\cdot P=\emptyset$ let uv be an arc in X from a point u in $f^{-1}(p)$ to a point $v\in P-p$ such that $uv\cdot f^{-1}(p)=u$ and $uv\cdot P=v$. In either case there exists an arc I from a point $u\in f^{-1}(p)$ to $p\in Y$ such that $I\cdot f^{-1}(p)=u$ and $I\cdot Y=p$. Let $w\in I-(u+p)$ and let uw be the subarc of I from u to w. By hypothesis, $f^{-1}f(uw)$ is compact and hence if a is the last point of I in $f^{-1}f(uw)$, the subarc ua lies entirely in X. Furthermore, since a lies between u and p, $a\notin f^{-1}(p)$. Let $b=f(a)$ and let J denote the unique arc of Y from b to p. Then each of the locally connected continua $f(ua)$ and $r(ap)=p+f(ap-p)$ contains J. Hence if $\{y_i\}$ $(i=1,2,\ldots)$ is a sequence of points in J that converge to p, we may choose a sequence $\{x_i\}$ $(i=1,2,\ldots)$ in X such that for each $i=1,2,\ldots$, $x_i\in f^{-1}(y_i)\cdot(ap-p)$. The set $H=\sum x_i$ $(i=1,2,\ldots)$ is also a subset of the compact set $f^{-1}f(ua)$ and hence has a limit point y in X. Since f is continuous, $y\in f^{-1}(p)$. But since $ap-p$ is closed in X, y is also in $ap-p$. This is a contradiction, because $(ap-p)\cdot f^{-1}(p)$ is empty by construction. Thus f is a compact mapping.

EXAMPLE. The compactness of the boundaries of point inverses is necessary in Theorem (3.4). Let $X=\{(x,y)\colon 0<x\leqslant 1 \text{ and } y=\sin 1/x\}$ and let $Y=[-1,1]$. Define $f\colon X\to Y$ by $f(x,y)=y$. Then Z is homeomorphic to the closure of X in the plane and is not locally connected at any point of Y. However, point inverses of f do not have compact boundaries.

EXAMPLE. Corollary (3.5) cannot be extended to regular spaces. Let X be the interval $[0,1)$ and let Y be the unit circle in the complex plane. Define $f\colon X\to Y$ by $f(x)=e^{2\pi ix}$. Then f is 1-1 and hence satisfies the conditions of Corollary (3.5), but f is not compact.

(3.6) THEOREM. *Suppose that X and Y are locally connected generalized continua and that f is a closed mapping of X onto Y. Then Z is locally connected.*

Proof. By Theorem 1 of [4] the singular set S of f in Y is a totally disconnected set. Furthermore, $Y-S$ is open in Z, hence Z is locally connected at all points of $X+(Y-S)$. Since a locally compact, connected Hausdorff space cannot fail to be locally connected on just a totally disconnected set, Z must be locally connected.

(3.7) DEFINITIONS. A mapping $f\colon X\to Y$ is said to be *quasi-open* provided for any $y\in Y$ and any open set U of X containing a compact component of $f^{-1}(y)$, y is interior to $f(U)$. See [16], p. 9. A mapping $f\colon X\to Y$ is said to be *quasi-monotone* provided for any continuum K in Y with a non-empty interior, $f^{-1}(K)$ has just finitely many components and each of these maps onto K under f. See [20], p. 152.

(3.8) LEMMA. *Suppose that W and V are locally compact Hausdorff spaces and suppose that g is a compact quasi-open mapping of W onto V. Then if K is any continuum in V every component of $g^{-1}(K)$ maps onto K under g.*

Proof. Let H be any component of $g^{-1}(K)$. By theorem (10.4) of [16], g can be factored into the form $g=lm$ where m is a compact monotone mapping of W onto a locally compact Hausdorff space M and l is a compact, light and open mapping of M onto V. Since m is compact and monotone, $I=m(H)$ is a component of $l^{-1}(K)$. By Theorem (11.1) of [16], I maps onto K under l. (In Theorem 11.1 M is assumed to be separable and metric in addition to being locally compact, however the proof given is sufficient for the locally compact Hausdorff case). Then $g(H) = lm(H) = l(I) = K$ as required.

