this decomposition is core with respect to satisfying McAuley's condition K_3 ([10], p. 9), but is not core (or atomic) with respect to giving an aposyndetic hyperspace, nor is it core with respect to having T-closed elements.

References

[1] H. S. Davis, D. P. Stadtlander and P. M. Swingle, *Semigroups, continua and the set functions T^n*, Duke Math. Jour. 29 (1962), pp. 265-280.

[2] — *Properties of the set function T^n*, Portugal. Math. 21 (1962), pp. 113-133.

[3] H. S. Davis and P. M. Swingle, *The iteration and recursion of set functions*, Portugal. Math. 25 (1964), pp. 103-129.

[4] P. C. Hammer, *Extended topology: Set-valued set-functions*, Nieuw Archief voor Wiskunde (3), X, (1962), pp. 55-77.

[5] F. Hausdorff, *Mengenlehre*, Berlin 1927.

[6] F. B. Jones, *Aposyndetic continua and certain boundary problems*, Amer. Jour. Math. 67 (1941), pp. 545-553.

[7] C. Kuratowski, *Théorie des continus irréductibles entre deux points* II, Fund. Math. 10 (1927), pp. 225-275.

[8] — *Les décompositions semi-continues d'espaces métriques compacts*, Fund. Math. 11 (1928), pp. 169-185.

[9] — *Topologie, II*, Warszawa 1961.

[10] L. F. McAuley, *An atomic decomposition of continua into aposyndetic continua*, Trans. Amer. Math. Soc. 88 (1958), pp. 1-11.

[11] R. L. Moore, *Foundations of point set theory*, Amer. Math. Soc. Colloquium Publications 13 (1932).

[12] G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloquium Publications 28 (1942).

[13] — *Semi-locally-connected sets*, Amer. J. Math. 61 (1939), p. 738.

[14] R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloquium Publications 32 (1949).

[15] W. A. Wilson, *On upper semi-continuous decompositions of compact continua*, Amer. Jour. Math. 54 (1932), pp. 377-386.

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Spaces in which sequences suffice II

by

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4. Introduction. In this paper we continue the work begun in [6] presenting some new facts on sequential and Fréchet spaces (Sections 5 and 6) and paying particular attention to those sequential spaces which are not Fréchet spaces (Section 7).

5. Sequential spaces II. The category of sequential spaces fails to have two important permanence properties; it is neither hereditary ([6], Example 1.8) nor productive ([6], Example 1.11). There is another example of a non-sequential subspace of a sequential space (due essentially to Arens [1]) which plays a critical role in what follows.

5.1. Example. *There is a countable, normal sequential space M with a non-sequential subspace.*

Proof. Let $M = (\boldsymbol{N}\times\boldsymbol{N}) \cup \boldsymbol{N} \cup \{0\}$ with each $(m, n) \in \boldsymbol{N}\times\boldsymbol{N}$ an isolated point, where $\boldsymbol{N}$ denotes the set of natural numbers. For a basis of neighborhoods at $n_0 \in \boldsymbol{N}$, take all sets of the form $\{n_0\} \cup \{(m, n_0) \mid m \geqslant m_0\}$. U will be a neighborhood of 0 if and only if $0 \in U$ and U is a neighborhood of all but finitely many $n \in \boldsymbol{N}$. One verifies routinely that M is normal and sequential. We shall show that $\{0\}$ is sequentially open but not open in $M\setminus\boldsymbol{N}$.

Since $0 \in \mathrm{cl}_M(\boldsymbol{N}\times\boldsymbol{N})$, $\{0\}$ is not open in $M\setminus\boldsymbol{N}$. If $\{(m_i, n_i)\}$ is any sequence in $\boldsymbol{N}\times\boldsymbol{N}$, either there is some $n_0 \in \boldsymbol{N}$ such that $n_i = n_0$ for infinitely many i, or there is no such n_0. In the first case, $\{(m_i, n_i)\}$ has a cluster point in the set $\{n_0\} \cup \{(m, n_0) \mid m \in \boldsymbol{N}\}$ and hence does not converge to 0. In the second case one easily constructs a neighborhood of 0 disjoint from $\{(m_i, n_i)\}$.

