Unified spaces and singular sets for mappings of locally compact spaces

by

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1. Introduction. This paper may be considered as a continuation of the research started by G. T. Whyburn in [13], [14] and [15]. The results stated in this section are due to him and may be found in these papers.

Let \( X \) and \( Y \) be disjoint Hausdorff spaces and let \( f : X \to Y \) be a continuous map of \( X \) into \( Y \). In [13] and [14] Whyburn defined a unified space for the mapping \( f \) as follows:

Let \( W \) denote the set-theoretic union of \( X \) and \( Y \) and define a set \( Q \) in \( W \) to be open if it satisfies:

(i) \( Q \cdot X \) and \( Q \cdot Y \) are open in \( X \) and \( Y \) respectively; and

(ii) for any compact set \( K \) in \( Q \cdot Y \), \( f^{-1}(K) \cdot (X - Q \cdot X) \) is compact in \( X \).

The set \( W \) together with the collection of open sets so defined is a \( T_1 \)-topological space which is called the unified space of \( f \) and which we denote by \( Z \). (Actually in [13] Whyburn did not require \( X \) and \( Y \) to be disjoint, but took disjoint copies of \( X \) and \( Y \) for his construction. He did, however, require that \( f \) be an onto mapping. In [14] he simplified the treatment by assuming that \( X \) and \( Y \) were disjoint and by allowing \( f \) to be an onto mapping.)

The injections of \( X \) and \( Y \) into \( Z \) are open and closed respectively, thus \( X \) is embedded in \( Z \) as an open set and \( Y \) is embedded in \( Z \) as a closed set.

Associated with \( Z \) is a retraction \( r : Z \to Y \) of \( Z \) onto \( Y \) defined by \( r(z) = f(z) \) for \( z \in X \) and \( r(z) = z \) for \( z \in Y \). This retraction is continuous and compact. A mapping \( g \) from a topological space \( W \) into a topological space \( V \) is said to be compact provided for every compact set \( K \) in \( V \),

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Let $g^{-1}(K)$ be compact in $W$. Thus when $Z$ is (locally) compact, $Z$ is also (locally) compact. Furthermore, whenever $f$ is an open mapping, so also is $r$.

In [13] it was shown that when $X$ and $Y$ are both locally compact Hausdorff spaces, so also is $Z$ and that when $X$ and $Y$ are both locally compact separable metric spaces, $Z$ is also a locally compact separable metric space. Henceforth we shall assume that $X$ and $Y$ are at least locally compact Hausdorff spaces and thus $Z$ will be locally compact and Hausdorff.

We adopt Whyburn's notation and use $\mathcal{F}$ to denote the closure in $Z$ of any subset $E$ of $Z$. For any subset $A$ of $X$ (or $Y$), the closure of $A$ in $X$ (or in $Y$ respectively) will be denoted by $\overline{A}$, the interior of $A$ by $\text{int}A$ and the boundary of $A$ by $\partial A$. Thus, since $Y$ is closed in $Z$, for any set $A$ in $Y$, $\overline{A} = A$.

The restriction $r|X$ of $r$ to $X$ is topologically equivalent to $f$ and thus $r|\mathcal{F}$ is a compact mapping which extends $f$. Furthermore, since $Y$ is locally compact, $r$ and $r|\mathcal{F}$ are closed mappings [13], Section 4, p. 106].

Heinz Bauer in his study of conservative maps in [1] showed that for every mapping $f: X \to Y$, where $X$ and $Y$ are locally compact Hausdorff spaces, there exists a locally compact Hausdorff space $X_0$ and a mapping $f_0$ satisfying:

(i) $X_0$ is a dense subset of $X_0$,
(ii) $f_0|X = f$,
(iii) $f_0|X$ is a compact mapping,
(iv) $f_0|X$ is 1-1 on $X_0 - X$.

Furthermore, he showed that every space satisfying (i)-(iv) is homeomorphic to $X_0$ under a homeomorphism that leaves points of $X$ fixed. Thus his $X_0$ is homeomorphic to Whyburn's $\mathcal{F}$ and $f_0$ is topologically equivalent to $r|\mathcal{F}$.

Momentarily leaving the unified space $Z$, let us consider the mapping $f: X \to Y$ and define $Q$ to be the union of the interiors of sets in $Y$ having a compact inverse under $f$. Then if $P = f^{-1}(Q)$, $P: X \to Q$ is a compact mapping. The complement $S$ of $Q$ in $X$ is closed and contains all of the points of $Y$ at which $f$ is not compact, i.e. $S$ is the set of those points in $Y$ which are not interior to any set with a compact inverse under $f$. The set $S$ is called the singular set of $f$ in $Y$ and the set $T = f^{-1}(S)$ is called the singular set of $f$ in $X$. We see immediately that $f$ is compact if and only if $S$ is empty.

The singular set $S$ of $f$ in $Y$ is the boundary $\overline{X} \times Y$ of $X$ in $Z$. In order to see this let $y \in Y$ and suppose that $y \in \overline{X} \times Y$. Since $Z$ is locally compact and Hausdorff, there exist conditionally open sets $U$ and $V$ of $Z$ containing $y$ such that $\overline{V} \cap U \subset Z - X$. Then $y$ is an element of the interior relative to $Y$ of the compact set $\overline{V} (= V)$ and since $U$ is open in $Z$, $f^{-1}(V) = f^{-1}(V) \cap (X - U \times X)$ is compact in $X$. Thus $y \notin S$. Now suppose that $y \notin S$. Then there exists a compact set $K$ of $Y$ such that $y \notin \text{int}K$ (relative to $Y$) and $f^{-1}(K)$ is compact. But this implies that $\text{int}K$ is open in $Z$ and since $\text{int}K$ misses $X$, $y \notin \overline{X} \times Y$. Thus $\overline{X} \times Y = S$. This equality provides a clue to the relationship of the mapping properties of $f$ and the topological properties of $Z$. For example we have that when $X$ and $Y$ are connected, $Z$ is connected if and only if $f$ is not compact. Whyburn has shown in [14] that the mapping $f: X \to f(X)$ is compact if and only if $r(x) = x = r^{-1}(x)$ for every $x \in \overline{X} \times Y$ or equivalently if and only if $\overline{X} \times f(X)$ is empty.

A fundamental aim of this paper is to relate the topological properties of $X$ and $Y$ and the mapping properties of $f$ to the topological properties of $Z$. For example we have that when $X$ and $Y$ are connected, $Z$ is connected if and only if $f$ is not compact. Whyburn has shown in [14] that the mapping $f: X \to f(X)$ is compact if and only if $r(x) = x = r^{-1}(x)$ for every $x \in \overline{X} \times Y$ or equivalently if and only if $\overline{X} \times f(X)$ is empty.

A fundamental aim of this paper is to relate the topological properties of $X$ and $Y$ and the mapping properties of $f$ to the topological properties of $Z$. In Section 3 we show that $Z$ is paracompact if and only if $Y$ is paracompact. In this same section we give necessary and sufficient conditions for the metrizability of $Z$ and obtain a bound for the covering dimension of $Z$ in terms of the strong inductive dimension of $Y$ and the covering dimension of point inverses of $f$. We also show that the domain of any compact retraction is a unified space of a mapping.

In Section 3 the local connectedness of $Z$ is investigated and we give a necessary and sufficient condition for $Z$ to be locally connected in terms of a mapping property of $f$. Several interesting maps are shown to have a locally connected unified space. In Section 4 we give the definitions of two topological properties and show how they are related to unicoherence. Necessary and sufficient conditions for the unicoherence of a compact and locally connected unified space are then given in terms of the topological properties of $X$ and $Y$. In the last section of this paper we investigate the compactness and the connectedness of the singular sets $S$ and $T$ in $Y$ and $X$ respectively.

