# Note on metric-dependent dimension functions

by

**Richard E. Hodel*** (Durham, N.C.)

**1. Introduction.** Let $(X, \varrho)$ be a metric space, let $\dim X$ be the covering dimension of $X$, and let $\mu\dim(X, \varrho)$ be the metric dimension of $X$. Let $d_2$, $d_3$, and $d_4$ denote the dimension functions for metric spaces introduced by Nagami and Roberts in [5]. A summary of the relation among these various dimension functions for $(X, \varrho)$ is as follows.

$$d_2(X, \varrho) \leqslant d_3(X, \varrho) \leqslant \mu\dim(X, \varrho) \leqslant d_4(X, \varrho) = \dim X \leqslant 2\mu\dim(X, \varrho) .$$

In this paper we continue the study of dimension functions for metric spaces which by their definition appear to depend upon the particular metric. In § 2 we introduce a new dimension function $d_5$ and show that for any metric space $(X, \varrho)$, $d_3(X, \varrho) \leqslant d_5(X, \varrho) \leqslant \mu\dim(X, \varrho)$ and if $X$ is separable $d_5(X, \varrho) = \mu\dim(X, \varrho)$. In § 4 we prove the relation $\dim X \leqslant 2d_3(X, \varrho)$. This result sharpens the inequality $\dim X \leqslant 2\mu\dim(X, \varrho)$ first obtained by Katětov in [2] and gives a partial solution to Problem 1 in [5]. Finally, in § 5 we give several characterizations of covering dimension for metric spaces.

**2. The dimension function $d_5$.** The reader is referred to Nagami and Roberts' paper [5] for definitions of the dimension functions $\mu\dim$, $d_2$, $d_3$, and $d_4$. The dimension function $d_5$, a "uniform" $d_4$ function, is defined as follows.

DEFINITION 2.1. Let $(X, \varrho)$ be a metric space. If $X = \emptyset$, then $d_5(X, \varrho) = -1$. Otherwise, $d_5(X, \varrho) \leqslant n$ if $(X, \varrho)$ satisfies this condition: given any countable number of pairs of closed sets $C_1, C_1'$; $C_2, C_2'$; ... such that for all $i$, $\varrho(C_i, C_i') \geqslant \delta > 0$, there exist open sets $W_1, W_2, \ldots$ such that

(1) $C_i \subset W_i \subset \overline{W}_i \subset (X - C_i')$, for all $i$.

(2) $\operatorname{ord}\{\overline{W}_i - W_i\colon i = 1, 2, \ldots\} \leqslant n$.

If $d_5(X, \varrho) \leqslant n$ is true and $d_5(X, \varrho) \leqslant n-1$ is false, then $d_5(X, \varrho) = n$.

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It is clear that for any metric space $(X, \varrho)$, $d_3(X, \varrho) \leqslant d_5(X, \varrho)$. Furthermore, the proof in [5] that $d_3(X, \varrho) \leqslant \mu\dim(X, \varrho)$ can be easily modified to show that $d_5(X, \varrho) \leqslant \mu\dim(X, \varrho)$. We thus have the following theorem.

THEOREM 2.1. *Let $(X, \varrho)$ be a metric space. Then $d_3(X, \varrho) \leqslant d_5(X, \varrho) \leqslant \mu\dim(X, \varrho)$.*

In [5] Nagami and Roberts prove that if $(X, \varrho)$ is a metric space with $\varrho$ a totally bounded metric then $d_3(X, \varrho) = \mu\dim(X, \varrho)$. We use a similar technique to prove that for separable metric spaces $d_5$ and $\mu\dim$ are equivalent. A major unsolved problem in the theory of metric-dependent dimension functions is the following. Is $d_3(X, \varrho) = \mu\dim(X, \varrho)$ for (separable) metric spaces?