We are left by these examples with the problem of characterizing those subspaces of a sequential space X which are themselves sequential. Such a characterization can be effected in terms of X as a quotient (under the quotient map φ_X) of X^*, the topological sum of its convergent sequences (see 1.12, [6]) as follows.

5.2. Proposition. *A subspace Y of a sequential space X is sequential iff $\varphi_X|\overset{-1}{\varphi_X}(Y)$ is a quotient map.*

Proof. Let $Y_1 = \varphi_X^{-1}(Y)$ and $\varphi_1 = \varphi_X|Y_1$. Denote by σY the set Y topologized by the *sequential closure* of the relative topology from X (i.e. all sequentially open sets are open), and let $\varphi_Y\colon Y^* \to Y$ be the map which, with Y replaced by σY, becomes the quotient map 1.12, [6] (σY is a sequential space).

If $\varphi_1^{-1}(U)$ is open in Y_1, where $U \subseteq Y$, then $\varphi_Y^{-1}(U) = \varphi_1^{-1}(U) \cap Y^*$ is open in Y^*. Conversely, we have $\varphi_1^{-1}(U) = \varphi_X^{-1}(U) = \bigcup \{\varphi_X^{-1}(U \cap S| \ S$ is a convergent sequence in $X\}$, and if $\varphi_X^{-1}(U)$ is open in Y^*, then each $U \cap S$ is open in S, and so $\varphi_1^{-1}(U)$ is open in X^*. Thus φ_1 is a quotient map iff φ_Y is, and 5.2 follows from 1.2 and 1.12, [6].

Sequential subspaces will be used in Section 6 to characterize Fréchet spaces.

T. K. Boehme, in [3], asked whether or not a sequential space with unique sequential limits need be Hausdorff. Several examples showing that they need not have been given (see Fréchet [8] (1), Dudley [5], Franklin [7]). The following example, which is both countable and compact was discovered independently by A. Arhangel'skiĭ and the author.

5.3. Example. *There is a countable, compact, sequential space M_1 with unique sequential limits which is not Hausdorff.*

Proof. Let p be some point not in M (see 5.1) and let $M_1 = M \cup \{p\}$ with M open in M_1 and where basic neighborhoods of p are of the form $\{p\} \cup \big((N\times N)\setminus F\big)$ with F the union of the ranges of a finite number of convergent sequences in $N\times N$ (2). Clearly M_1 is countable and, since 0 and p have no disjoint neighborhoods, is not Hausdorff. Since M is Hausdorff and a convergent sequence in M cannot also converge to p, M_1 has unique sequential limits. The compactness of M_1 follows immediately from the covering definition. Every sequentially open subset of M is open in M_1. If S is sequentially open in M_1 and $p \in S$, clearly S is a neighborhood of each $x \in S$ different from p, since $S\setminus\{p\}$ is sequentially open in M. Since any subset $\{(m, n)| \ m \in A \subseteq N, n \in B \subseteq N\}$ contains a sequence converging to p whenever B is infinite, $X\setminus S$ must contain points of $N\times N$ having only finitely many second indices. Hence S contains a basic neighborhood of p and is therefore open in M_1. Thus M_1 is sequential.

After a preliminary result we shall see that the situation described in Example 5.3 cannot occur in the presence of suitable compactness conditions.

5.4. Proposition. *A sequential space has unique sequential limits iff each countably compact subset is closed (and hence sequential).*

(1) The author is indebted to C. E. Aull for this reference.
(2) This idea was gleaned from H. F. Cullen [4].

Proof. If $\{x_n\}$ is a sequence converging to two distinct points x and x', then $\{x\} \cup \{x_n| \ n \in N\}$ is a non-closed compact set. Conversely, if X is sequential and has unique sequential limits, X is T_1. Let $K \subseteq X$ be countably compact. If $\{x_n\}$ is a sequence in K and $\{x_n\} \to x_0$ then $\{x_0\} \cup \cup \{x_n| \ n \in N\}$ is sequentially closed and hence closed. Thus x_0 is the only possible accumulation point of $\{x_n| \ n \in N\}$. If $\{x_n| \ n \in N\}$ is infinite, $x_0 \in K$. If not for some n, $x_0 = x_n \in K$. Hence K is closed.