Summary of notation. As stated above $f: X \to Y$ will always denote a continuous map of $X$ into $Y$ where $X$ and $Y$ are at least locally compact Hausdorff spaces. When $f$ is an onto mapping it will be stated explicitly. The singular sets of $f$ in $X$ and $Y$ will be denoted by $T$ and $S$ respectively. We shall always use $Z$ to denote the unified space of $f$ and use $r: Z \to Y$ to denote the retraction of $Z$ onto $Y$ induced by $f$.

A region is an open connected set, a continuum is a compact connected set and a generalized continuum is a locally compact, connected and separable metric space. By a mapping we will mean a continuous transformation.

2. Metrizability, dimension and characterizations of the unified spaces. In this section we show that $Z$ is paracompact if and only if $Y$ is paracompact and prove a theorem giving a necessary and sufficient
condition for the metrizability of $Z$. The covering dimension of $Z$ is shown to be related to the covering dimension of point inverses of $f$ and the strong inductive dimension of $Y$. Finally we show that the domain of any compact retraction is homeomorphic to a unified space for a mapping. In this section we assume that $X$ and $Y$ are at least locally Hausdorff spaces.

(2.1) Lemma. Let $g$ be a closed mapping of the topological space $W$ onto the topological space $V$ such that point inverses of $g$ are compact. If $V$ is paracompact, so also is $W$.

Proof. Let $(U_{a} \quad (a \epsilon A)$ be an open covering of $W$. For each $y \epsilon V$ select a finite union $H_{y}$ of elements of $(U_{a} \quad (a \epsilon A)$ that contains the compact set $g^{-1}(y)$. Since $g$ is closed, $H_{y} = V - g(W - H_{y})$ is an open set containing $y$ and $(H_{y} \quad (y \epsilon V)$ is an open covering of $V$. Since $V$ is paracompact, there exists a locally finite open cover $(K_{y} \quad (y \epsilon V)$ of $V$ which refines $(H_{y} \quad (y \epsilon V)$. We now assert that for each $y \epsilon V$, $g^{-1}(K_{y})$ is contained in $H_{y}$. Thus for each $K_{y}$ we may and do choose a finite subset $I_{y}$ of $I$ such that $g^{-1}(K_{y}) \subseteq \sum_{a \epsilon I_{y}} U_{a} \quad (y \epsilon V)$. We now assert that $(g^{-1}(K_{y}) \cup_{a \epsilon I_{y}} \cup_{a \epsilon I_{y}})$ is locally finite covering of $W$. In order to see this let $x \epsilon W$. Since $(K_{y} \quad (y \epsilon V)$ is locally finite, there exists an open set $P$ of $V$ containing $x$ such that $P$ meets only finitely many $K_{y}$. Then $g^{-1}(P)$ is an open subset of $W$ containing $x$ that meets only finitely many of the sets $g^{-1}(K_{y})$ and thus meets only finitely many of the sets $g^{-1}(K_{y}) \cup_{a \epsilon I_{y}} \cup_{a \epsilon I_{y}}$. Thus every open cover of $W$ has a locally finite open refinement and $W$ is paracompact.

(2.2) Theorem. A necessary and sufficient condition that $Z$ be paracompact is that $Y$ be paracompact.

Proof. Since $X$ is locally compact and Hausdorff, $r: Z \rightarrow Y$ is a closed mapping ([19], Section 4, p. 106). Thus the sufficiency is a consequence of Lemmas (2.1). The necessity follows from a well-known result of E. Michael [9].

(2.3) Remark. Note that whenever $Z$ is paracompact, the closed set $X$ is paracompact. Thus in this case even when $X$ is not normal, $X$ is embedded as a dense subset of the paracompact, locally compact Hausdorff space $X$ on which $f$ has a compact extension.

Theorem. Suppose that $Y$ is a metric space. A necessary and sufficient condition for $Z$ to be a metric space is that inverse image under $f$ of any compact sets in $Y$ be a separable metric space.

Proof. The necessity. Let $X$ be a compact subset of $Y$; then $r^{-1}(X)$ is a compact subset of $Z$ and as such is a separable metric space. Hence $f^{-1}(X)$ is also a separable metric space.

Proof of the sufficiency. By a result of A. H. Stone, $Y$ is paracompact. Thus there exists a locally finite covering $(K_{a} \quad (a \epsilon A)$ of $Y$ by compact sets. By our assumption each $H_{a} = f^{-1}(K_{a}) \subseteq \epsilon Y$, is a separable metric space. Whyburn has shown that when the range and domain of a mapping are both separable metric spaces, so also is the unified space [13]. Hence the unified space $Z_{a}$ of the mapping $f|H_{a}: H_{a} \rightarrow K_{a}$ is a separable metric space. One can easily show that for each $a \epsilon I$, $Z_{a}$ is homeomorphic to $r^{-1}(K_{a})$. Thus $(r^{-1}(K_{a}) \quad (a \epsilon I)$ is a locally finite closed covering of $Z$ by metric spaces and by a result of Nagata [10], $Z$ is metrizable.

(2.5) Remark. As an immediate consequence of Theorem (2.4) we see that unless $Z$ is a metric space, $X$ is separable if and only if $Y$ is separable.

V. V. Proivkov in [12], Cor. 1, p. 154, has shown that the domain of any mapping whose point inverses have weak-inductive dimension zero is a metric space whenever it is locally connected and locally compact and the range is a metric space. Thus if we consider the case where $f$ is a light mapping (i.e. point inverses are totally disconnected) of $X$ onto $Y$ and $X$ and $Y$ are metric spaces, $Z$ is also a metric space whenever it is locally connected. This is because the weak-inductive dimension of $f^{-1}(y)$ is not increased by the addition of the point $y$ in $Z$, so that the weak-inductive dimension of $r^{-1}(y) = y + f^{-1}(y)$ is still zero. The local connectedness of $Z$ is studied in Section 3.

(2.6) Definitions. The empty set and only the empty set has strong-induction (covering) dimension $-1$. A space $W$ has strong-inductive dimension $\leq n$ (n $\geq 0$), written $\text{Ind} W \leq n$, if for every pair of disjoint closed sets in $W$ can be separated by a closed set of strong-inductive dimension $\leq n - 1$. We say that the strong-inductive dimension of $W$ is $n$ written $\text{Ind} W = n$, if $\text{Ind} W \leq n$ is true and $\text{Ind} W \leq n - 1$ is false.

The order of an open covering is the largest integer $n$ such that there are $n + 1$ members of the covering with a non-empty intersection. A space $W$ has covering dimension $\leq n$ (n $\geq 0$), written $\text{dim} W \leq n$, if for every open covering of $W$ has an open refinement of order $\leq n$. We say that the covering dimension of $W$ is $n$, written $\text{dim} W = n$, if $\text{dim} W \leq n$ is true and $\text{dim} W \leq n - 1$ is false.

(2.7) Theorem. Suppose that $Y$ is paracompact and that there is an integer $k \geq 0$ such that for every point $y \epsilon Y$, $\text{dim} f^{-1}(y) \leq k$. Then $\text{dim} Y \leq \text{dim} Z \leq \text{Ind} Y + k$.

Proof. By theorems (2.2), $Z$ is paracompact and hence normal and by Theorem (2.5) of [13], $\text{dim} f^{-1}(y) \geq \text{dim} r^{-1}(y)$ for every $y \epsilon Y$. Since $Z$ is locally compact and Hausdorff, the compact map $r: Z \rightarrow Y$ is a closed map ([19], Section 4, p. 106). Thus applying Theorem VII.7 of [11], p. 106 we have $\text{dim} Z \leq \text{Ind} Y + k$. Furthermore, since $Y$ is closed in $Z$, $\text{dim} Y \leq \text{dim} Z$ and this completes the proof.

(2.8) Remark. If we knew that $\text{dim} X \leq \text{dim} Z$, Theorem (2.7) would extend Theorem VII.7 of [11] which considers closed maps on
normal spaces to continuous maps on locally compact Hausdorff spaces. However, there are not any theorems in dimension theory that will insures that \( \dim X \leq \dim Z \). In fact, there are examples of compact Hausdorff spaces \( E \) with subsets \( W \) where \( \dim W > \dim E \); this is due to the lack of complete normality. The best we can say at present is that \( X \) is imbedded as a dense open subset of \( \bar{X} \) where \( \dim \bar{X} \leq \dim Y + k \) and \( \bar{X} \) is a compact extension of \( J \) with \( \dim J \leq \dim \bar{J} \) for all \( y \in Y \).