(3.9) THEOREM. *Suppose that X and Y are connected and locally connected and that Y satisfies the first axiom of countability. Suppose further that f is a quasi-open mapping of X into Y. Then Z is locally connected if and only if the retraction $r\colon Z\to Y$ is quasi-monotone. Furthermore whenever $f\colon X\to Y' = f(X)$ is also a compact mapping, Z is locally connected and thus r is a quasi-monotone map.*

Proof. We first argue that whenever f is a quasi-open mapping, so also is r. To this end let $y \in Y$, let C be a compact component of $r^{-1}(y)$ and let U be any open set in Z containing C. Now if $C \subset X$, C is a component of $f^{-1}(y)$ and since f is quasi-open, y is interior to $f(U\cdot X)$ and hence to $r(U) = f(U\cdot X) + U\cdot Y$. On the other hand if $C \not\subset X$, C contains y and y is then interior to $U\cdot Y$ and hence to $r(U)$. Thus r is quasi-open.

If Z is locally connected, r is a compact quasi-open mapping on a locally connected space into a first countable space and such maps can be shown to be quasi-monotone. See Theorem (10.41) of [16]. On the other hand, if r is quasi-monotone, f must satisfy the conditions of Theorem (3.1) and hence Z is locally connected.

Now let us assume that $f\colon X\to Y' = f(X)$ is compact. We will show that Z is locally connected by showing that f satisfies the conditions of Theorem (3.1). To this end let $y \in Y$ and let U be any open set in Y containing y. Now if $y \in Y'$, choose a conditionally compact region R about y such that $\bar{R} \subset U\cdot Y'$. This is possible since $f\colon X\to Y'$ is compact and so $f(X)\cdot S = \emptyset$ and Y' is open in Y. Then $f^{-1}(\bar{R})$ has only finitely many components since $f\colon X\to Y'$ is a compact quasi-open mapping on a locally connected space and as such is a quasi-monotone map. If $y \notin Y'$, let R be a conditionally compact region in Y about y such that $\bar{R}\subset U$. Then $f^{-1}(\bar{R})$ does not have any compact components. For suppose that C is such a component. Then C would be a component of $r^{-1}(\bar{R})$ also. By Lemma (3.8), C maps onto $\bar{R}$ under the compact quasi-open map r. But $r(C) = f(C)$ and $y \notin f(C)$. This is a contradiction and so f satisfies the conditions of Theorem (3.1).

(3.10) DEFINITION. A mapping $f\colon X\to Y$ is said to be *monotone* provided for every $y \in Y$, $f^{-1}(y)$ is continuum.

(3.11) THEOREM. *Suppose that X and Y are locally generalized continua and suppose that f is a non-compact monotone map of X onto Y. Then if $f|T$ is a compact mapping, Z is locally connected and both Y and Z are multicoherent.*

Proof. We wish to show that f satisfies the conditions of Theorem (3.1). To this end let $y \in Y$ and let U be any open set in Y containing y. We consider case: (1) If $y \notin S$, let R be any region of Y containing y such that $\bar{R} \subset U\cdot(Y-S)$. Then since f is compact and monotone on $X-T$, $f^{-1}(\bar{R})$ is a continuum and f satisfies the conditions of Theorem (3.1) in this case. Case (2): If $y \in S$, let V be a conditionally compact open set in Y containing y such that $\bar{V}\subset U$. By hypothesis $f|T$ is compact so $H = f^{-1}(\bar{V}\cdot S)$ is compact in X. Let W be any conditionally compact open set in X containing H and let $\{U_i\}$ $(i = 1, 2, \ldots)$ be a sequence of conditionally compact regions in Y closing down on y such that $\overline{U}_1 \subset V\cdot\big(Y-f(\mathrm{Fr}\,W)\big)$ and for each $i = 1, 2, \ldots$, $\overline{U}_{i+1}\subset U_i$. Then each of the sets $f^{-1}(\overline{U}_i)$ misses $\mathrm{Fr}\,W$ so that $f^{-1}({}_i)\overline{U}\cdot W$ is a compact set in X. We now show by an argument similar to that of the proof of the necessity of Theorem (3.1) that some $(f^{-1}\overline{U}_i)$ has only finitely many components that lie in W. For if each $f^{-1}(\overline{U}_i)\cdot W$ has infinitely many components we may, as in the proof of Theorem (3.1), select a sequence $\{C_i\}$ $(i = 1, 2, \ldots)$ of distinct components of $f^{-1}(\overline{U}_1)\cdot W$ such that for each $i = 1, 2, \ldots$, $C_i\cdot f^{-1}(\overline{U}_i) \neq \emptyset$. Then if C is the component of $f^{-1}(\overline{U}_1)\cdot W$ that contains $f^{-1}(y)$, $f^{-1}(y)$ is interior to C and since $\limsup C_i \neq \emptyset$, C must intersect infinitely many of the C_i's which is impossible. Thus we may and do assume that $f^{-1}(\overline{U}_1)\cdot W$ has just finitely many components.