5.5. Corollary (Aull). *A sequential space has unique sequential limits iff each sequentially compact subset is closed.*

Proof. If K is a sequentially compact subset of a sequential space X with unique sequential limits, then K is countably compact and, by 5.4, closed. To converse is clear.

5.6. Proposition. *A locally countably compact (locally sequential compact, locally compact) sequential space X with unique sequential limits is Hausdorff.*

Proof. Each countably compact subset of X is closed (by 5.4), sequential (by [6], 1.9), and, hence, sequentially compact (by [6], 1.10). (Note that the Hausdorff hypothesis there, while used in the proof, is not needed.) The proof is the following: Let $\{x_n\}$ be a sequence in a countably compact space and y one of its cluster points. If for infinitely many n, $x_n = y$, some subsequence of $\{x_n\}$ converges to y. If not, suppose that $n > n_0$ implies $x_n \neq y$. Then if $T = \{x_n| \ n > n_0\}$, $y \in \mathrm{cl}\, T\setminus T$ and thus T is not sequentially closed. Hence some sequence in T converges to a point (not necessarily y) outside of T and $\{x_n\}$ has a convergent subsequence. Thus X is locally sequentially compact and (by Boehme's [3] Theorem 1) $X\times X$ is sequential. Hence X is Hausdorff (see [3], page 7 or [7], footnote (3)).

Note. The apparent contradiction between 5.6 and 5.3 results from the fact that Boehme's Theorem 1 ([3]) requires local sequential compactness in the sense that each point has a basis of sequentially compact neighborhoods. The space M_1 of 5.3 is compact but not locally (sequentially) compact.

The next proposition, a lemma for the following one, is possibly of independent interest.

Proposition. *Every locally sequential space X is sequential.*

Proof. If x belongs to a sequentially open subset U of X and N is any sequential neighborhood of x, then $\mathrm{int}\, N$ is sequential (by [6], 1.9), and $U' = (\mathrm{int}\, N) \cap U$ is open in $\mathrm{int}\, N$ and hence in X. Thus U is open and X is sequential.

We close this section with the companion piece to 5.2.

5.8. Proposition. *The product of two sequential spaces X and Y is sequential iff $\varphi_X\times\varphi_Y$ is a quotient map.*

Proof. By Boehme's Theorem 1 ([3]) each cartesian product of a convergent sequence (including its limit point) in X with another in Y is sequential. Hence, by 5.7, $X^* \times Y^*$ is sequential. Thus if $\varphi_X \times \varphi_Y$ is a quotient map, by 1.3, [6], $X \times Y$ is sequential.

Conversely, suppose that $X \times Y$ is sequential, and that $W \subseteq X \times Y$ is such that $(\varphi_X \times \varphi_Y)^{-1}(W)$ is open in $X^* \times Y^*$. One verifies routinely that W is sequentially open, and hence, open, so that $\varphi_X \times \varphi_Y$ is a quotient map.

6. Fréchet spaces II. In this section we consider the possible cardinality of compact Hausdorff Fréchet spaces, and the uniqueness of sequential limits in Fréchet spaces.

A well-known, unanswered question of Alexandroff asks whether or not there is a first countable compact Hausdorff space with cardinality greater than c. The corresponding question for Fréchet spaces is trivially answered by

6.1. PROPOSITION. *The one-point compactification of any discrete space is a Fréchet space.*

In [7] an example of a Fréchet space, not Hausdorff, but with unique sequential limits was given. Another such (discovered also by A. Arhangel'skiĭ) which is countable and compact follows. (See also 5.3. M_1 is not Fréchet.)

6.2. EXAMPLE. *There is a countable, compact, Fréchet space with unique sequential limits which is not Hausdorff.*

Proof. Let $X = (N \times N) \cup \{p, q\}$ with $p \neq q$ and $\{p, q\} \cap (N \times N) = \emptyset$. Each $(i, j) \in N \times N$ will be discrete. Basic neighborhoods of p will be of the form $\{p\} \cup \bigcup_{i \geqslant k} \{(i, j)|\ j \in N\}$ for each $k \in N$, and those of q of the form $\{q\} \cup \bigcup_{i} \{(i, j)|\ j \geqslant j_i \in N\}$. One verifies routinely that X is compact but not Hausdorff. A sequence $\{(i_n, j_n)\}$ in $N \times N$ converges to p iff $\{i_n\}$ is unbounded, and to q iff $\{i_n\}$ is eventually constant and $\{j_n\}$ is unbounded. Hence sequential limits are unique.