(2.9) Theorem. Suppose that \( g : W \to Y \) is a compact retraction where \( W \) is a locally compact Hausdorff space and \( Y \) is a proper subset of \( W \). Then if \( x \) denotes the set \( W - Y \) and \( f \) is the restriction of \( g \) to \( X \), the identity transformation \( i : Z \to W \) of the unified space \( Z \) of \( f : X \to Y \) onto \( W \) is a homeomorphism. Thus the domain of any compact retraction onto a proper subset is a unified space for a mapping.

Proof. I first wish to show that \( x \) is continuous. To this end let \( U \) be any open set in \( W \). Clearly \( U \cap X \) and \( U \cap Y \) are open in \( X \) and \( Y \) respectively. Now if \( K \) is any compact set in \( U \cap X \), then \( g^{-1}(K) \) is compact in \( W \) as \( g^{-1}(K) = (W - U) \). But \( g^{-1}(K) \cap (W - U) = f^{-1}(K) \cap (X - X \cap U) \) and so \( U \cap X \) is open in \( X \) and \( x \) is a continuous map.

Since \( W \) is a locally compact Hausdorff space, every compact mapping onto \( W \) is a closed mapping ([19], Section 4, p. 690). Thus in order to show that \( x \) is a homeomorphism it suffices to show that \( x \) is a compact mapping. To this end let \( X \) be a compact set in \( W \). Then \( g(X) \) is a compact set in \( Y \) and since \( x(Y) \) is a homeomorphism, \( \gamma(x(Y)) \) is a compact subset of \( Y \) considered as a subset of \( Z \). The retraction \( r : Z \to Y \) is a compact mapping, so \( H = r^{-1}(Y) \) is a compact subset of \( Z \). Furthermore, since \( x \) is continuous, \( \gamma^{-1}(K) \) is a closed subset of \( H \) and is compact. Thus \( x \) is a compact mapping and is a homeomorphism.

(2.10) Remark. Note that the unified space of any constant map on a non-compact, locally compact Hausdorff space \( W \) is homeomorphic to the one-point compactification of \( W \). This fact together with the fact that the retraction of any unified space is a compact mapping yields:

A necessary and sufficient condition that a non-degenerate topological space be a compact Hausdorff space is that it be homeomorphic to the unified space of a mapping of a locally compact Hausdorff space into a compact Hausdorff space.

3. Local connectedness of the unified space. In this section necessary and sufficient conditions for the local connectedness \( Z \) are given in terms of a mapping property of \( f \) and certain interesting maps are shown to have a locally connected unified space. Throughout this section we will assume that \( X \) and \( Y \) are locally connected and connected in addition to being locally compact and Hausdorff.

(3.1) Theorem. Suppose that \( X \) and \( Y \) are locally connected and connected and suppose also that \( Y \) satisfies the first axiom of countability. Then a necessary and sufficient condition for \( Z \) to be locally connected is that for every point \( y \in Y \) and every open set \( U \) of \( Y \) containing \( y \) there is a region \( R \) of \( Y \) containing \( y \) such that \( R \subset U \) and \( f^{-1}(R) \) has just finitely many compact components.

Proof of the sufficiency. Suppose that \( Z \) is not locally connected. Then \( f \) is not compact. For if it were, \( Z \) would be the sum of two disjoint open and locally connected sets and would be locally connected contrary to our supposition. Hence \( Z \) is connected. By theorem (2.1) of [22], p. 102, there exist conditionally compact open sets \( U \) and \( V \) of \( Z \) where \( \bar{U} \cap \bar{V} \) \( \neq \emptyset \) and \( \bar{U} \setminus \bar{V} \), \( \bar{V} \setminus \bar{U} \) \( \neq \emptyset \) for each \( \alpha \in \Gamma \). Let \( \gamma(\bar{U} \setminus \bar{V}) \) meet both \( \bar{U} \setminus \bar{V} \) and \( \bar{V} \setminus \bar{U} \) and the component \( C \) of \( \bar{U} \setminus \bar{V} \) that contains \( L' \) \( \neq \emptyset \) for any point \( L' \in (\bar{U} \setminus \bar{V}) \) and hence since \( X \) is locally connected and open in \( \bar{U} \setminus \bar{V} \) and \( \bar{U} \setminus \bar{V} \) must lie entirely in \( Y \).

Let \( P \) and \( Q \) be open subsets of \( Z \) such that \( \bar{P} \subset \bar{C} \subset \bar{Q} \subset \bar{Q} \subset U \). For each \( \alpha \in \Gamma \), let \( K_\alpha \) be a component of \( \bar{C}_\alpha(\bar{Q} \setminus \bar{P}) \) such that \( \bar{C}_\alpha(\bar{P} \setminus \bar{Q}) \) and \( L = \lim sup K_\alpha \). Then \( L \) meets both \( \bar{P} \setminus \bar{Q} \) and \( \bar{Q} \setminus \bar{L} \).

We assert that \( K_\alpha \cap Y = \emptyset \) for all but finitely many \( \alpha \in \Gamma \). For suppose that \( K_\alpha \cap Y \neq \emptyset \) for all \( \beta \in \Lambda \) where \( \beta \) is an infinite subset of \( \Gamma \). For each \( \alpha \in \Lambda \), let \( y_\alpha \in K_\alpha \cap Y \). Then \( \sum y_\alpha(\beta \in \Lambda) \) would have a limit point \( y \) in \( Y \) \( \cap \bar{Q} \setminus \bar{P} \). Let \( W \) be a region of \( Y \) containing \( y \) that is contained in \( \bar{U} \setminus \bar{V} \). Then \( W \) must contain infinitely many \( y_\alpha \) and hence intersect \( K_\alpha \) for infinitely many \( \beta \in \Lambda \). But this is a contradiction for \( W \) must be a subset of \( C \), the component of \( \bar{U} \setminus \bar{V} \) that contains \( L' \), and hence \( W \) cannot meet any of the \( K_\alpha \), \( \beta \in \Lambda \). Therefore we may and do assume that \( K_\alpha \cap Y = \emptyset \) for all \( \alpha \in \Lambda \).

Let \( \alpha \in \Lambda \). For hypothesis there exists a region \( R \) of \( Y \) containing \( s \) such that \( \bar{R} \subset X(\bar{Q} \setminus \bar{P}) \) and \( f^{-1}(R) \) has only finitely many compact components. We note that only one component of \( r^{-1}(R) \) meets \( Y \) since \( r^{-1}(R) \cap Y = \emptyset \). Furthermore, if \( B \) is a component of \( r^{-1}(R) \) that misses \( Y \), \( B \) is a compact component of \( f^{-1}(R) \). Thus since \( f^{-1}(R) \) has only finitely many compact components, \( r^{-1}(R) \) has only finitely many components, say \( A_1, A_2, \ldots, A_n \). Since \( r \) is continuous and since \( s \in \lim sup K_\alpha \), \( r(s) \in \lim sup K_\alpha \). Also since \( s \in X \), and each \( K_\alpha \subset X \) and \( r(s) = s \) and \( r(K_\alpha) = f(K_\alpha) \) and so \( s \in \lim sup f(K_\alpha) \). Therefore \( R \) must intersect infinitely many of the sets \( f(K_\alpha) \) and some \( A_i \), say \( A_i \), must meet infinitely many of the sets \( K_\alpha \), \( \alpha \in \Lambda \). Let \( \alpha \) be an infinite subset of \( \Gamma \) such that \( K_\alpha(A_1) \neq \emptyset \) for each \( \alpha \in \Lambda \). If \( A_i \) is a subset of the \( \bar{Q} \setminus \bar{P} \). For if it were, \( A_i \) would not be connected.
Let $P$ denote the set $\{(q, 0) + (p - p)\}$. For each $a \in A$, $A_1 \not= \emptyset$ and $A_1$ must meet the boundary of $K_a$. Since each $K_a$, $a \in A_1$, is a subset of $X$ and $X$ is locally connected, the boundary of each $K_a$ is the set $K_a \cap P$. Thus for each $a \in A_1$, $A_1, K_a \not= \emptyset$. For each $a \in A_1$ let $x_a \in A_1, K_a \not= \emptyset$. Then $\sum x_a \in (a \in A_1)$ is a subset of $f^{-1}(R) \setminus X \setminus X (0 - P)$ and the latter is compact in $X$ by the definition of the topology in $Z$. Thus $\sum x_a \in (a \in A_1)$ has a limit point $x$ in $X$. But this is a contradiction since $x$ would then be in $X \setminus (U \setminus (\{0 - \bar{V}\}$ and this set is empty. Hence $Z$ is locally connected.

