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Reçu par la Rédaction 18. 7. 1966

The hypothesis $2^{\aleph_0} \leqslant \aleph_n$ and ambiguous points of planar functions

by

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Let P be the set of all points in the Euclidean plane provided with a Cartesian coordinate system having a horizontal x-axis and a vertical y-axis. By a line with direction θ we shall mean a straight line in the plane P whose angle of inclination is θ, where $0 \leqslant \theta < \pi$. Suppose that n is a natural number and that $0 \leqslant \theta_1 < \theta_2 < \ldots < \theta_n < \pi$ (¹). We define the relation

$$P = E_1(\theta_1; K_1) \cup E_2(\theta_2; K_2) \cup \ldots \cup E_n(\theta_n; K_n)$$

to mean that P is the union of n sets, $E_1, E_2, \ldots, E_n$, where E_j $(j = 1, 2, \ldots, n)$ intersects every line with direction θ_j in a subset of that line satisfying the condition K_j. In this paper, K_j will take one of the following forms: (i) $< \aleph_a$, (ii) $\leqslant \aleph_a$, (iii) n.d., where $E_j(\theta_j; K_j)$ then means, respectively, that E_j intersects every line with direction θ_j in a set of power less than $\aleph_a$, in a set of power less than or equal to $\aleph_a$, in a linear nowhere dense set of points.

We shall be concerned with the following specific propositions:

(H_n) $$2^{\aleph_0} \leqslant \aleph_n ,$$

(Q_n) $$P = E_1(\theta_1; < \aleph_0) \cup E_2(\theta_2; < \aleph_0) \cup E_3(\theta_3; < \aleph_0) \cup \ldots \cup E_{n+2}(\theta_{n+2}; < \aleph_0) ,$$

(B_n) $$P = E_1(\theta_1; \text{n.d.}) \cup E_2(\theta_2; \leqslant \aleph_0) \cup E_3(\theta_3; \leqslant \aleph_1) \cup \ldots \cup E_{n+2}(\theta_{n+2}; \leqslant \aleph_n) .$$

It is evident that $(\mathrm{Q}_n) \Rightarrow (\mathrm{B}_n)$. I showed [1] that $(\mathrm{B}_1) \Rightarrow (\mathrm{H}_1)$, and Davies showed [4] that $(\mathrm{H}_1) \Rightarrow (\mathrm{Q}_1)$. Subsequently Davies proved [5] that $(\mathrm{H}_n) \Rightarrow (\mathrm{Q}_n)$ and $(\mathrm{Q}_n) \Rightarrow (\mathrm{H}_n)$ for every n.

I shall prove that $(\mathrm{B}_n) \Rightarrow (\mathrm{H}_n)$ for every n, and I shall then apply this result to show that the existence of a function with a certain kind of ambiguous behavior (this term will be defined in the next paragraph) implies (H_n) (whereas the result $(\mathrm{Q}_n) \Rightarrow (\mathrm{H}_n)$ is insufficient to show this).

Let $\zeta \in P$. By a *segment* Λ at ζ we mean a rectilinear segment extending from a point $\zeta' \in P$, with $\zeta' \neq \zeta$, to the point ζ; Λ is regarded

(¹) What is essential here is not that the thetas be in this particular order, but that they be distinct.

as containing ζ' but not ζ. Suppose that $f(z)$ is an arbitrary single-valued complex-valued function of $z \in P$. If Λ is a segment at ζ, then the *cluster set* of f at ζ along Λ, denoted by $C_\Lambda(f, \zeta)$, is defined to be the set of all points ω on the Riemann sphere with the property that, for some sequence of points $\{z_n\}$ on Λ for which $\lim_{n\to\infty} z_n = \zeta$, we have $\lim_{n\to\infty} f(z_n) = \omega$. We say that a point $\zeta \in P$ is an *ambiguous point* of f (more precisely, a *rectilinearly oppositely ambiguous point* of f; see [2]), provided that there exist collinear segments Λ_1 and Λ_2 at ζ with $\Lambda_1 \cap \Lambda_2 = \emptyset$ such that

$$C_{\Lambda_1}(f, \zeta) \cap C_{\Lambda_2}(f, \zeta) = \emptyset .$$

If the line through ζ that contains Λ_1 and Λ_2 has direction θ, then θ is called a *corresponding direction of ambiguity* of f at ζ.