Let A be any subset of $X \backslash \{q\}$. If for each i, $A \cap (\{i\} \times N)$ is finite, $q \notin \mathrm{cl} A$. If for some i it is infinite, there is a sequence in A converging to q. Since each point of $X \backslash \{q\}$ has a countable basis of neighborhoods, X is a Fréchet space.

7. When is a sequential space Fréchet? Example 2.2 [6] shows that sequential spaces need not be Fréchet. An early draft of [6] asserted that locally compact sequential spaces must be. An error in the proof was pointed out by C. E. Aull. A counter-example follows.

7.1. EXAMPLE. *There is a sequential compact Hausdorff space which is not Fréchet.*

Proof. The example is the one-point compactification of the space Ψ of Isbell (see [9], 5I, page 79). We shall briefly describe the construction for the convenience of the reader.

Let $\mathcal{F}$ be an infinite maximal pairwise almost disjoint collection of infinite subsets of N and let $\Psi = \mathcal{F} \cup N$ with points of N discrete and neighborhoods of $F \in \mathcal{F}$ those subsets of Ψ containing F and all but finitely many points of F. Let $\Psi^* = \Psi \cup \{\infty\}$ be the one-point compactification of Ψ. Since each $F \cup \{F\}$ is compact, Ψ is locally compact and hence Ψ^* is compact Hausdorff.

Now any sequence of distinct points in F converges to F and any sequence of distinct points in $\mathcal{F}$ converges to ∞. Hence any sequentially open subset of Ψ^* is a neighborhood of each of its points and so Ψ^* is sequential. But no sequence in N converges to ∞ even though $\infty \in \mathrm{cl} N$.

Note that $\Psi^* \backslash \mathcal{F}$ is a non-sequential subspace of Ψ^*. The following proposition (a corollary to results of Arhangel'skiĭ) shows that such a subspace must always exist.

7.2. PROPOSITION. *A sequential space is Fréchet iff it is hereditarily sequential.*

Proof. By 2.1, [6], every subspace of a Fréchet space is Fréchet and hence sequential.

Conversely, if X is hereditarily sequential, by 5.2 φ_X is a hereditary quotient map and hence pseudo-open ([2]). Thus by 2.3, [6], X is Fréchet (the Hausdorff hypothesis is not needed here).

The space M of 5.1 can be used to give a characterization (in the Hausdorff case) which is more useful than 7.2.

7.3. PROPOSITION. *A Hausdorff sequential space is Fréchet iff it contains no subspace which, with the sequential closure topology* (see the proof of 5.2), *is homeomorphic to M.* (3)

Proof. Let Y be a subspace of a sequential space X and h: $M \to \sigma Y$ (see the proof of 5.2) be a homeomorphism. Then if $Y \neq \sigma Y$, Y is a non-sequential subspace of X and, by 7.2, X is not Fréchet. If $Y = \sigma Y$, $Y \backslash h(N)$ is a non-sequential subspace of X.

Conversely, suppose that X is not Fréchet and, for any $A \subseteq X$, let A' be the set of all limits of sequences in A. Then for some $B \subseteq X$, $B' \neq \mathrm{cl} B$. Since X is sequential, there is a sequence $\{x_i\}$ in B' converging to some $x_0 \in \mathrm{cl} B \backslash B'$. Without loss of generality take the x_i distinct and $\{x_i\} \subseteq B \backslash B'$. Then for each i, there is a sequence $\{x_{ij}\} \subseteq B$ which converges to x_i. Since X is Hausdorff and $\{x_0\} \cup \{x_i|\ i \in N\}$ is compact, the x_{ij} may be taken all distinct. But since no sequence of x_{ij}'s converges to x_0, the

(3) The author is greatly indebted to the referee, whose skepticism led to the correction of an error in the original version of this proposition.

space σY where $Y = \{x_0\} \cup \{x_i|\ i \in N\} \cup \{x_{ij}|\ i, j \in N\}$ is easily seen to be homeomorphic to M.

Note that Ψ^* of example 7.1 contains no subspace homeomorphic to M.