LEMMA 2.2. *Let $(X, \varrho)$ be a separable metric space and let $\varepsilon > 0$ be given. Then there exist open collections $U_i$ and $V_i$, $i = 1, 2, \ldots$, satisfying these conditions.*

(1) *$U_i$, $i = 1, 2, \ldots$, covers $X$.*

(2) *$V_i$, $i = 1, 2, \ldots$, is locally finite.*

(3) *$\varrho(\overline{U}_i, X - V_i) \geqslant \frac{1}{8}\varepsilon$, for all $i$.*

(4) *$\operatorname{mesh}\{V_i:\ i = 1, 2, \ldots\} < \varepsilon$.*

Proof. Let $x_i$, $i = 1, 2, \ldots$, be a dense subset of $X$. For $i = 1, 2, \ldots$ let

$$A_i = \{p:\ \varrho(p, x_i) < 4\varepsilon/8\}, \qquad C_i = \{p:\ \varrho(p, x_i) < 6\varepsilon/8\},$$

$$B_i = \{p:\ \varrho(p, x_i) < 5\varepsilon/8\}, \qquad D_i = \{p:\ \varrho(p, x_i) < 7\varepsilon/8\}.$$

Finally, for $i = 1, 2, \ldots$, let $V_i = D_i - \bigcup_{j<i} \overline{A}_j$ and $U_i = C_i - \bigcup_{j<i} \overline{B}_j$. Clearly each $U_i$ and $V_i$ is open and $\operatorname{mesh}\{V_i:\ i = 1, 2, \ldots\} < \varepsilon$. It remains to prove these assertions.

ASSERTION 1. *The collection $U_i$, $i = 1, 2, \ldots$, covers $X$.*

Proof. Let $p$ be a point in $X$. Let $i$ be smallest integer such that $\varrho(p, x_i) < 3\varepsilon/4$. Then $p$ is in $C_i$ and $p$ is not in $\overline{B}_j$ for $j < i$. Hence $p$ is in $U_i$.

ASSERTION 2. *The collection $V_i$, $i = 1, 2, \ldots$, is locally finite.*

Proof. Let $p$ be a point in $X$, and let $i$ be such that $p$ is in $A_i$. Then $A_i$ is an open neighborhood of $p$ and for $j > i$ $A_i \cap V_j = \emptyset$.

ASSERTION 3. *For all $i$, $\varrho(\overline{U}_i, X - V_i) \geqslant \varepsilon/8$.*

Proof. Suppose that for some $i$ $\varrho(\overline{U}_i, X - V_i) < \varepsilon/8$. Let $p$ and $q$ be points of $\overline{U}_i$ and $(X - V_i)$ respectively such that $\varrho(p, q) < \varepsilon/8$. Now $\overline{U}_i$ is contained in $\overline{C}_i$, and so $p$ is in $\overline{C}_i$. Since $\varrho(C_i, X - D_i) \geqslant \varepsilon/8$, it follows that $q$ is not in $X - D_i$. Thus the point $q$ is in $D_i$ but not in $V_i$. Therefore there is a $j < i$ such that $q$ is in $\overline{A}_j$. Since $\varrho(p, q) < \varepsilon/8$, $p$ is in $\overline{B}_j$, which is a contradiction since $p$ is in $\overline{U}_i$.



THEOREM 2.3. *If $(X, \varrho)$ is a separable metric space, then $d_5(X, \varrho) = \mu\dim(X, \varrho)$.*

Proof. It suffices to show that $\mu\dim(X, \varrho) \leqslant d_5(X, \varrho)$. So let $d_5(X, \varrho) \leqslant n$ and let $\varepsilon > 0$ be given. We shall construct an open cover $\mathfrak{L}$ of $X$ such that $\operatorname{mesh}\mathfrak{L} < \varepsilon$ and $\operatorname{ord}\mathfrak{L} \leqslant n+1$. It then follows that $\mu\dim(X, \varrho) \leqslant n$.

For $\varepsilon > 0$ let $U_i$ and $V_i$, $i = 1, 2, \ldots$, be collections of open sets satisfying the conditions in Lemma 2.2. Since $\varrho(\overline{U}_i, X - V_i) \geqslant \varepsilon/8 > 0$ for all $i$, and since $d_5(X, \varrho) \leqslant n$, there exist open sets $W_i$, $i = 1, 2, \ldots$, such that

(1) $\overline{U}_i \subset W_i \subset \overline{W}_i \subset V_i$, for all $i$.

(2) $\operatorname{ord}\{\overline{W}_i - W_i:\ i = 1, 2, \ldots\} \leqslant n$.