Let $0 \leqslant \theta_1 < \theta_2 < \theta_3 < \pi$. It was shown in [2], Theorem 11, that (H_1) implies the existence of a function f whose range is an at most enumerable set, such that every point of P is an ambiguous point of f with θ_1 or θ_2 or θ_3 as direction of ambiguity. This result was improved in [3], Theorem 7, by showing that the number of points in the range of f can be reduced to four; it cannot, however, be reduced to three ([3], Theorem 8). Conversely ([3], Theorem 9), the existence of a function f whose range is an at most enumerable set, such that every point of P is an ambiguous point of f with θ_1 or θ_2 or θ_3 as direction of ambiguity, implies (H_1). These results will be generalized in the present paper.

We begin by proving

THEOREM 1. *Let n be a natural number, and suppose that $0 \leqslant \theta_1 < \theta_2 < \ldots < \theta_{n+2} < \pi$. Then $(B_n) \Rightarrow (H_n)$.*

Proof. The argument we use is a modification of that given by Davies ([5], pp. 277-278) to prove that $(Q_n) \Rightarrow (H_n)$. Assume to the contrary that (B_n) is true and (H_n) is false. Then

$$1 < \aleph_0 < \aleph_1 < \ldots < \aleph_{n+1} \leqslant 2^{\aleph_0} .$$

We define $n+3$ subsets $C_0, C_1, C_2, \ldots, C_{n+2}$ of P as follows. Let C_0 consist of some single point $\zeta \in P$. Let C_1 be an enumerable everywhere dense subset of the line with direction θ_1 that contains ζ. For $j = 2, \ldots, n+2$, let C_j be the union of $\aleph_{j-1}$ disjoint sets, each of which is a translation of C_{j-1} in the direction θ_j, where C_j lies on $\aleph_{j-2}$ lines with direction θ_j. That these sets exist is evident from the analysis given by Davies. Thus we have

$$|C_0| = 1, \quad |C_1| = \aleph_0, \quad |C_2| = \aleph_1, \quad \ldots, \quad |C_{n+2}| = \aleph_{n+1} .$$

Now the set C_{n+2} lies on $\aleph_n$ lines with direction θ_{n+2}. Each of these lines, according to (B_n), contains at most $\aleph_n$ points of the set E_{n+2}. Consequently, these $\aleph_n$ lines contain altogether at most $\aleph_n^2 = \aleph_n$ points of E_{n+2}.

But C_{n+2} is the union of $\aleph_{n+1}$ disjoint congruent sets, and if each of these sets contained a point of E_{n+2}, we should have at least $\aleph_{n+1}$ points of E_{n+2} on the $\aleph_n$ lines in question, which is impossible. Hence, at least one of the $\aleph_{n+1}$ disjoint sets congruent to C_{n+1}, call it C'_{n+1}, contains no point of E_{n+2}. Pursuing a similar argument with the set C'_{n+1} now, we arrive at the conclusion that one of the $\aleph_n$ disjoint sets congruent to C_n of which C'_{n+1} is the union, call it C'_n, contains no point of E_{n+1}. Continuing in this manner, we obtain a descending sequence of sets

$$C'_{n+1} \supset C'_n \supset \ldots \supset C'_1 ,$$

where C'_j is congruent to C_j and contains no point of E_{j+1} $(j = 1, 2, \ldots, n+1)$. Consider, finally, the set C'_1. Since it is congruent by translation to C_1, the set C'_1 is an enumerable everywhere dense subset of some line with direction θ_1. But by (B_n), this line intersects E_1 in a linear nowhere dense set of points, and hence C'_1 contains a point ζ' that does not belong to E_1. The point ζ', however, does not belong to any of the sets E_j $(j = 2, 3, \ldots, n+2)$ either, which contradicts (B_n). Our assumption is therefore untenable, and the theorem is proved.

We are now in a position to establish

THEOREM 2. *Let n be a natural number, and suppose that $0 \leqslant \theta_1 < \theta_2 < \ldots < \theta_{n+2} < \pi$. Then the existence of a function f with an at most enumerable range such that every point of P is an ambiguous point of f with θ_1 or θ_2 or ... or θ_{n+2} as direction of ambiguity, implies (H_n).*

Proof. Assume that a function f possessing the indicated property exists. Define E_j $(j = 1, 2, \ldots, n+2)$ to be the set of all those points of P that are ambiguous points of f with θ_j as direction of ambiguity. Then $P = \bigcup_{j=1}^{n+2} E_j$. According to [3], Theorem 3, E_1 intersects every line with direction θ_1 in a linear nowhere dense set of points, and by [2], Theorem 7, E_j $(j = 2, \ldots, n+2)$ intersects every line with direction θ_j in an at most enumerable set. Hence (B_n) is true, and this, according to Theorem 1, implies (H_n).