Example 1.11, [6], shows that the product of Fréchet spaces need not be Fréchet. This is also an immediate consequence of 6.2 (see [3], page 7 or [7], footnote (3)). In each of these cases, the product is not even sequential. The next example shows that this need not always be the case.

7.4. Example. *The product of two Hausdorff Fréchet spaces can be sequential without being Fréchet.*

Proof. Let $X = R/Z$, the real line with the integers identified, and $I = [0, 1]$ the closed unit interval. I is first countable and hence Fréchet. The quotient map φ: $R \to X$ is pseudo-open and hence by 2.3, [6], X is Fréchet. Since I is compact, by Boehme's Theorem 1, [3], $X \times I$ is sequential. For each $n \in N$ let $A_n = \{(n - 1/k, 1/n)|\ k \in N\}$ and let $A = \bigcup \{A_n|\ n \in N\}$. Then $(0, 0) \in \mathrm{cl} A$ but no sequence in A converges to $(0, 0)$. Hence $X \times I$ is not Fréchet.

7.5. Example. *The product of two hereditarily quotient (pseudo-open) maps may be a quotient map without being hereditarily quotient (pseudo-open).*

Proof. The natural identifications φ_X: $X^* \to X$ and φ_I: $I^* \to I$ (see 5.2) are pseudo-open by 2.3, [6] but $\varphi_X \times \varphi_I$ is not, since $X^* \times I^*$ is a Fréchet space. However by 5.8 $\varphi_X \times \varphi_I$ is a quotient map.

References

[1] R. Arens, *Note on convergence in topology*, Math. Mag. 23 (1950), pp. 229-234.

[2] A. Arhangel'skiĭ, *Some types of factor mappings and the relations between classes of topological spaces*, Soviet Math. Dokl. 4 (1963), pp. 1726-1729.

[3] T. K. Boehme, *Linear s-spaces*, Symposium on Convergence Structures, University of Oklahoma, 1965.

[4] H. F. Cullen, *Unique sequential limits*, Bull. Unione. Mat. Ital., III-20 (1965), pp. 123-124.

[5] R. M. Dudley, *On sequential convergence*, Trasn. Amer. Math. Soc. 112 (1964), pp. 483-507.

[6] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. 57 (1965), pp. 107-115.

[7] — *On unique sequential limits*, Nieuw Archief voor Wiskunde 14 (1966), pp. 12-14.

[8] M. Fréchet, *Les espaces abstracts*, Paris, 1928. pp, 212-3.

[9] L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton, 1960.

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On bundles over a sphere with fibre Euclidean space

by

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The origin for this work is a paper of S. P. Novikov [17] on the topological invariance of rational Pontrjagin classes. His paper gives the first method (beyond mere homotopy theory) for proving topological invariance of certain properties. The object of this paper is to consider the special case of (topological) bundles over a sphere with fibre Euclidean space, and to compare the piecewise-linear (hereafter written as PL) and topological classifications. Perhaps the most interesting of the results obtained is that topological equivalence of two such bundles implies (stable) piecewise linear equivalence; however, we go on to extract all the information we can from the method.

I am indebted to Steve Gersten and Larry Siebenmann for pointing out that results from the latter's thesis can be used to fill an apparent gap in the argument of [17]: Novikov's recently published detailed proof [28] appears to use the same reasoning.

Our main result is the following

Theorem. *The natural homomorphism*

$$j\colon \pi_i(G, \mathrm{PL}) \to \pi_i(G, \mathrm{Top})$$

has a left inverse, for all $i > 0$, except possibly for $i = 2$ or 4. Even in these cases, j is injective.

In the first paragraph we establish our notation. The next is devoted to the lemmas which are needed at the key place in the argument. We then prove the main theorems. A final section is devoted to discussion of special features of low dimensional cases, to which the proofs do not apply without modification.

§ 1. Structure groups and classifying spaces. First it will be convenient to establish our notation and recall some known results.

By O_n we denote the usual *orthogonal group acting on* $\boldsymbol{R}^n$. PL_n is the *group of piecewise linear homeomorphisms of* $\boldsymbol{R}^n$ *onto itself*, leaving the origin fixed. It is necessary to define PL_n as a *semi-simplicial group* [14].

Top_n will denote the *group of all homeomorphisms of* $(\boldsymbol{R}^n, O)$ *onto itself*, with the compact-open topology.