Proof of the necessity. Suppose that $Z$ is locally connected and suppose that there exists a sequence $(U_i) (i = 1, 2, ...)$ of conditionally compact regions of $Y$ closing down on a point $y \in Y$ such that for each $i$, $U_i \in C U_i$ and $f^{-1}(U_i)$ has infinitely many compact components. Since every compact component of $f^{-1}(U)$ is a component of $r^{-1}(U)$, each $r^{-1}(U_i)$ must have infinitely many components.

Since $r^{-1}(U)$ is compact and every component of $r^{-1}(U)$ is open in $Z$, $r^{-1}(U)$ is covered by a finite collection of components of $r^{-1}(U)$. Let $C_1$ be a component of $r^{-1}(U)$ that misses $r^{-1}(U_2)$. Note that $C_1$ must lie entirely in $X$ since $U_2 = r^{-1}(U_1)$. $Y$ is a continuum that intersects $r^{-1}(U)$. Let $K_1$ be a component of $r^{-1}(U)$ that misses $r^{-1}(U_2)$ and let $C_1$ be the component of $r^{-1}(U)$ that contains $K_1$. Since $C_1, r^{-1}(U_2) = \emptyset$, $C_1 \not= C_2$. Continue in this manner and select a sequence $(C_i) (i = 1, 2, ...)$ of distinct components of $r^{-1}(U)$ such that for each $i = 1, 2, ...$, $r^{-1}(C_i) \not= \emptyset$. For each $i = 1, 2, ...$, let $x_i \in r^{-1}(U_i) \setminus C_i$. Since $r^{-1}(U_i)$ is compact, $\sum x_i \in (i = 1, 2, ...)$ has a limit point $p$ and since $r$ is continuous, $p \in r^{-1}(y)$. But this is a contradiction since $Z$ is locally connected and $p$ is interior to some component of $r^{-1}(U)$. This component must intersect infinitely many $C_i (i = 1, 2, ...)$ and this is absurd. This completes the proof.

(3.3) Remark. We note that in the proof of the necessity of Theorem (3.1) we only used the fact that for some sequence $(U_i) (i = 1, 2, ...)$ of regions closing down on $y$, $f^{-1}(U_i)$ has a compact component that missed $f^{-1}(U)$, $i = 1, 2, ...$, in order to obtain a contradiction. Thus if $Z$ is locally connected, about every point $y \in Y$ there is a conditionally compact region $Z$ such that and $W$ are any regions in $Y$ containing $y$ where $U \subset W \subset \emptyset$, then every compact component of $f^{-1}(W)$ contains a component of $f^{-1}(U)$ and thus every compact component of $f^{-1}(W)$ meets $f^{-1}(y)$.

Example. In this example $Z$ is locally connected and $Y$ contains a point $y$ such that for every positive integer $n$ there exists an open set $U$ of $Y$ such that the inverse under $f$ of every region $R$ of $Y$ containing $y$ with $R \subset U$ has at least $n$ compact components. Thus there is no upper bound on the number of compact components of $f^{-1}(R)$ in Theorem (3.1).

Let $L_0$ be the segment in the plane joining the origin and the point $(1, 1)$ and for each positive integer $i$, let $L_i$ be the segment from $(0, 0)$ to $(1, 1)$. Let $Y$ be the set $\sum L_i (i = 0, 1, 2, ...)$ minus the origin and let $X$ be the interval $[0, 1]$ on the $x$-axis. Define $f: X \rightarrow Y$ by $f(x, y) = x$. Then the unified space $Z$ of $f$ is merely the set $X + Y$ with the usual topology induced from the plane. The inverse image of the set $R = \{0, 1, 2, ...\}$ in $Y$ containing $y = 0$ has exactly $n$ compact components.

Example. We cannot weaken the hypothesis of Theorem (3.1) to that of $Z$ being locally connected at a point $y \in Z$. In this example every sufficiently small region $R$ of $Y$ about $y$ has infinitely many compact components in $f^{-1}(R)$ and $Z$ is locally connected at $y$. We construct this example in the plane.

Let $L_0$ and $M_0$ denote the line segments from the origin to the points $(1, 1)$ and $(1, -1)$ respectively. For every positive integer $i$, let $L_i$ denote the segment from $(2^{-i}, 0)$ to $(2^{-i}, 0)$ minus the latter endpoint and let $M_i$ denote the segment from $(3^{-i}, -3^{-i})$ to $(3^{-i}, 0)$ minus the latter endpoint. For every pair of positive integers $(i, j)$ where $i > j$, let $P(i, j)$ denote the segment from $(2^{-i}, 2^{-j})$ to $(2^{-i}, 2^{-j})$ and let $Q(i, j)$ denote the segment from $(3^{-i} - 3^{-j})$ to $(3^{-i} - 3^{-j})$.

Let $A$ denote the right half of the circle centered at $(1, 0)$ with radius 1. Finally let $X$ denote the set

$$A + \sum_{i=0}^{\infty} L_i + \sum_{i=0}^{\infty} M_i + \sum_{i,j \in \mathbb{Z}} P(i, j) + \sum_{i,j \in \mathbb{Z}} Q(i, j) \setminus \text{the origin}.$$
of $Y$ containing $y$ such that every region $R$ of $Y$ containing $y$ with $R \subseteq U$ has infinitely many compact components in $f^{-1}(R)$. Then since $Y$ is regular and satisfies the first axiom of countability, we may select a sequence $(U_i)$ ($i = 1, 2, ...$) of conditionally compact regions of $Y$ closing down on $y$ such that for each $i$, $U_i \subseteq U_i \subseteq U$, and $f^{-1}(U_i)$ has infinitely many compact components. First of all we note that for each $i$, $H_i = f^{-1}(U_i)$ is a subset of $f^{-1}(f(U_i))$ and since $f$ is finite and point inverses of $f$ have compact boundaries, $H_i$ is a compact subset of $X$.

Since $H_i$ is compact and every component of $f^{-1}(U_i)$ is open in $X$, $H_i$ is covered by a finite collection of components of $f^{-1}(U_i)$. Let $C_i$ be a component of $f^{-1}(U_i)$ that misses $H_i$. Let $K_i$ be a component of $f^{-1}(U_i)$ that misses $H_i$, and let $C_i$ be the component of $f^{-1}(U_i)$ that contains $K_i$. Since $K_i$ meets $H_i$, $C_i \not\subseteq C_i$. Continue in this manner and select a sequence $(G_i)$ ($i = 1, 2, ...$) of distinct components of $f^{-1}(U_i)$ such that for each $i$, $G_i$ contains a component of $f^{-1}(U_i)$ and thus meets $H_i$. For each $i > 2$, let $x_i \in G_i$. Then $\sum x_i$ ($i = 2, 3, ...$) has a limit point $p$ in the compact set $H_i$. But this is a contradiction; $X$ is locally connected and $H_i \subseteq f^{-1}(U_i)$ and thus $p$ is interior to some component of $f^{-1}(U_i)$. This component must then intersect infinitely many $G_i$ ($i = 2, 3, ...$) which is impossible. Thus $f$ satisfies the condition of Theorem (3.1) and $Z$ is locally connected.