Since $f^{-1}(\overline{U}_1)\cdot W$ contains $T\cdot f^{-1}(\overline{U}_1)$ and since $\overline{U}_1\subset\bar{V}$, $f^{-1}(\overline{U}_1)$ has just finitely many compact components that intersect T. We next argue that every component of $f^{-1}(\overline{U}_1)$ that misses T is non-compact. For suppose that Q is such a component and suppose that Q is compact. Then if F denote the mapping $f|X-T$: $X-T\to Y$, F is a compact monotone mapping and as such is quasi-open. Let Z_0 be the unified space of $F\colon X-T\to Y$ and let $u\colon Z_0\to Y$ be the retraction of Z_0 onto Y induced by F. Since Q is a compact subset of $X-T$, Q is a component of $F^{-1}(\overline{U}_1)$ and hence of $u^{-1}(\overline{U}_1)$. By Lemma (3.8), Q maps onto $\overline{U}_1$ under the compact quasi-open map U. But this is impossible since $u(Q) = F(Q) = f(Q)$ and Q misses $f^{-1}(y)$. Therefore Q cannot be compact and $f^{-1}(\overline{U}_1)$ has just finitely many compact components. Thus f satisfies the conditions of Theorem (3.1) and Z is locally connected.

By Theorem 6 of [15], p. 1430, Y is multicoherent. Whyburn has argued in Theorem 2 of Section 6 of [14] that whenever Z is connected and locally connected and Y is multicoherent, so also is Z. This completes the proof.

EXAMPLE. In the following example $f\colon X\to Y$ is a 1-1 mapping of X onto Y where X and Y are locally connected generalized continua. The restricted mapping $f|T$ is not compact nor Z is locally connected. Both Z and Y are unicoherent.

We construct this example in E^3. For each positive integer i, let L_i be the segment from $(0, 1/i, 0)$ to $(1, 1/i, 0)$, let L_0 denote the non-negative x-axis and let A denote the segment from $(0, 1, 0)$ to the origin. For each pair of positive integers (i, j) let $P(i, j)$ denote the solid triangle with vertices $(1/(i+1), 1/j, 0)$, $(1/(i+1), 1/j, 1/i-1/(i+1))$ and $(1/(i+1), 1/(j+1), 0)$ Let P denote the solid triangle with vertices $(0, 0, 0)$, $(-1, 0, 0)$ and $(0, 0, 1)$ and for each positive integer i, let Q_i denote the solid triangle with vertices $(i, 0, 0)$, $(i, 0, 1)$ and $(i, -1, 0)$. Let W denote the xy-plane. Finally let X be the set $W+P+\sum\limits_{i,j>0} P(i,j)+\sum\limits_{i>0} Q_i$ minus the closed set $A+\sum\limits_{i\geqslant 0} L_i$.