Now $(\overline{W}_i - W_i)$ is contained in $V_i$, for all $i$, and $V_i$, $i = 1, 2, \ldots$, is locally finite so by a Theorem of Morita ([4], p. 17) there exist open sets $H_i$, $i = 1, 2, \ldots$, such that

(1) $(\overline{W}_i - W_i) \subset H_i \subset V_i$, for all $i$.

(2) $\operatorname{ord}\{H_i:\ i = 1, 2, \ldots\} \leqslant n$.

For $i = 1, 2, \ldots$, let $K_i = W_i - \bigcup_{j<i} \overline{W}_j$; note that each $K_i$ is an open set and for $i \neq l$, $K_i \cap K_l = \emptyset$. Finally, let

$$\mathfrak{L} = \{H_i:\ i = 1, 2, \ldots\} \cup \{K_i:\ i = 1, 2, \ldots\}.$$

Then $\mathfrak{L}$ is an open cover of $X$ such that $\operatorname{mesh}\mathfrak{L} < \varepsilon$ and $\operatorname{ord}\mathfrak{L} \leqslant n+1$.

**3. An important lemma.** The following lemma plays an important role in the proofs of Theorem 4.1 and Theorem 5.1.

LEMMA 3.1. *Let $(X, \varrho)$ be a metric space, and let $\mathcal{G} = \{G_1, \ldots, G_m\}$ be a finite open cover of $X$. Then there exist open collections*

$$\mathcal{U} = \{U_i^j:\ i = 1, \ldots, m;\ j = 1, 2, \ldots\} \quad \textit{and} \quad \mathcal{V} = \{V_i^j:\ i = 1, \ldots, m;\ j = 1, 2, \ldots\}$$

*satisfying these conditions.*

(1) *$\mathcal{U}$ covers $X$.*

(2) *$\mathcal{V}$ is locally finite and refines $\mathcal{G}$.*

(3) *$\varrho(\overline{U}_i^j, X - V_i^j) \geqslant 1/2^{2j+2}$, for all $i, j$.*

(4) *Let $j = 1, 2, \ldots$, $k > j+1$, $1 \leqslant i \leqslant m$, and $1 \leqslant l \leqslant m$. Then $V_i^j \cap V_l^k = \emptyset$.*

Proof. For $i = 1, \ldots, m$, $j = 1, 2, \ldots$, let

$$C_i^j = \{p:\ \varrho(p, X - G_i) > 1/2^j\}, \qquad D_i^j = \{p:\ \varrho(p, X - G_i) > 3/2^{j+2}\}.$$

Put

$$U_i^j = C_i^{2j} - \bigcup_{k=1}^{m} \overline{D}_k^{2j-3} \quad \text{and} \quad V_i^j = D_i^{2j} - \bigcup_{k=1}^{m} \overline{C}_k^{2j-3} \quad (D_k^{-1} = C_k^{-1} = \emptyset).$$

Finally, put $\mathcal{U} = \{U_i^j\colon i = 1, \ldots, m; j = 1, 2, \ldots\}$ and $\mathcal{V} = \{V_i^j\colon i = 1, \ldots, m; j = 1, 2, \ldots\}$. It is clear that $\mathcal{U}$ and $\mathcal{V}$ are open collections and that $\mathcal{V}$ refines $\mathcal{G}$. Moreover, condition (4) implies that $\mathcal{V}$ is a star finite collection, and so $\mathcal{V}$ is certainly locally finite. The proof is complete if we prove the following assertions.

Assertion 1. *$\mathcal{U}$ covers $X$.*

Proof. Let $p$ be a point in $X$. Pick $i$, $1 \leqslant i \leqslant m$, such that $\varrho(p, X - G_i) \geqslant \varrho(p, X - G_k)$, $k \neq i$. Since $\mathcal{G}$ covers $X$, $\varrho(p, X - G_i) > 0$. Pick the smallest integer $j$ such that $p$ is in $C_i^{2j}$. If $j = 1$, then $p$ is in $U_i^j$ and we are finished. If $j > 1$, then $p$ is not in $C_i^{2j-2}$ and so $\varrho(p, X - G_i) \leqslant 1/2^{2j-2}$. We now show that $p$ is in $U_i^j$. Since $\overline{D}_i^{2j-3}$ is contained in $C_i^{2j-2}$, $p$ is not in $\overline{D}_i^{2j-3}$. Suppose, however, that for some $k \neq i$ $p$ is in $\overline{D}_k^{2j-3}$. Then $\varrho(p, X - G_k) \geqslant 3/2^{2j-1}$, and so $\varrho(p, X - G_k) > \varrho(p, X - G_i)$, a contradiction of the choice of $i$. Hence we conclude that $p$ is in $U_i^j$.