**Corollary.** Suppose that $Y$ is a dendrite and $X$ is a locally connected generalized continuum and further suppose that $f$ is a mapping of $X$ onto $Y$ such that for any compact set $K$ in $X$, $f^{-1}(K)$ is also compact in $X$. Then $f$ is a compact mapping.

**Proof.** Suppose that $f$ is not compact. It is well known that $Y$ is a regular space and so by Theorem (3.4), $Z$ is a connected and locally connected space. Whyburn has shown that $Z$ is also a separable metric space and thus $Z$ is arcwise connected. Let $x \in X$ and let $P$ be an arc in $Z$ from $x$ to a point $p \in Y$ such that $P \cap X = \emptyset$. $f^{-1}(p) \not\subseteq \emptyset$ and let $u$ be the last point of $P$ in $f^{-1}(p)$ in the order from $x$ to $p$. If $f^{-1}(p) \cap P = \emptyset$ and let $v$ be an arc in $Z$ from $x$ to a point in $f^{-1}(p)$ to a point $v$ such that $w = f^{-1}(p) = v$ and $w \in P$. In either case there exists an arc $I$ from a point $u \in f^{-1}(p)$ to a point $v$ such that $I \subseteq f^{-1}(p)$ contains a point in $P$. Then $w \in I \cap (u \cup v)$ and let $w$ be the subarc of $I$ from $u$ to $w$. By hypothesis, $f^{-1}(I)$ is compact and hence if $a$ is the last point of $I$ in $f^{-1}(I)$, the subarc $aV$ lies entirely in $X$. Furthermore, since $a$ lies between $u$ and $v$, $a \in f^{-1}(p)$. Let $b = f(a)$ and let $J$ denote the unique arc of $Y$ from $b$ to $p$. Then each of the locally connected continua $f(u)$ and $r(ap) = p + f(ap) = p + f(ap)$ contains $J$. Hence if $(y_i)$ ($i = 1, 2, ...$) is a sequence of points in $J$ that converge to $p$, we may choose a sequence $(x_i)$ ($i = 1, 2, ...$) in $X$ such that for each $i = 1, 2, ..., x_i \in f^{-1}(y_i)$ and $x_i \not\subseteq (ap)$. Then $H = \sum x_i$ ($i = 1, 2, ...$) is also a subset of the compact set $f^{-1}(f(u))$ and hence has a point $p_0 \in X$. Since $f$ is continuous, $y \in f^{-1}(p)$, but since $ap \not\subseteq (ap)$ is closed in $X$, $y$ is also in $(ap)$. This is a contradiction, because $(ap) \subseteq f^{-1}(p)$ is empty by construction. Thus $f$ is a compact mapping.

**Example.** The compactness of the boundaries of point inverses is necessary in Theorem (3.4). Let $X = \{(x, y) : 0 < x < 1$ and $y = \sin(1/x)$ and let $Y = [-1, 1]$. Define $f: X \rightarrow Y$ by $f(x, y) = y$. Then $Z$ is homeomorphic to the closure of $X$ in the plane and $X$ is not locally connected at any point of $X$. However, point inverses of $f$ do not have compact boundaries.

**Example.** Corollary (3.5) cannot be extended to regular spaces. Let $X$ be the interval $[0, 1]$ and let $Y$ be the unit circle in the complex plane. Define $f: X \rightarrow Y$ by $f(x) = e^{ix}$. Then $f$ is 1-1 and hence satisfies the conditions of Corollary (3.5), but $f$ is not compact.

**Theorem.** Suppose that $X$ and $Y$ are locally connected continua and that $f$ is a closed mapping of $X$ onto $Y$. Then $Z$ is locally connected.

**Proof.** By Theorem 1 of [4] the singular set $S$ of $f$ is a totally disconnected set. Furthermore, $Y - S$ is open in $Z$, hence $Z$ is locally connected at all points of $X \cap (Y - S)$. Since a locally compact, connected Hausdorff space cannot fail to be locally connected on a totally disconnected set, $Z$ must be locally connected.

**Definiton.** A mapping $f: X \rightarrow Y$ is said to be quasi-open provided for any $y \in Y$ and any open set $U$ of $X$ containing a compact component of $f^{-1}(y)$, $U$ is interior to $f(U)$. See [16], p. 9. A mapping $f: X \rightarrow Y$ is said to be quasi-monotone provided for any continuum $K$ in $Y$ with a non-empty interior, $f^{-1}(K)$ has just finitely many components and each of these maps onto $X$ under $f$. See [20], p. 152.

**Lemma.** Suppose that $W$ and $V$ are locally compact Hausdorff spaces and suppose that $g$ is a compact quasi-open mapping of $W$ onto $V$. Then if $K$ is any continuum in $W$ every component of $g^{-1}(K)$ maps onto $K$ under $g$.

**Proof.** Let $H$ be any component of $g^{-1}(K)$. By theorem (10.6) of [16], $g$ can be factored into the form $g = im$ where $m$ is a compact monotone mapping of $W$ onto a locally compact Hausdorff space $M$ and $i$ is a compact, light and open mapping of $M$ onto $V$. Since $m$ is compact and monotone, $I = m(H)$ is a component of $f^{-1}(K)$. By Theorem (11.1) of [16], $I$ maps onto $K$ under $l$. (In Theorem 11.1 $M$ is assumed to be separable and metric in addition to being locally compact, however the proof given

...
is sufficient for the locally compact Hausdorff case). Then \( g(B) = \text{lim}(B) = \{U\} = \mathbb{K} \) as required.

(3.9) THEOREM. Suppose that \( X \) and \( Y \) are connected and locally connected and that \( Y \) satisfies the first axiom of countability. Suppose further that \( f \) is a quasi-open mapping of \( X \) into \( Y \). Then \( Z \) is locally connected if and only if the retraction \( r : Z \rightarrow Y \) is quasi-monotone. Furthermore whenever \( f : X \rightarrow Y = f(X) \) is also a compact mapping, \( Z \) is locally connected and thus \( r \) is a quasi-monotone map.

Proof. We first argue that whenever \( f \) is a quasi-open mapping, so also is \( r \). To this end let \( y \in Y \), let \( C \) be a compact component of \( r^{-1}(y) \), and let \( U \) be any open set in \( Z \) containing \( C \). Now if \( C \subseteq X \), \( C \) is a component of \( f^{-1}(y) \) and since \( f \) is quasi-open, \( y \) is interior to \( f(U - X) \) and hence to \( r(U) = f(U - X) + U \). On the other hand if \( C \subseteq X \), \( C \) contains \( y \) and \( y \) is then interior to \( U \). \( Y \) and hence to \( r(U) \). Thus \( r \) is quasi-open.

If \( Z \) is locally connected, \( r \) is a compact quasi-open mapping on a locally connected space into a first countable space and such maps can be shown to be quasi-monotone. See Theorem (10.41) of [16]. On the other hand, if \( r \) is quasi-monotone, \( f \) must satisfy the conditions of Theorem (3.1) and hence \( Z \) is locally connected.

Now let us assume that \( f : X \rightarrow Y = f(X) \) is compact. We will show that \( Z \) is locally connected by showing that \( f \) satisfies the conditions of Theorem (3.1). To this end let \( y \in Y \) and let \( U \) be any open set in \( Y \) containing \( y \). Now if \( y \in Y \), choose a conditionally compact region \( R \) about \( y \) such that \( R \subseteq U \). This is possible since \( f = f(U - X) \) is compact and \( f(X) = f(U - X) + U \). Then \( f^{-1}(R) \) has only finitely many components since \( f \) is quasi-open mapping on a locally connected space and as such is a quasi-monotone map. If \( y \in Y \), \( R \) is a conditionally compact region about \( y \) such that \( R \subseteq U \). Then \( f^{-1}(R) \) does not have any compact components. For suppose that \( C \) is such a component. Then \( C \) would be a component of \( r^{-1}(y) \) also. By Lemma (3.81), \( C \) maps onto \( R \) under the compact quasi-open map \( r \). But \( r(C) = f(C) \) and \( y \in f(C) \). This is a contradiction and so \( f \) satisfies the conditions of Theorem (3.1).