We describe the mapping f and the range space Y at the same time. Let f be a 1-1 mapping on X such that:

(i) the side of the triangle $P(i, j)$ with endpoints $(1/(i+1), 1/j, 0)$ and $(1/(i+1), 1/j, 1/i-1/(i+1))$ is mapped onto the missing interval from $(1/(i+1), 1/j, 0)$ to $(1/i, 1/j, 0)$ on L_i;

(ii) the side of the triangle P with endpoints $(0, 0, 0)$ and $(0, 0, 1)$ is mapped onto the missing line A;

(iii) the side of triangle Q_i with endpoints $(i, 0, 0)$ and $(i, 0, 0)$ is mapped onto the interval from $(i-1, 0, 0)$ to $(i, 0, 0)$ on the x-axis.

(iv) f is the identity elsewhere.

The resultant space (the space X with the missing set $A+\sum L_i$ filled in by the sides of the vertical triangles) is the space Y. It is unicoherent as is the unified space Z of f.

We see that Z is not locally connected since the inverse image of the closure of any sufficiently small region R of Y containing $f(2, 0, 1)$ ($=$ the missing point $(1, 0, 0)$) would intersect all but finitely many of the triangles $P(1, j)$ and so would have infinitely many components in $f^{-1}(\bar{R})$.

4. Unicoherence of the unified space.

In this section we give necessary and sufficient conditions for Z to be unicoherent whenever it is a locally connected metric continuum. We also show that even when Z is not compact there are certain necessary conditions that X and Y must satisfy in order for Z to be unicoherent.

We first state some definitions and lemmas that will be useful in this and the following section. The statements of the lemmas are included for completeness and the proofs appear in [2].

(4.1) DEFINITIONS. A connected space W is said to be *unicoherent* provided whenever $W = H+K$ where H and K are closed and connected sets, $H\cdot K$ is also connected. The space W is said to be *weakly-unicoherent* provided whenever $W = H+K$ where H is a closed and connected set and K is a continuum, $H\cdot K$ is also a continuum.

A connected set W is said to have the *Complementation Property* provided for every compact set K in W, $W-K$ has at most one non-conditionally compact component.

(4.2) LEMMA. *Suppose that W is a locally connected generalized continuum. A necessary and sufficient condition that W be weakly-unicoherent is that every conditionally compact component of the complement of a closed and connected set has the Complementation Property.*

(4.3) LEMMA. *Let W be a non-compact locally connected generalized continuum. Then a necessary and sufficient condition for W to be weakly-unicoherent and have the Complementation Property is that W satisfy:*

(∗) *For any continuum K in W and any open set U containing K, there is a conditionally compact region R of W about K such that $\operatorname{Fr} R$ is a continuum lying entirely in U.*

(4.4) LEMMA. *Let W be a locally connected weakly-unicoherent generalized continuum and let A be a closed subset of W. Then* (1) *every component of $W-A$ is weakly-unicoherent and* (2) *if every non-empty component of A is non-compact and if W has the Complementation Property, every component of $W-A$ is weakly-unicoherent and has the Complementation Property.*

(4.5) LEMMA. *Let W be a non-compact locally connected generalized continuum. Then a necessary and sufficient condition for $W_\infty = W+\{\infty\}$, the one-point compactification of W, to be locally connected and unicoherent is that W be weakly-unicoherent and have the Complementation Property.*

(4.6) LEMMA. *Let W be a continuum and let A be a unicoherent and locally connected subcontinuum of W such that $B = W-A$ is connected and locally connected. Then W is unicoherent if and only if B is weakly-unicoherent and has the Complementation Property.*

(4.7) THEOREM. *Let X and Y be locally connected generalized continua and let f be a non-compact mapping of X into Y. Suppose that Z is locally connected and unicoherent. Then* (1) *X is weakly-unicoherent and Y is unicoherent;* (2) *S is connected; and* (3) *if $\overline{f(X)}$ is compact, X has the Complementation Property.*

Proof. (1). Whyburn has shown in the proof of Theorem 2 of Section 6 of [14] that when Z is a locally connected generalized continuum and Y is multicoherent, so also is Z. Hence Y must be unicoherent. The weak-unicoherence of X is a direct consequence of Part (1) of Lemma (4.4).

(2). Recall that $\widetilde{X}\cdot Y = S$, thus S must be connected by the unicoherence of Z.

(3). If $\overline{f(X)}$ is compact, $\bar{X}$ is compact. Thus X is a conditionally compact component of the complement of the closed and connected set Y and by Lemma (4.2), X has the Complementation Property.