THEOREM 3. *Let n be a natural number, and suppose that $0 \leqslant \theta_1 < \theta_2 < \ldots < \theta_{n+2} < \pi$. Then (H_n) implies the existence of a function f whose range is a set of at most $2^{n+2}-1$ values, such that every point of P is an ambiguous point of f with θ_1 or θ_2 or ... or θ_{n+2} as direction of ambiguity.*

Proof. Assume (H_n) to be true. According to Davies ([5], p. 278), P is the union of $n+2$ mutually exclusive sets E_j $(j = 1, 2, \ldots, n+2)$ such that every line with direction θ_j intersects E_j in only a finite number of points. If L is a straight line in the plane and (x_0, y_0) is a point of L, we say that a point (x, y) of L different from (x_0, y_0) precedes (x_0, y_0) if $y < y_0$ or both $y = y_0$ and $x < x_0$, but succeeds (x_0, y_0) otherwise. If

$z_0 \in L$, and if $z \in L$ precedes z_0, we write $z \prec z_0$, whereas if z succeeds z_0, we write $z \succ z_0$.

We first define $n+2$ functions $f_j(z)$ $(j = 1, 2, \ldots, n+2)$ for $z \in P$ as follows. Given $z \in P$, denote by $L^{(j)}$ the unique line with direction θ_j that contains the point z, and let E_k be the unique one of the aforementioned $n+2$ sets that contains z. First of all, we put $f_k(z) = 0$. Next, let $\{z_1^{(j)}, z_2^{(j)}, \ldots, z_{m(L^{(j)})}^{(j)}\}$ be the (possibly empty) set of finitely many points of E_j that lie on $L^{(j)}$ $(j = 1, 2, \ldots, n+2;\ j \neq k)$, where

$$z_1^{(j)} \prec z_2^{(j)} \prec \ldots \prec z_{m(L^{(j)})}^{(j)} .$$

If this set is empty, put $f_j(z) = 0$. If this set is not empty, we continue in the following way. If $z \prec z_1^{(j)}$, put $f_j(z) = 0$. If $z \succ z_{m(L^{(j)})}^{(j)}$, put $f(z) = 0$ or $2/3^j$ according as $m(L^{(j)})$ is even or odd. If neither of these two relations holds, then since $z \notin E_j$, there exists a unique natural number r such that $z_r^{(j)} \prec z \prec z_{r+1}^{(j)}$. Put $f_j(z) = 0$ or $2/3^j$ according as r is even or odd. This completes the definition of the functions $f_j(z)$ $(j = 1, 2, \ldots, n+2)$. Now define

$$f(z) = \sum_{j=1}^{n+2} f_j(z) \qquad (z \in P) .$$

It is easy to verify that the range of f is a set of at most $2^{n+2}-1$ values, since every value of f is of the form

$$\sum_{j=1}^{n+2} \frac{t_j}{3^j} ,$$

where each t_j is either 0 or 2 and at least one t_j is 0. Suppose finally that $\zeta \in P$. Then there is a unique natural number j_0 $(1 \leqslant j_0 \leqslant n+2)$ such that $\zeta \in E_{j_0}$, as well as a unique line $L^{(j_0)}$ with direction θ_{j_0} such that $\zeta \in L^{(j_0)}$. If z and z' are points of $L^{(j_0)}$ on opposite sides of ζ and sufficiently close to ζ, then the distance between the points $f(z)$ and $f(z')$ is at least $1/3^{j_0}$, so that ζ is an ambiguous point of f with θ_{j_0} as direction of ambiguity, and the proof of the theorem is complete.

By combining Theorems 2 and 3, we obtain the following

COROLLARY. *If n is a natural number and $0 \leqslant \theta_1 < \theta_2 < \ldots < \theta_{n+2} < \pi$, then (H_n) is equivalent to the existence of a function f whose range is a set of at most $2^{n+2}-1$ values, such that every point of P is an ambiguous point of f with θ_1 or θ_2 or ... or θ_{n+2} as direction of ambiguity.*

Remark. In connection with Theorem 3, let $\varrho(n)$ be the smallest possible number of values in the range of a function f with the indicated ambiguous behavior whose existence is implied by (H_n). It has been shown in [3], Theorems 7 and 8, that $\varrho(1) = 4$, whereas the value of the expression $2^{n+2}-1$ appearing in Theorem 3 is 7 for $n = 1$. It is very likely, therefore, that $\varrho(n) < 2^{n+2}-1$ for $n > 1$. The proof that $\varrho(1) = 4$ might lead one to conjecture that $\varrho(n) = 2^{n+1}$ for every natural number n, but this remains an open problem.

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Reçu par la Rédaction le 25. 7. 1966