(3.10) DEFINITION. A mapping \( f : X \rightarrow Y \) is said to be monotone provided for every \( y \in Y \), \( f^{-1}(y) \) is continuum.

(3.11) THEOREM. Suppose that \( X \) and \( Y \) are locally generalized continua and suppose that \( f \) is a non-compact monotone map of \( X \) onto \( Y \). Then \( f \) is a compact mapping, \( Z \) is locally connected and both \( Y \) and \( Z \) are multi-coherent.

Proof. We wish to show that \( f \) satisfies the conditions of Theorem (3.3). To this end let \( y \in Y \) and let \( U \) be any open set in \( Y \) containing \( y \). We consider case: (1) If \( y \in S \), let \( R \) be any region of \( Y \) containing \( y \) such that \( R \subseteq U \). \( Y - S \) is a continuum and \( f \) satisfies the conditions of Theorem (3.1) in this case. Case (2) If \( y \in S \), let \( V \) be a conditionally compact open set in \( Y \) containing \( y \) such that \( V \subseteq U \). By hypothesis \( f \) is a continuum and \( R \) is a continuum in \( Y \). Then \( f^{-1}(V) \subseteq f^{-1}(Y - S) \) is a continuum and \( f \) satisfies the conditions of Theorem (3.1) in this case. Case (3) If \( y \in S \), let \( V \) be a conditionally compact open set in \( Y \) containing \( y \) such that \( V \subseteq U \). By hypothesis \( f \) is a continuum and \( R \) is a continuum in \( Y \). Then \( f^{-1}(V) \subseteq f^{-1}(Y - S) \) is a continuum and \( f \) satisfies the conditions of Theorem (3.1) in this case. Case (4) If \( y \in S \), let \( V \) be a conditionally compact open set in \( Y \) containing \( y \) such that \( V \subseteq U \). By hypothesis \( f \) is a continuum and \( R \) is a continuum in \( Y \). Then \( f^{-1}(V) \subseteq f^{-1}(Y - S) \) is a continuum and \( f \) satisfies the conditions of Theorem (3.1) in this case. Case (5) If \( y \in S \), let \( V \) be a conditionally compact open set in \( Y \) containing \( y \) such that \( V \subseteq U \). By hypothesis \( f \) is a continuum and \( R \) is a continuum in \( Y \). Then \( f^{-1}(V) \subseteq f^{-1}(Y - S) \) is a continuum and \( f \) satisfies the conditions of Theorem (3.1) in this case.
with vertices \(\{1/i+1\}, \{j/0\}, \{1/i+1, 0\}\) and \(1/j+1, 0\). Let \(P\) denote the solid triangle with vertices \((0, 0, 0), (-1, 0, 0)\) and \((0, 0, 1)\) and for each positive integer \(n\), let \(Q_n\) denote the solid triangle with vertices \((i, 0, 0), (i, 0, 1)\) and \((i, -1, 0)\). Let \(W\) denote the \(xy\)-plane. Finally let \(X\) be the set \(W + P + \sum_{n=0}^{\infty} P(i, j) + \sum_{n=0}^{\infty} Q_n\) minus the closed set \(A + \sum_{n=0}^{\infty} I_n\).

We describe the mapping \(f\) and the range space \(Y\) at the same time. Let \(f\) be a 1-1 mapping on \(X\) such that:

(i) the side of the triangle \(P(i, j)\) with endpoints \((1/i+1, 1/j, 0)\) and \((1/i+1, 1/j, 1/i+1)\) is mapped onto the missing interval from \((1/i+1, 1/j, 0)\) to \((1/i, 1/j, 0)\) on \(J_i\);

(ii) the side of the triangle \(P\) with endpoints \((0, 0, 0)\) and \((0, 0, 1)\) is mapped onto the missing line \(A_i\);

(iii) the side of the triangle \(Q_n\) with endpoints \((i, 0, 0)\) and \((i, 0, 0)\) is mapped onto the interval from \((i-1, 0, 0)\) to \((i, 0, 0)\) on the \(x\)-axis.

(iv) \(f\) is the identity elsewhere.

The resultant space (the space \(X\) with the missing set \(A + \sum_{n=0}^{\infty} I_n\) filled in by the sides of the vertical triangles) is the space \(Y\). It is unicoherent as is the unified space \(Z\) of \(f\).

We see that \(Z\) is not locally connected since the inverse image of the closure of any sufficiently small region \(E\) of \(Y\) containing \(f(2, 0, 1)\) (= the missing point \((1, 0, 0)\)) would intersect all but finitely many of the triangles \(P(1, j)\) and so would have infinitely many components in \(f^{-1}(E)\).

4. Unicoherence of the unified space. In this section we give necessary and sufficient conditions for \(Z\) to be unicoherent whenever it is a locally connected metric continuum. We also show that even when \(Z\) is not compact there are certain necessary conditions that \(X\) and \(Y\) must satisfy in order for \(Z\) to be unicoherent.

We first state some definitions and lemmas that will be useful in this and the following section. The statements of the lemmas are included for completeness and the proofs appear in [2].

4.1 Definitions. A connected space \(W\) is said to be unicoherent provided whenever \(W = W + K\) where \(W\) and \(K\) are closed and connected sets, \(W\) is also connected. The space \(W\) is said to be weakly-unicoherent provided whenever \(W = W + K\) where \(W\) is a closed and connected set and \(K\) is a continuum, \(W\) is also a continuum.

A connected set \(W\) is said to have the Complementation Property provided for every compact set \(K\) in \(W\), \(W - K\) has at most one non-continuum component.

(4.2) Lemma. Suppose that \(W\) is a locally connected generalized continuum. A necessary and sufficient condition that \(W\) be weakly-unicoherent is that every conditionally compact component of the complement of a closed and connected set has the Complementation Property.

(4.3) Lemma. Let \(W\) be a non-compact locally connected generalized continuum. Then a necessary and sufficient condition for \(W\) to be weakly-unicoherent and have the Complementation Property is that \(W\) satisfy:

(a) for any continuum \(K\) in \(W\) and any open set \(U\) containing \(K\), there is a conditionally compact region \(B\) of \(W\) about \(K\) such that \(B\setminus K\) is a continuum lying entirely in \(U\).

(4.4) Lemma. Let \(W\) be a locally connected weakly-unicoherent generalized continuum and let \(A\) be a closed subset of \(W\). Then (1) every component of \(W\) is weakly-unicoherent and (2) if every non-empty component of \(A\) is non-compact and if \(W\) has the Complementation Property, every component of \(W\) is weakly-unicoherent and has the Complementation Property.

(4.5) Lemma. Let \(W\) be a non-compact locally connected generalized continuum. Then a necessary and sufficient condition for \(W = W + \{\infty\}\), the one-point compactification of \(W\), to be locally connected and unicoherent is that \(W\) be weakly-unicoherent and have the Complementation Property.

(4.6) Lemma. Let \(W\) be a continuum and let \(A\) be a non-empty locally and connected subcontinuum of \(W\) such that \(B = W - A\) is connected and locally connected. Then \(W\) is unicoherent if and only if \(B\) is weakly-unicoherent and has the Complementation Property.

(4.7) Theorem. Let \(X\) and \(Y\) be locally connected generalized continua and let \(f\) be a non-compact mapping of \(X\) into \(Y\). Suppose that \(Z\) is locally connected and unicoherent. Then (1) \(X\) is weakly-unicoherent and \(Y\) is unicoherent; (2) \(S\) is connected; and (3) if \(f(X)\) is compact, \(X\) has the Complementation Property.