EXAMPLES. (a) In this example f is a 1-1 mapping of X into Y, where X, Y and Z are all locally connected unicoherent generalized continua and S is a continuum. However, $\overline{f(X)}$ is not compact nor does X have the Complementation Property. Let X be the open interval $(0, 1)$, let Y

by the half open interval $[0,1)$ and let f be the identity map of X into Y. Then Z is homeomorphic to an open interval and is unicoherent and $\overline{f(X)} = Y$ is not compact and X does not have the Complementation Property.

(b) This example shows that even when (i) X is unicoherent and has the Complementation Property, (ii) S is connected, and (iii) Y is unicoherent, Z need not be unicoherent. Thus conditions (i), (ii) and (iii) are not sufficient for the unicoherence of Z. Let $X = \{(x, y)\colon 0 \leqslant x < 1$ and $0 < y < 1\}$ and let $Y = [0,1)$. Define $f\colon X \to Y$ by $f(x, y) = x$. Then Z is homeomorphic to the cylinder $S^1 \times [0,1)$ and is not unicoherent.

We note that in these two examples Z is not compact. When Z is compact, conditions (i) and (iii) are necessary and sufficient for the unicoherence of Z.

(4.8) Theorem. *Let X and Y be locally connected generalized continua and let f be a non-compact mapping of X into Y. Suppose further that Z is locally connected and compact. Then Z is unicoherent if and only if X is weakly-unicoherent and has the Complementation Property and Y is unicoherent.*

Proof. Follows from Theorem (4.7) and Lemma (4.6).

(4.9) Lemma. *Suppose that Z is not compact. Let Z_∞ and Y_∞ denote the one-point compactifications of Z and Y respectively and let Z' denote the unified space of the mapping $f\colon X \to Y_\infty$. Then Z' and Z_∞ are homeomorphic.*

Proof. Let u denote the retraction of Z' onto Y_∞ associated with $f\colon X \to Y_\infty$ and let $Q = u^{-1}(Y)$. (Here we are regarding Y as a subset of Y_∞). Then Q is a locally compact Hausdorff space and $u|Q\colon Q \to Y$ is a compact retraction of Q onto Y. Thus by Theorem (2.9) Q is homeomorphic to the unified space of the mapping $u|X\colon X \to Y$. But $u|X = f$ so Q is homeomorphic to Z, thus we may consider Z_∞ to be a one-point compactification of Q. We note that Z' is also a one-point compactification of Q. It then follows from the topological uniqueness of the one-point compactification of a locally compact Hausdorff space that Z' and Z_∞ are homeomorphic.

(4.10) Corollary. *Let $f\colon X \to Y$ be a non-compact mapping where X and Y are locally connected generalized continua and suppose that Z is locally connected. Then a necessary and sufficient condition for Z to be weakly-unicoherent and have the Complementation Property is that both X and Y are weakly-unicoherent and have the Complementation Property.*

Proof. The necessity. Suppose that Z is weakly-unicoherent and has the Complementation Property. If Y is compact, Z is compact and unicoherent and the necessity in that case follows from Theorem (4.8). Thus we may assume that Y is not compact. By Lemma (4.5), Z_∞, the one-point compactification of Z, is unicoherent. Let Z' denote the unified space of the mapping $f\colon X \to Y_\infty$ where Y_∞ is the one-point compactification of Y. By Lemma (4.9), Z' is homeomorphic to Z_∞, hence Z' is unicoherent. Thus applying Theorem (4.8) to Z' we have that X is weakly-unicoherent and has the Complementation Property and Y_∞ is unicoherent. By Lemma (4.5), Y is weakly-unicoherent and has the Complementation Property and this completes the proof of the necessity.

The sufficiency. If Y is compact, Y is unicoherent and the sufficiency follows from Theorem (4.8). If Y is not compact, Y_∞ is unicoherent and by Theorem (4.8) the unified space Z' of the mapping $f\colon X \to Y_\infty$ is unicoherent. Since Z' is homeomorphic to Z_∞, Z_∞ is unicoherent. Thus by Lemma (4.5), Z is weakly unicoherent and has the Complementation Property.