Assertion 2. *For $i = 1, \ldots, m$, $j = 1, 2, \ldots$, $\varrho(\overline{U}_i^j, X - V_i^j) \geqslant 1/2^{2j+2}$.*

Proof. Suppose that for some $i, j$ we have $\varrho(\overline{U}_i^j, X - V_i^j) < 1/2^{2j+2}$. Let $p$ and $q$ be points in $U_i^j$ and $X - V_i^j$ respectively such that $\varrho(p, q) < 1/2^{2j+2}$. Now $U_i^j$ is contained in $C_i^{2j}$, and so $p$ is in $C_i^{2j}$. Since $\varrho(C_i^{2j}, X - D_i^{2j}) \geqslant 1/2^{2j+2}$, it follows that $q$ is not in $X - D_i^{2j}$. Thus the point $q$ is in $D_i^{2j}$ but not in $V_i^j$. Therefore there is a $k$, $1 \leqslant k \leqslant m$, such that $q$ is in $\overline{C}_k^{2j-3}$. Since $\varrho(p, q) < 1/2^{2j+2}$, it easily follows that $p$ is in $\overline{D}_k^{2j-3}$, a contradiction since $p$ is in $U_i^j$.

Assertion 3. *Let $j = 1, 2, \ldots$, $k > j+1$, $1 \leqslant i \leqslant m$, and $1 \leqslant l \leqslant m$. Then $V_i^j \cap V_l^k = \emptyset$.*

Proof. Suppose that the point $p$ is in $V_i^j$ and $V_l^k$, where $k > j+1$. Since $p$ is in $V_l^k$, $p$ is not in $\overline{C}_i^{2k-3}$. But if $p$ is in $V_i^j$, then $p$ is in $D_i^{2j}$, and since $D_i^{2j}$ is contained in $\overline{C}_i^{2k-3}$, it follows that $p$ is in $\overline{C}_i^{2k-3}$, a contradiction.

**4. The relation** $\dim X \leqslant 2 \cdot d_3(X, \varrho)$. In [5] Nagami and Roberts pose the following question. Is it true that for any metric space $(X, \varrho)$, $\dim X \leqslant 2 \cdot d_2(X, \varrho)$? In this section we prove that $\dim X \leqslant 2 \cdot d_3(X, \varrho)$, thus generalizing Katětov's result [2] that $\dim X \leqslant 2\mu\dim(X, \varrho)$. The inequality $\dim X \leqslant 2 \cdot d_2(X, \varrho)$ seems quite difficult. In fact, the following seems to be unknown. If $d_2(X, \varrho) = n < \infty$, is $\dim X$ finite?

Theorem 4.1. *Let $(X, \varrho)$ be a metric space. Then $\dim X \leqslant 2 \cdot d_3(X, \varrho)$.*

Proof. Let $d_3(X, \varrho) \leqslant n$ and let $\mathcal{G} = \{G_1, \ldots, G_m\}$ be a finite open cover of $X$. We shall construct an open refinement $\mathcal{L}$ of $\mathcal{G}$ such that $\operatorname{ord}\mathcal{L} \leqslant 2n+1$. It then follows that $\dim X \leqslant 2n$.

Given the open cover $\mathcal{G}$, let $\mathcal{U} = \{U_i^j\colon i = 1, \ldots, m;\ j = 1, 2, \ldots\}$ and $\mathcal{V} = \{V_i^j\colon i = 1, \ldots, m;\ j = 1, 2, \ldots\}$ be open collections satisfying the conditions of Lemma 3.1. Fix $j$, and consider the collection $\{\overline{U}_i^j, X - V_i^j\colon i = 1, \ldots, m\}$. Since $\varrho(\overline{U}_i^j, X - V_i^j) > 0$, $i = 1, \ldots, m$, and $d_3(X, \varrho) \leqslant n$, there is an open collection $\mathcal{W}_j = \{W_i^j\colon i = 1, \ldots, m\}$ such that

(1) $\overline{U}_i^j \subset W_i^j \subset \overline{W}_i^j \subset V_i^j$, $i = 1, \ldots, m$.