Proof. (1). Whyburn has shown in the proof of Theorem 2 of Section 6 of [14] that when \(Z\) is a locally connected generalized continuum and \(Y\) is multi-continuous, so also is \(Z\). Hence \(Y\) must be unicoherent. The weak-unicoherence of \(X\) is a direct consequence of Part (1) of Lemma (4.4).

(2). Recall that \(X\) is compact, and \(S\) must be connected by the unicoherence of \(Z\).

(3). If \(f(X)\) is compact, \(X\) is compact. Thus \(X\) is a conditionally compact component of the complement of the closed and connected set \(Y\) and by Lemma (4.3), \(X\) has the Complementation Property.

Examples. (a) In this example \(f\) is a 1-1 mapping of \(X\) into \(Y\), where \(X\), \(Y\) and \(Z\) are all locally connected unicoherent continua and \(S\) is a continuum. However, \(f(X)\) is not compact nor does \(X\) have the Complementation Property. Let \(X\) be the open interval (0, 1), let \(Y\)
by the half open interval \((0, 1]\) and let \(f\) be the identity map of \(X\) into \(Y\). Then \(Z\) is homeomorphic to an open interval and is unicoherent and \(f(X) = Y\) is not compact and \(X\) does not have the Complementation Property.

(b) This example shows that even when (i) \(X\) is unicoherent and has the Complementation Property, (ii) \(S\) is connected, and (iii) \(Y\) is unicoherent, \(Z\) need not be unicoherent. Thus conditions (i), (ii) and (iii) are not sufficient for the unicoherence of \(Z\). Let \(X = (a, b) : 0 < a < 1\) and \(0 < y < 1\) and let \(Y = [0, 1]\). Define \(f: X \rightarrow Y\) by \(f(x, y) = a\). Then \(Z\) is homeomorphic to the cylinder \(S^1 \times [0, 1]\) and is not unicoherent.

We note that in these two examples \(Z\) is not compact. When \(Z\) is compact, conditions (i) and (iii) are necessary and sufficient for the unicoherence of \(Z\).

(1.8) Theorem. Let \(X\) and \(Y\) be locally connected generalized continua and let \(f\) be a non-compact mapping of \(X\) into \(Y\). Suppose further that \(Z\) is locally connected and compact. Then \(Z\) is unicoherent if and only if \(X\) is weakly-unicoherent and has the Complementation Property and \(Y\) is unicoherent.

Proof. Follows from Theorem (4.7) and Lemma (4.6).

(4.9) Lemma. Suppose that \(Z\) is not compact. Let \(Z_m\) and \(Z_m\) denote the one-point compactifications of \(Z\) and \(Z\) respectively and let \(Z\) denote the unified space of the mapping \(f: X \rightarrow Y_m\). Then \(Z_m\) and \(Z_m\) are homeomorphic.

Proof. Let \(u\) denote the retraction of \(Z\) onto \(Z_m\) associated with \(f: X \rightarrow Y_m\) and let \(Q = u^{-1}(Y)\). (Here we are regarding \(Y\) as a subset of \(Y_m\)). Then \(Q\) is a locally compact Hausdorff space and \(u\): \(Q \rightarrow Y\) is a compact retraction of \(Q\) onto \(Y\). Thus by Theorem (2.9) \(Q\) is homeomorphic to the unified space of the mapping \(u\): \(X \rightarrow Y\). But \(u\): \(X \rightarrow f\) so \(Q\) is homeomorphic to \(Z\), thus we may consider \(Z_m\) to be a one-point compactification of \(Q\). We note that \(Z\) is also a one-point compactification of \(Q\). It then follows from the topological uniqueness of the one-point compactification of a locally compact Hausdorff space that \(Z_m\) and \(Z_m\) are homeomorphic.

(4.10) Corollary. Let \(f: X \rightarrow Y\) be a non-compact mapping where \(X\) and \(Y\) are locally connected generalized continua and suppose that \(Z\) is locally connected. Then a necessary and sufficient condition for \(Z\) to be weakly-unicoherent and have the Complementation Property is that both \(X\) and \(Y\) are weakly-unicoherent and have the Complementation Property.

Proof. The necessity. Suppose that \(Z\) is weakly-unicoherent and has the Complementation Property. If \(Y\) is compact, \(Z\) is compact and unicoherent and the necessity in that case follows from Theorem (4.8). Thus we may assume that \(Y\) is not compact. By Lemma (4.5), \(Z_m\), the one-point compactification of \(Z\), is unicoherent. Let \(Z\) denote the unified space of the mapping \(f: X \rightarrow Y_m\) where \(Y_m\) is the one-point compactification of \(Y\). By Lemma (4.9), \(Z\) is homeomorphic to \(Z_m\), hence \(Z\) is unicoherent. Thus applying Theorem (4.8) to \(Z\) we have that \(X\) is weakly-unicoherent and has the Complementation Property and \(Y\) is unicoherent. By Lemma (4.5), \(Y\) is weakly-unicoherent and has the Complementation Property and this completes the proof of the necessity.

The sufficiency. If \(Y\) is compact, \(Y\) is unicoherent and the sufficiency follows from Theorem (4.8). If \(Y\) is not compact, \(Y_m\) is unicoherent and by Theorem (4.8) the unified space \(Z\) of the mapping \(f: X \rightarrow Y_m\) is unicoherent. Since \(Z\) is homeomorphic to \(Z_m\), \(Z_m\) is unicoherent. Thus by Lemma (4.5), \(Z\) is weakly unicoherent and has the Complementation Property.

(4.11) Corollary. Let \(f: E^n \rightarrow F^n\) be a non-compact mapping of \(E^n\) into \(F^n\) where \(n, m > 2\) and let \(Z(u, m)\) denote the unified space of \(f\). Then if \(Z(u, m)\) is locally connected, it is weakly-unicoherent and has the Complementation Property.

(4.12) Remark. Note that the proof of the sufficiency of Theorem (4.8) depends on Lemma (4.6) which does not require \(Z\) to be locally connected. Thus whenever \(Z\) is a weakly-unicoherent, locally connected generalized continuum that has the Complementation Property and \(Y\) is a locally connected unicoherent continuum, \(Z\) is unicoherent even if it is not locally connected. In particular we have:

(4.13) Corollary. Let \(f: E^n \rightarrow S^n\) be a mapping of \(E^n\) into \(S^n\) where \(n, m > 2\) and let \(Z(u, m)\) denote the unified space of \(f\). Then \(Z(u, m)\) is unicoherent.

5. Singular sets for mappings. In this section we investigate the relationship between the connectedness and compactness properties of the singular sets of a non-compact mapping and certain topological properties of the range and domain. G. T. Whyburn in [14] has shown that when \(f\) is a monotone map of \(X\) onto \(Y\) where \(X\) and \(Y\) are locally connected generalized continua and \(Y\) is unicoherent, the singular set \(T\) of \(f\) in \(X\) cannot have any non-empty compact components. In [6] E. Duda has shown that every monotone mapping of a locally connected generalized continuum having the Complementation Property onto the plane is a compact mapping. Similar results are obtained here where we show that when \(f\) is a mapping of a locally connected and connected space having the Complementation Property into a space \(Y\), the singular set \(S\) of \(f\) in \(Y\) cannot have any non-empty compact components if \(f(X)\) is not compact; and if \(f(X)\) is compact, \(S\) is a continuum. Furthermore we show that every closed mapping of a locally connected, connected and paracompact space having the Complementation Property onto a non-
compact space must be a compact mapping. Finally we show that under certain conditions the Complementation Property and weak-unicoherence are inherited by the components of the complements of $S$ and $T$ in $Y$ and $X$ respectively.

We first prove two lemmas.

(5.1) **Lemma.** Let $W$ be a locally connected, connected, locally compact Hausdorff space and let $V$ be an open subset of $W$ such that $V$ is not compact but $\overline{V} = Fr V$ is compact. Then $V$ has a non-conditionally compact $(in W)$ component.

