Metric dimension and equivalent metrics*

by

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Given a metric space (X, ϱ), the *metric dimension* of (X, ϱ), indicated $\mu\dim(X, \varrho)$, is the smallest integer m such that for all $\varepsilon > 0$ there exists an open cover $\mathcal{U}_\varepsilon$ of X such that (i) ϱ-mesh $\mathcal{U}_\varepsilon < \varepsilon$ and (ii) ord $\mathcal{U}_\varepsilon \leqslant m+1$. It is trivial that $\mu\dim(X, \varrho) \leqslant \dim X$, where dim is covering dimension. In the other direction, Katětov [2] has shown that $2\mu\dim(X, \varrho) \geqslant \dim X$. In [3], examples are given (for all $n > 1$) of spaces (X_n, ϱ) with $\dim X_n = n$ and $\mu\dim(X_n, \varrho) = [(n+1)/2]$ (the biggest integer in $(n+1)/2$).

The purpose of the present paper is to prove the following theorem, which answers a question raised in [3]. Readers are referred to [3] for an extensive bibliography.

THEOREM. *Let (X, ϱ) be a metric space with $\mu\dim(X, \varrho) = m$ and $\dim X = n$. Then for any integer k such that $m \leqslant k \leqslant n$, there exists a metric ϱ_k for X such that* (i) *ϱ_k is topologically equivalent to ϱ and* (ii) $\mu\dim(X, \varrho_k) = k$.

To facilitate the proof of the theorem, we introduce and prove three lemmas.

LEMMA 1. *Let (X, ϱ) be a metric space, r a positive integer, and let $\mathcal{U}$ be a locally finite open cover of X such that* ord $\mathcal{U} \leqslant r+1$. *Then there exists an open cover $\mathcal{V}$ of X, refining $\mathcal{U}$, such that $\mathcal{V} = \bigcup_{i=0}^{r} \mathcal{V}_i$, where each $\mathcal{V}_i$ is a disjoint open collection of subsets of X. Thus* ord $\mathcal{V} \leqslant r+1$.

Proof. Let $\mathcal{U}$ be indexed by a set A, so $\mathcal{U} = \{U_\alpha\colon \alpha \in A\}$. For each $\alpha \in A$ define the real function $g_\alpha\colon X \to [0, 1]$ by the formula

$$g_\alpha(x) = \frac{\varrho(x, X - U_\alpha)}{\sum_{\beta \in A} \varrho(x, X - U_\beta)}. \tag{1}$$

Since $\mathcal{U}$ is a cover, for fixed x there is at least one β such that $x \in U_\beta$ so the denominator is not zero. Also, since $\mathcal{U}$ is locally finite, there exists

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an open neighborhood W_x of x which intersects only a finite number of elements of $\mathfrak{U}$. Since $\varrho(y, X-U_\beta)$ is a continuous function of y, for all β, it then follows that g_α is continuous, for all $\alpha \in A$. We list the following properties for reference later on.

$$\text{(2)}\quad \begin{array}{ll} \text{(i)} & g_\alpha(x) > 0 \text{ if and only if } x \in U_\alpha\,, \\ \text{(ii)} & 0 \leqslant g_\alpha(x) \leqslant 1\,, \\ \text{(iii)} & \text{for fixed } x,\ \sum_{\alpha\in A} g_\alpha(x) = 1\,. \end{array}$$

We define the collections $\mathfrak{V}_0, \mathfrak{V}_1, \ldots, \mathfrak{V}_r$ by use of the functions g_α as follows. $\mathfrak{V}_0 = \{V_\alpha\colon \alpha \in A\}$, where $V_\alpha = \{x\colon g_\alpha(x) > g_\beta(x) \text{ if } \beta \neq \alpha\}$. For $0 < i \leqslant r$, let B be a set of $i+1$ *distinct* elements of A and define

$$V_B = \{x\colon \min_{\alpha\in B}\{g_\alpha(x)\} > g_\beta(x), \text{ for all } \beta \notin B\}\,.$$

Let $\mathfrak{V}_i$ be the set of all V_B, where B is a set of $i+1$ distinct elements of A, and let $\mathfrak{V} = \bigcup_{i=0}^{r} \mathfrak{V}_i$.