(4.11) Corollary. *Let $f\colon E^n \to E^m$ be a non-compact mapping of E^n into E^m where $n, m \geqslant 2$ and let $Z(n, m)$ denote the unified space of f. Then if $Z(n, m)$ is locally connected, it is weakly-unicoherent and has the Complementation Property.*

(4.12) Remark. Note that the proof of the sufficiency of Theorem (4.8) depends on Lemma (4.6) which does not require Z to be locally connected. Thus whenever X is a weakly-unicoherent, locally connected generalized continuum that has the Complementation Property and Y is a locally connected unicoherent continuum, Z is unicoherent even if it is not locally connected. In partiuclar we have:

(4.13) Corollary. *Let $f\colon E^n \to S^m$ be a mapping of E^n into S^m where $n, m \geqslant 2$ and let $Z'(n, m)$ denote the unified space of f. Then $Z'(n, m)$ is unicoherent.*

5. Singular sets for mappings.

In this section we investigate the relationship between the connectedness and compactness properties of the singular sets of a non-compact mapping and certain topological properties of the range and domain. G. T. Whyburn in [14] has shown that when f is a monotone map of X onto Y where X and Y are locally connected generalized continua and Y is unicoherent, the singular set T of f in X cannot have any non-empty compact components. In [6] E. Duda has shown that every monotone mapping of a locally connected generalized continuum having the Complementation Property onto the plane is a compact mapping. Similar results are obtained here where we show that when f is a mapping of a locally connected and connected space having the Complementation Property into a space Y, the singular set S of f in Y cannot have any non-empty compact components if $\overline{f(X)}$ is not compact; and if $\overline{f(X)}$ is compact, S is a continuum. Furthermore we show that every closed mapping of a locally connected, connected and paracompact space having the Complementation Property onto a non-compact space must be a compact mapping. Finally we show that under certain conditions the Complementation Property and weak-unicoherence are inherited by the components of the complements of S and T in Y and X respectively.

We first prove two lemmas.

(5.1) LEMMA. *Let W be a locally connected, connected, locally compact Hausdorff space and let V be an open subset of W such that $\bar{V}$ is not compact but $K = \mathrm{Fr}\, V$ is compact. Then V has a non-conditionally compact (in W) component.*

Proof. Suppose to the contrary that every component of V is conditionally compact. Let Q be a conditionally compact open subset of W containing K. Since $\bar{Q}$ is compact and $\bar{V}$ is not compact, we may choose a component N_1 of V such that $N_1 \not\subset \bar{Q}$. Then since $\bar{Q} + \bar{N}_1$ is compact, we may choose a component N_2 of V such that $N_2 \not\subset \bar{Q} + \bar{N}_1$. Continuing in this manner we obtain a sequence $\{N_1, N_2, \ldots\}$ of distinct components of V such that for each $i = 1, 2, \ldots$, $N_i \cdot (W - Q) \neq \emptyset$. We note that since W is connected and locally connected, the closure of every component of V meets K. Thus for each $i = 1, 2, \ldots$, $N_i \cdot Q \neq \emptyset$. This implies that each N_i meets the boundary of Q and thus $\mathrm{Fr}\, Q$ contains a limit point p of $\sum N_i$ $(i = 1, 2, \ldots)$. We note that each N_i is also a component of $W - K$ and since W is locally connected, p is interior to some component N of $W - K$. But this is a contradiction for then N would have to meet infinitely many N_i which is absurd. This completes the proof.