(2) $\operatorname{ord}\{\overline{W}_i^j - W_i^j\colon i = 1, \ldots, m\} \leqslant n$.



By a Theorem of Morita ([4], p. 17) there is an open collection $\mathcal{H}_j = \{H_i^j\colon i = 1, \ldots, m\}$ such that

(1) $(\overline{W}_i^j - W_i^j) \subset H_i^j \subset V_i^j$, $i = 1, \ldots, m$.

(2) $\operatorname{ord}\mathcal{H}_j \leqslant n$.

Now let $\mathcal{H} = \bigcup_{j=1}^{\infty} \mathcal{H}_j$; then $\operatorname{ord}\mathcal{H} \leqslant 2 \cdot n$. For, let $p$ be a point of $X$ which is covered by $\mathcal{H}$ and let $j_0$ be the smallest integer such that $p$ is covered by $\mathcal{H}_{j_0}$. By condition (4) of Lemma 3.1, $p$ is not covered by $\mathcal{H}_j$ for $j > j_0+1$. Since $\operatorname{ord}\mathcal{H}_{j_0} \leqslant n$ and $\operatorname{ord}\mathcal{H}_{j_0+1} \leqslant n$, it follows that $p$ is contained in at most $2 \cdot n$ elements of $\mathcal{H}$.

Let $\mathcal{K} = \bigwedge_{\substack{i=1,\ldots,m \\ j=1,2,\ldots}} \{W_i^j, X - \overline{W}_i^j\}$; $\mathcal{K}$ is a mutually disjoint collection of open sets (see [4], p. 17). Finally, let $\mathcal{L} = \mathcal{H} \cup \mathcal{K}$; then $\mathcal{L}$ is an open refinement of $\mathcal{G}$ and $\operatorname{ord}\mathcal{L} \leqslant 2n+1$.

**5. Characterizations of covering dimension.** Consider the following conditions for a metric space $(X, \varrho)$.

$(\alpha_n)$: Given any countable locally finite closed collection $F_1, F_2, \ldots$ and any open collection $V_1, V_2, \ldots$ such that for all $i$, $\varrho(F_i, X - V_i) > 0$, there is an open collection $W_1, W_2, \ldots$ such that

(1) $F_i \subset W_i \subset \overline{W}_i \subset V_i$, for all $i$.

(2) $\operatorname{ord}\{\overline{W}_i - W_i\colon i = 1, 2, \ldots\} \leqslant n$.

$(\beta_n)$ Given any locally finite closed collection $\{F_\alpha\colon \alpha \text{ in } A\}$ and any open collection $\{V_\alpha\colon \alpha \text{ in } A\}$ such that for all $\alpha$ in $A$, $\varrho(F_\alpha, X - V_\alpha) > 0$, there is an open collection $\{W_\alpha\colon \alpha \text{ in } A\}$ such that

(1) $F_\alpha \subset W_\alpha \subset \overline{W}_\alpha \subset V_\alpha$, for all $\alpha$.

(2) $\operatorname{ord}\{\overline{W}_\alpha - W_\alpha\colon \alpha \text{ in } A\} \leqslant n$.

$(\gamma_n)$: Given any closed collection $\{F_\alpha\colon \alpha \text{ in } \bigcup_{i=1}^{\infty} A_i\}$, where for each $i$, $\{F_\alpha\colon \alpha \text{ in } A_i\}$ is locally finite, and any open collection $\{V_\alpha\colon \alpha \text{ in } \bigcup_{i=1}^{\infty} A_i\}$ such that for all $\alpha$ in $\bigcup_{i=1}^{\infty} A_i$, $\varrho(F_\alpha, X - V_\alpha) > 0$, there is an open collection $\{W_\alpha\colon \alpha \text{ in } \bigcup_{i=1}^{\infty} A_i\}$ such that

(1) $F_\alpha \subset W_\alpha \subset \overline{W}_\alpha \subset V_\alpha$ for all $\alpha$ in $\bigcup_{i=1}^{\infty} A_i$.

(2) $\operatorname{ord}\{\overline{W}_\alpha - W_\alpha\colon \alpha \text{ in } \bigcup_{i=1}^{\infty} A_i\} \leqslant n$.