1. *Proof that* $\mathfrak{V}$ *is an open collection.* Take $V \in \mathfrak{V}$, $y \in V$, and let B be the finite subset of A such that $V = V_B$ in the definition above. Let g be the continuous real function defined as follows: $g(x) = \min_{\alpha\in B}\{g_\alpha(x)\}$. Then $V = \{x\colon g(x) > g_\beta(x), \text{ for all } \beta \notin B\}$. Since $y \in V$, $g(y) > 0$, so, defining W_0 as the set of all x such that $g(x) > 0$, it follows that $y \in W_0$, and W_0 is an open set.

Let W_y be an open neighborhood of y which hits only a finite number of elements of $\mathfrak{U}$ ($\mathfrak{U}$ is locally finite). Let $C \subset A$ be the set of all α such that $W_y \cap U_\alpha \neq \emptyset$. Then C is finite, $C \supset B$, and we may write $C = B \cup \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$. For each i $(1 \leqslant i \leqslant s)$, let W_i be the open set consisting of all x such that $g(x) > g_{\alpha_i}(x)$. Finally, set $W = W_y \cap (\bigcap_{i=0}^{s} W_i)$. Then $y \in W$, W is open, and $W \subset V$. Thus V is open.

2. $\mathfrak{V}_i$ *is a disjoint collection.* Fix i and suppose that B_1 and B_2 are different $(i+1)$-subsets of A. Then there exist $\beta_1 \in B_1$ and $\beta_2 \in B_2$ such that $\beta_1 \notin B_2$ and $\beta_2 \notin B_1$. Suppose $x \in V_{B_1}$. Then $\min_{\alpha\in B_1}\{g_\alpha(x)\} > g_{\beta_2}(x)$, so $g_{\beta_1}(x) > g_{\beta_2}(x)$. If x is also in V_{B_2}, then $g_{\beta_2}(x) > g_{\beta_1}(x)$, a contradiction.

3. $\mathfrak{V}$ *is a cover of* X. Let $x \in X$ be given, and let $\alpha_0, \alpha_1, \ldots, \alpha_j$ be the set of all (at most $r+1$) elements β of A such that $x \in U_\beta$ (i.e. $g_\beta(x) > 0$), the integer subscripts being assigned so that $g_{\alpha_0}(x) \geqslant g_{\alpha_1}(x) \geqslant \ldots \geqslant g_{\alpha_j}(x)$. Determine i $(0 \leqslant i \leqslant j)$ as the greatest integer such that $g_{\alpha_i}(x) = g_{\alpha_0}(x)$, and let $B = \{\alpha_0, \ldots, \alpha_i\}$. Then we have $g_{\alpha_0}(x) = g_{\alpha_1}(x) = \ldots = g_{\alpha_i}(x) > g_\beta(x)$ if $\beta \notin B$, so $x \in V_B \in \mathfrak{V}_i$. This completes the proof of Lemma 1.

LEMMA 2. *Suppose that r is a positive integer and $\varepsilon > 0$. Then there exist $r+1$ finite open covers of* $[0,1]$, $\mathfrak{W}_0, \mathfrak{W}_1, \ldots, \mathfrak{W}_r$ *such that*

(i) mesh $\mathfrak{W}_i < \varepsilon$,

(ii) ord $\mathfrak{W}_i \leqslant 2$,

(iii) *for* $x \in [0,1]$ *and fixed* i, *if* $\operatorname{ord}_x \mathfrak{W}_i = 2$, *then for all* $j \neq i$, $\operatorname{ord}_x \mathfrak{W}_j = 1$.