(5.2) LEMMA. *Let f be a non-compact mapping of X into Y where X is a locally connected and connected space and suppose that S has $n \geqslant 1$ non-empty compact components. Then there exists a compact set K in X such that $X - K$ has at least n non-conditionally compact components and if in addition $\overline{f(X)}$ is not compact, K can be chosen so that $X - K$ has at least $n+1$ non-conditionally compact components.*

Proof. Let $C_1, C_2, \ldots, C_n$ be n distinct non-empty compact components of S and let $W_1, W_2, \ldots, W_n$ be a collection of pairwise disjoint conditionally compact open subsets of Y containing $C_1, C_2, \ldots, C_n$ respectively chosen so that $S \cdot \mathrm{Fr}\, W_i = \emptyset$ for $i = 1, 2, \ldots, n$. For each $i = 1, 2, \ldots, n$, let $V_i = f^{-1}(W_i)$. Then for each i, $\mathrm{Fr}\, V_i$ is compact since $\mathrm{Fr}\, V_i \subset f^{-1}(\mathrm{Fr}\, W_i)$ and $\mathrm{Fr}\, W_i$ misses S and $\bar{V}_i$ is not compact since $W_i \supset C_i$. Hence by Lemma (5.1) each V_i, $i = 1, 2, \ldots, n$, contains a non-conditionally compact component B_i of X.

Suppose also that $\overline{f(X)}$ is not compact. Then if $V_0 = X - \sum f^{-1}(\bar{W}_i)$ $(i = 1, 2, \ldots, n)$, $\bar{V}_0$ is not compact. But $\mathrm{Fr}(V_0) \subset \sum \mathrm{Fr}\, V_i$ $(i = 1, 2, \ldots, n)$ and so $\mathrm{Fr}\, V_0$ is compact. Thus by Lemma (5.1), V_0 contains a non-conditionally compact component B_0 of X. Then $B_0, B_1, \ldots, B_n$ is the desired collection. As an immediate consequence of Lemma (5.2) we have:

(5.3) THEOREM. *Let $f\colon X \to Y$ be a non-compact mapping where X is a connected and locally connected space having the Complementation Property. Then (1) if $\overline{f(X)}$ is not compact, the singular set S of f in Y cannot have any non-empty compact components; and (2) if $\overline{f(X)}$ is compact, S is a continuum.*

(5.4) THEOREM. *Suppose that X is a locally connected, connected and paracompact space having the Complementation Property. Then every closed mapping f of X onto a non-compact space Y is a compact mapping.*

Proof. By Theorem (1.1) of [8], point inverses of f have compact boundaries. I assert that this implies that every point inverse of f is compact. For suppose to the contrary that $\mathrm{Fr}\, f^{-1}(y) = K$ is compact but $f^{-1}(y)$ is not. By Lemma (5.1), $f^{-1}(y) - K$ must have a non-conditionally compact component which we denote by N. Then since X has the Complementation Property, N is the only non-conditionally compact component of $X - K$ and this implies that $X - N$ is compact. But then $f(X) = f(X-N) + f(N) = f(X-N) + y$ is a compact set and this contradicts our hypothesis that $f(X) = Y$ is not compact. Thus every point inverse of f is compact and a closed mapping with compact point inverses onto a locally compact Hausdorff space is compact. See [19].

(5.5) THEOREM. *Suppose that f is a non-compact mapping of X into Y where X and Y are locally connected generalized continua, X has the Complementation Property and Y is weakly-unicoherent. Then (1) if $\overline{f(X)}$ is not compact and Y has the Complementation Property, every component of $Y - S$ is weakly-unicoherent and has the Complementation Property; (2) if f is onto and Y is not compact, every component of $Y - S$ is weakly-unicoherent and has the Complementation Property; and (3) if f is a monotone onto mapping and if Y is not compact, every non-empty component of T is non-compact and every component of $X - T$ has the Complementation Property.*

Proof. (1). By Theorem (5.3) every non-empty component of S is non-compact and by Part (2) of Lemma (4.4) every component of $Y - S$ is weakly-unicoherent and has the Complementation Property.