Proof. Suppose to the contrary that every component of $V$ is conditionally compact. Let $Q$ be a conditionally compact open subset of $W$ containing $X$. Since $Q$ is compact and $V$ is not compact, we may choose a component $V_1$ of $V$ such that $V_1 \subseteq Q$. Then since $Q + V_1$ is compact, we may choose a component $V_2$ of $V$ such that $V_2 \subseteq Q + V_1$. Continuing in this manner we obtain a sequence $\{X_n, X_{i+1}\}$ of distinct components of $V$ such that for each $i = 1, 2, ..., n$, $X_i \subseteq X_{i+1}$. We note that since $W$ is connected and locally connected, the closure of every component of $V$ meets $K$. Thus for each $i = 1, 2, ..., n$, $X_i \subseteq Q$. This implies that each $X_i$ meets the boundary of $Q$ and thus $Fr Q$ contains a limit point of $\sum_{i=1}^{\infty} X_i$, a contradiction.

(5.3) **Theorem.** Let $f: X \rightarrow Y$ be a non-compact mapping where $X$ is a connected and locally connected space having the Complementation Property. Then $f(X)$ is compact, the singular set $S$ of $f$ in $X$ cannot have any non-empty component; and (2) if $f(X)$ is compact, $S$ is a continuum.

Proof. By Theorem (1.1) of (8), point inverses of $f$ have compact boundaries. It asserts that this implies that every point inverse of $f$ is compact. For suppose to the contrary that $Fr f^{-1}(y) = K$ is compact but $f^{-1}(y)$ is not. By Lemma (5.1), $f^{-1}(y) - K$ must have a non-conditionally compact component which we denote by $\Sigma$. Then since $X$ has the Complementation Property, $\Sigma$ is the only non-conditionally compact component of $X - K$ which implies that $X - \Sigma$ is compact. But then $f(X)$ is compact.

(5.4) **Theorem.** Suppose that $X$ is a locally connected, connected and paracompact space having the Complementation Property. Then every closed mapping $f : X \rightarrow Y$ is a compact mapping.

Proof. By Theorem (1.1) of (8), point inverses of $f$ have compact boundaries. It asserts that this implies that every point inverse of $f$ is compact. For suppose to the contrary that $Fr f^{-1}(y) = K$ is compact but $f^{-1}(y)$ is not. By Lemma (5.1), $f^{-1}(y) - K$ must have a non-conditionally compact component which we denote by $\Sigma$. Then since $X$ has the Complementation Property, $\Sigma$ is the only non-conditionally compact component of $X - K$ which implies that $X - \Sigma$ is compact. But then $f(X)$ is compact.

(5.5) **Theorem.** Suppose that $f$ is a non-compact mapping of $X$ into $Y$ where $X$ and $Y$ are locally connected generalized continua, $X$ has the Complementation Property and $Y$ weakly-unicoherent. Then (1) if $f(X)$ is not compact and $Y$ has the Complementation Property, every component of $Y - S$ is weakly-unicoherent and has the Complementation Property; (2) if $f$ is onto and $Y$ is not compact, every component of $Y - S$ is weakly-unicoherent and has the Complementation Property; and (3) if $f$ is a monotone mapping and $Y$ is not compact, every non-empty component of $T$ is non-compact.

Proof. By Theorem (1.5) of (3), every non-empty component of $S$ is non-compact and by Part (3) of Lemma (4.4) every component of $Y - S$ is weakly-unicoherent and has the Complementation Property.

(2). Let $Q$ be a component of $Y - S$. We wish to show that $Q$ satisfies condition (a) of Lemma (4.3). To this end let $K$ be a continuum lying in $Q$ and let $U$ be an open subset of $Q$ containing $K$. By Lemma (4.3) of (7), there exists a conditionally compact region $M$ of $Q$ containing $K$ such that $M \subseteq U$ and $Y - M$ has only finitely many components, say $X_1, X_2, ..., X_n$. Since $M$ misses $S$, $f(M)$ is compact and since $X$ has the Complementation Property, $X - M$ has only one non-conditionally compact component which we denote by $B$. Now $X - B$ is compact and thus the component of $Y - M$ that contains $f(B)$ is the only non-conditionally compact component of $Y - M$. Without loss of generality let
us suppose that $f(B) \subseteq N_i$. Then each of the sets $N_1, N_2, ..., N_p$ is conditionally compact. By Theorem (5.3), every non-empty component of $S$ is non-compact. Since the boundary of each $N_i$ meets $S$, $N_i \cap S = \emptyset$ for $i = 1, 2, ..., p$. Thus $R = M + N_1 + N_2 + ... + N_p$ is a conditionally compact region whose closure lies entirely in $Q$. Furthermore, since $X$ is weakly unicoherent, $FrR = R - X$ is a continuum lying in $U$. Thus $Q$ satisfies condition (a) and $Q$ is weakly unicoherent and has the Complementation Property.

(3). Suppose that $K$ is an $n$-non-empty compact component of $T$. Let $E$ be a conditionally compact region in $X$ containing $K$ so that $F = FrE$ misses $T$. Let $G = f(E)$ and let $V$ be a region in $X - C$ which intersects $f(K)$. Let $p$ be an arc in $V$ such that $pq$ meets $f(E)$ in exactly one point $p$. This is possible since every point in $f(K)$ is a limit point of $X - f(E)$. Then $p \not\in f(E)$ since $V - C = \emptyset$. Hence $p \not\in S$ as $p$ is not interior to $f(E)$.

Let $A_{n}$ be the union of all the components of $X - F$ which intersect the set $f^{-1}(pq - p)$ and let $B = X - A$. Then $A_{n} = A_{n}$ and $B$ are closed, $B$ and $f(A_{n})$ are connected, $X = A_{n} + B$ and $A_{n} \times B$. By our hypothesis, $X - F$ has exactly one non-conditionally compact component which we denote by $Y$. Now either $Y \cap A_{n}$ and $B$ is compact or $Y \cap B$ and $A$ is compact. But in either case if we let $a = f(A)$ and $b = f(B)$, $Y = a + b$ where $a$ and $b$ are closed and connected and either $a$ or $b$ is compact.

Since $A_{n} \times B$, $a \times C \times C$. For if a point $a \not\in C$ and $a + b$ is not in $C + S$, a small region $V$ in $X$ containing $p$ and $S$ misses $C + S$ would have a region $W$ as its inverse under $f$ which would meet both $A$ and $B$ but not $F$ and this is impossible. Also since $A_{n} = A_{n}$ and $B = B$, we see that $A_{n} \cap S = \emptyset$ so that $a \not\in C$. Furthermore, $pq - p \not\in f(E)$ so that $pq - p \in B$. Thus $a + b \not\in S$ since $p \not\in S$. Hence $a + b$ is the sum of two non-empty opened sets $\alpha - S$ and $\beta - C$ and is not connected. But this contradicts the weak-unicoherence of $Y$ since $Y = a + b$ and either $a$ or $b$ is compact. This proof closely parallels that of Theorem (2) of [14].

In order to see that components of $X - T$ have the Complementation Property let $Q$ be a component of $X - T$ and let $K$ be a compact set in $Q$. Since $fQ$ is a compact monotone mapping, $Q$ is an inverse set of $f$, i.e., $Q = f^{-1}(Q)$, and $H = f^{-1}(K)$ is a compact set lying entirely in $Q$. Furthermore, $P = f(Q)$ is a component of $X - S$ and every component of $Q - H$ maps onto a component of $P - f(K)$ under $f$. By part (2) of this Theorem, $P - f(K)$ has exactly one non-conditionally compact (in $P$) component. This implies that $Q - H$ has exactly one non-conditionally compact (in $Q$) component since $fQ$ is a compact monotone mapping of $Q$ onto $P$. It is then clear that $Q - K$ must then have only one non-conditionally compact (in $Q$) component.

(5.6) Remark. Note in part (1) of Theorem (5.4) the components of $Y - S$ inherit the Complementation Property from $Y$ but in part (2) they inherit the Complementation Property from $X$.

References