Proof. Let q_0 be an odd prime integer such that $1/q_0 < \varepsilon/2$, and let $q_1 < q_2 < \ldots < q_r$ be the next r primes. Let δ be the minimum of the finite set of positive numbers

$$\{|m/q_i - n/q_j|\colon i \neq j;\ m = 1, 2, \ldots, q_i-1;\ n = 1, 2, \ldots, q_j-1;\ i, j = 0, 1, \ldots, r\}\,.$$

Let $\mathfrak{W}_i$ be the set of q_i open intervals

$$\{(j/q_i - \delta/2,\ (j+1)/q_i + \delta/2)\colon j = 0, 1, \ldots, q_i-1\}\,.$$

The covers $\mathfrak{W}_0, \mathfrak{W}_1, \ldots, \mathfrak{W}_r$ have the desired properties.

LEMMA 3. *Suppose that (X, ϱ) is a metric space with $\mu\dim(X, \varrho) = r < n = \dim X$, $f\colon X \to [0,1]$ is continuous, and $\sigma(x, y) = \varrho(x, y) + |f(x) - f(y)|$.*

Then σ is a metric on X, topologically equivalent to ϱ, and $r \leqslant \mu\dim(X, \sigma) \leqslant r+1$.

Proof. It is well known (see [1], p. 199) that σ is a metric for X and is topologically equivalent to ϱ. Furthermore, $\sigma(x, y) \geqslant \varrho(x, y)$, which implies that $\mu\dim(X, \sigma) \geqslant \mu\dim(X, \varrho)$. We will prove that $\mu\dim(X, \sigma) \leqslant r+1$.

Let $\varepsilon > 0$ be given. Since $\mu\dim(X, \varrho) = r$, it follows by definition that there exists an open cover $\mathfrak{B}$ of X such that (i) ϱ-mesh $\mathfrak{B} < \varepsilon/2$, and (ii) ord $\mathfrak{B} \leqslant r+1$. We may index $\mathfrak{B}$ by an ordinal η, so that $\mathfrak{B} = \{B_\alpha\colon \alpha < \eta\}$. Now every metric space is paracompact [4] so there is a locally finite open cover $\mathfrak{C}$ which refines $\mathfrak{B}$. Let $\mathfrak{U}$ be the open cover of X obtained by *amalgamating* $\mathfrak{C}$ *relative to* $\mathfrak{B}$. That is, $\mathfrak{U} = \{U_\alpha\colon \alpha < \eta\}$ where U_α is the union of all elements of $\mathfrak{C}$ which are subsets of B_α, but are not subsets of any B_β for any $\beta < \alpha$. Thus

$$U_\alpha = \bigcup\{C\colon C \in \mathfrak{C},\ C \subset B_\alpha,\ C \not\subset B_\beta \text{ if } \beta < \alpha\}\,.$$

Then (i) $\mathfrak{U}$ is a locally finite cover of X, (ii) ord $\mathfrak{U} \leqslant r+1$, and (iii) ϱ-mesh $\mathfrak{U} < \varepsilon/2$.

Applying Lemma 1, let $\mathfrak{V} = \bigcup_{i=0}^{r} \mathfrak{V}_i$ be an open cover of X refining $\mathfrak{U}$, such that each $\mathfrak{V}_i$ is a disjoint collection. Let $\mathfrak{W}_0, \mathfrak{W}_1, \ldots, \mathfrak{W}_r$ be the collection of $r+1$ open covers of $[0,1]$ given by Lemma 2, except that ϱ-mesh $\mathfrak{W}_i < \varepsilon/2$. For fixed i $(0 \leqslant i \leqslant r)$, let $\mathfrak{U}_i^*$ be the set of all intersections

$V \cap f^{-1}(W)$, where $V \in \mathfrak{V}_i$ and $W \in \mathfrak{W}_i$, and set $\mathfrak{U}^* = \bigcup_{i=0}^{r} \mathfrak{U}_i^*$. We show that σ-mesh $\mathfrak{U}^* < \varepsilon$ and $\operatorname{ord} \mathfrak{U}^* \leqslant r+2$, from which it follows that $\mu\dim(X, \sigma) \leqslant r+1$.