(2). Let Q be a component of $Y - S$. We wish to show that Q satisfies condition $(*)$ of Lemma (4.3). To this end let K be a continuum lying in Q and let U be an open subset of Q containing K. By Lemma (4.13) of [7], there exists a conditionally compact region M of Q containing K such that $\bar{M} \subset U$ and $Y - M$ has only finitely many components, say $N_1, N_2, \ldots, N_p$. Since $\bar{M}$ misses S, $N = f^{-1}(\bar{M})$ is compact and since X has the Complementation Property, $X - N$ has only one non-conditionally compact component which we denote by B. Now $X - B$ is compact and thus the component of $Y - M$ that contains $f(B)$ is the only non-conditionally compact component of $Y - M$. Without loss of generality let us suppose that $f(B) \subset N_1$. Then each of the sets $N_2, N_3, \ldots, N_p$ is conditionally compact. By Theorem (5.3), every non-empty component of S is non-compact. Since the boundary of each N_i misses S, $N_i \cdot S = \emptyset$ for $i = 2, 3, \ldots, p$. Thus $R = M + N_2 + N_3 + \ldots + N_p$ is a conditionally compact region whose closure lies entirely in Q. Furthermore, since Y is weakly unicoherent, $\mathrm{Fr}\,R = \bar{R} \cdot N_1$ is a continuum lying in U. Thus Q satisfies condition $(*)$ and Q is weakly unicoherent and has the Complementation Property.

(3). Suppose that K is a non-empty compact component of T. Let R be a conditionally compact region in X containing K so that $F = \mathrm{Fr}\,R$ misses T. Let $C = f(F)$ and let V be a region in $Y - C$ which intersects $f(K)$. Let pq be an arc in V such that pq meets $f(\bar{R})$ in exactly the point p. This is possible since every point in $f(K)$ is a limit point of $Y - f(\bar{R})$. Then $p \in f(R)$ since $V \cdot C = \emptyset$. Hence $p \in S$ as p is not interior to $f(R)$.

Let A_0 be the union of all the components of $X - F$ which intersect the set $f^{-1}(pq - p)$ and let $B = X - A$. Then $A = \bar{A}_0$ and B are closed, B and $f(A)$ are connected, $X = A + B$ and $A \cdot B \subset F$. By our hypotheis, $X - F$ has exactly one non-conditionally compact component which we denote by N. Now either $N \subset A$ and B is compact or $N \subset B$ and A is compact. But in either case if we let $\alpha = \overline{f(A)}$ and $\beta = \overline{f(B)}$, $Y = \alpha + \beta$ where α and β are closed and connected and either α or β is compact.

Since $A \cdot B \subset F$, $\alpha \cdot \beta \subset C + S$. For if a point $p \in \alpha \cdot \beta$ is not in $C + S$, a small region U in Y containing p but missing $C + S$ would have a region W as its inverse under f which would meet both A and B but not F and this is impossible. Also since $A \cdot F \neq \emptyset$ and $B \supset F$, we have that $\beta \supset C$ and $\alpha \cdot C \neq \emptyset$ so that $\alpha \cdot \beta \cdot C \neq \emptyset$. Furthermore, $pq - p \subset f(A)$ so that $p \in \alpha$; and $p \in \beta$ since $p \in f(R) \subset f(B) \subset \beta$. Thus $\alpha \cdot \beta \cdot S \neq \emptyset$ since $p \in S$. Hence $\alpha \cdot \beta$ is the sum of two non-empty separated sets $\alpha \cdot \beta \cdot S$ and $\alpha \cdot \beta \cdot C$ and is not connected. But this contradicts the weak-unicoherence of Y since $Y = \alpha + \beta$ and either α or β is compact. This proof closely parallels that of Theorem (2) of [14].

In order to see that components of $X - T$ have the Complementation Property let Q be a component of $X - T$ and let K be a compact set in Q. Since $f|Q$ is a compact monotone mapping, Q is an inverse set of f, i.e., $Q = f^{-1}f(Q)$, and $H = f^{-1}f(K)$ is a compact set lying entirely in Q. Furthermore, $P = f(Q)$ is a component of $Y - S$ and every component of $Q - H$ maps onto a component of $P - f(K)$ under f. By part (2) of this Theorem, $P - f(K)$ has exactly one non-conditionally compact (in P) component. This implies that $Q - H$ has exactly one non-conditionally compact (in Q) component since $f|Q$ is a compact monotone map of Q onto P. It is then clear that $Q - K$ must then have only one non-conditionally compact (in Q) component.

(5.6) **Remark.** Note in part (1) of Theorem (5.4) the components of $Y - S$ inherit the Complementation Property from Y but in part (2) they inherit the Complementation Property from X.

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