Each of these conditions is a possible candidate for a dimension function which would appear to be metric dependent. We shall now show, however, that each of these conditions characterizes covering dimension in metric spaces.

THEOREM 5.1. *The following are equivalent in a metric space* $(X, \varrho)$.

(1) $\dim X \leqslant n$.

(2) $(X, \varrho)$ *satisfies* $(\alpha_n)$.

(3) $(X, \varrho)$ *satisfies* $(\beta_n)$.

(4) $(X, \varrho)$ *satisfies* $(\gamma_n)$.

Proof. It is clear that $(4)\to(3)\to(2)$. The proof that $(2)\to(1)$ is similar to the proof of Theorem 4.1 and is outlined as follows. Let $\mathcal{G} = \{G_1, \dots, G_m\}$ be a finite open cover of $X$; we shall construct an open refinement $\mathcal{L}$ of $\mathcal{G}$ such that $\operatorname{ord}\mathcal{L} \leqslant n+1$. Given the open cover $\mathcal{G}$, let $\mathcal{U} = \{U_i^j: i = 1, \dots, m;\ j = 1, 2, \dots\}$ and $\mathcal{V} = \{V_i^j: i = 1, \dots, m;\ j = 1, 2, \dots\}$ be open collections satisfying the conditions of Lemma 3.1. Since $X$ satisfies $(\alpha_n)$, there is an open collection $\mathcal{W} = \{W_i^j: i = 1, \dots, m;\ j = 1, 2, \dots\}$ such that

(1) $\overline{U}_i^j \subset W_i^j \subset \overline{W}_i^j \subset V_i^j$, $i = 1, \dots, m;\ j = 1, 2, \dots$

(2) $\operatorname{ord}\{\overline{W}_i^j - W_i^j: i = 1, \dots, m;\ j = 1, 2, \dots\} \leqslant n$.

By a Theorem of Morita ([4], p. 17) there is an open collection $\mathcal{H} = \{H_i^j: i = 1, \dots, m;\ j = 1, 2, \dots\}$ such that

(1) $(\overline{W}_i^j - W_i^j) \subset H_i^j \subset V_i^j$, $i = 1, \dots, m;\ j = 1, 2, \dots$

(2) $\operatorname{ord}\mathcal{H} \leqslant n$.

Let $\mathcal{K} = \bigwedge_{\substack{i=1,\dots,m\\ j=1,2,\dots}} \{W_i^j, X - \overline{W}_i^j\}$, and let $\mathcal{L} = \mathcal{H} \cup \mathcal{K}$; then $\mathcal{L}$ is an open refinement of $\mathcal{G}$ and $\operatorname{ord}\mathcal{L} \leqslant n+1$.

It remains to prove $(1)\to(4)$. So let $\dim X \leqslant n$, let $\{F_a: a \text{ in } \bigcup_{i=1}^{\infty} A_i\}$ be a closed collection in $X$ such that for all $i$, $\{F_a: a \text{ in } A_i\}$ is locally finite, and let $\{V_a: a \text{ in } \bigcup_{i=1}^{\infty} A_i\}$ be an open collection such that for all $a$ in $\bigcup_{i=1}^{\infty} A_i$, $\varrho(F_a, X - V_a) > 0$. Given the locally finite closed collection $\{F_a: a \text{ in } A_i\}$, there is, by a Lemma due to Morita ([4], p. 22), a locally finite open collection $\{U_a: a \text{ in } A_i\}$ such that for all $a$ in $A_i$, $F_a$ is contained in $U_a$. For each $a$ in $\bigcup_{i=1}^{\infty} A_i$, let $G_a = U_a \cap V_a$. Now apply [6], p. 25, to the collections $\{F_a: a \text{ in } \bigcup_{i=1}^{\infty} A_i\}$ and $\{G_a: a \text{ in } \bigcup_{i=1}^{\infty} A_i\}$. It easily follows that $(X, \varrho)$ satisfies $(\gamma_n)$.

COROLLARY 5.2. (Nagami–Roberts.) *For any metric space* $(X, \varrho)$, $\dim X = d_4(X, \varrho)$.



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DUKE UNIVERSITY
Durham, N.C.; USA

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