1. σ-mesh $\mathfrak{U}^* < \varepsilon$. Suppose $U \in \mathfrak{U}^*$, so there exists i, $V \in \mathfrak{V}_i$ and $W \in \mathfrak{W}_i$ such that $U = V \cap f^{-1}(W)$. Since $V \in \mathfrak{V}$ and $\mathfrak{V}$ refines $\mathfrak{U}$, ϱ-diameter$(V) < \varepsilon/2$. Also for x and y in $f^{-1}(W)$ we have $|f(x)-f(y)| < \varepsilon/2$. Thus for x and y in U we have $\sigma(x, y) \leqslant \varrho(x, y) + |f(x)-f(y)| < \varepsilon$.

2. $\operatorname{ord} \mathfrak{U}^* \leqslant r+2$. Take $x \in X$. There may be an i (at most one) such that $f(x)$ is in two elements of $\mathfrak{W}_i$. There is at most one $V \in \mathfrak{V}_i$ such that $x \in V$, so x can be in at most two elements of $\mathfrak{U}_i^*$. For all j such that $f(x)$ is in only one element of $\mathfrak{W}_j$ it follows that x is in at most one element of $\mathfrak{U}_j^*$. Since $\mathfrak{U}^* = \bigcup_{i=0}^{r} \mathfrak{U}_i^*$, it follows that $\operatorname{ord} \mathfrak{U}^* \leqslant r+2$. This completes the proof of Lemma 3.

Proof of the theorem. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_t\}$ be a finite open cover of X such that every open cover $\mathcal{H}$ refining $\mathfrak{U}$ has $\operatorname{ord} \mathcal{H} \geqslant n+1$. (Such a cover exists since $\dim X \geqslant n$.) There exists a closed cover $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ with $F_i \subset G_i$, $i = 1, 2, \ldots, t$. (This is true even for all normal spaces.) For each i ($1 \leqslant i \leqslant t$), let $f_i\colon X \to [0, 1]$ be continuous, and such that $f_i(x) = 1$ for $x \in F_i$, $f_i(x) = 0$ if $x \in X - G_i$ (Urysohn's Lemma). For $1 \leqslant i \leqslant t$ define $\sigma_i\colon X \times X \to$ Real Numbers by the formula

$$\sigma_i(x, y) = \varrho(x, y) + \sum_{j=1}^{i} |f_j(x) - f_j(y)| .$$

We prove that $\mu\dim(X, \sigma_t) \geqslant n$. For if $\mathfrak{U}$ is any open cover of X with σ_t-mesh $\mathfrak{U} < 1$, then $\mathfrak{U}$ refines $\mathcal{G}$, hence order $\mathfrak{U} \geqslant n+1$. To prove this, take $U \in \mathfrak{U}$ and $x \in U$. Since $\mathcal{F}$ is a cover of X, there exists i such that $x \in F_i$. Then

$$\sigma_t(x, X - G_i) \geqslant |f_i(x) - f_i(X - G_i)| = 1 - 0 = 1 , \quad \text{so} \quad U \subset G_i .$$

Setting $\sigma_0(x, y) = \varrho(x, y)$, note that

$$\sigma_{i+1}(x, y) = \sigma_i(x, y) + |f_{i+1}(x) - f_{i+1}(y)| ,$$

so by Lemma 3,

$$\mu\dim(X, \sigma_i) \leqslant \mu\dim(X, \sigma_{i+1}) \leqslant \mu\dim(X, \sigma_i) + 1 ,$$

and

$$\mu\dim(X, \sigma_t) \geqslant n .$$

Thus, starting with $\mu\dim(X, \sigma_0) = m$, the metric dimension goes up at most one when σ_i is replaced by σ_{i+1}, and $\mu\dim(X, \sigma_t) \geqslant n$. Thus all values k ($m \leqslant k \leqslant n$) are assumed, and the theorem is proved.

References

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