References

[1] R. D. Anderson, *The Hilbert cube as a product of dendrons*, Notices of Amer. Math. Soc. 11 (1964), p. 522.

[2] P. Alexandroff and H. Hopf, *Topologie*, Berlin, 1935.

[3] T. Ganea, *On the homogeneity of infinite products of manifolds*, Proc. Amer. Math. Soc. 17 (1966), pp. 1128-1130.

[4] R. J. Koch and L. F. McAuley, *Semigroups on continua ruled by arcs*, Fund. Math. 56 (1964), pp. 1-8.

[5] P. Hilton, *Introduction to homotopy theory*, Cambridge, 1953.

Reçu par la Rédaction le 11. 2. 1967

A counter-example in dimension theory

by

T. H. Walton (Swansea)

In proving results involving the *strong inductive dimension* 'Ind' of a topological space one frequently uses the following

LEMMA. *Let A be a subset of the hereditarily normal space X. If* $\dim A \leqslant 0$ *then for any pair of a closed set F and an open set G with $F \subseteq G$ there exists an open set V such that*

$$F \subseteq V \subseteq G, \quad b(V) \cap A = \emptyset$$

where $b(V) = \overline{V} \setminus V$ is the boundary of V (see, e.g. Morita [1], Hurewicz and Wallman [2], Nagata [3]).

That this is not true for every normal space is shown by the construction which follows.

Let ω_0 be the first infinite ordinal and let ω_1 be the first uncountable ordinal and provide each of the sets

$$N = \{k\colon k \text{ and ordinal}, \ 0 \leqslant k \leqslant \omega_0\},$$
$$P = \{a\colon a \text{ an ordinal}, \ 0 \leqslant a \leqslant \omega_1\}$$

with the order topology.

Let the set $I = \{t\colon t \text{ real}, \ 0 \leqslant t \leqslant 1\}$ be provided with the usual topology. Form the topological product

$$Z = P \times I \times N.$$

Then since P, I, N are all compact Hausdorff spaces, so is their product Z. Finally, form the quotient space X by identifying all points of Z with the same t-coordinate for $a = \omega_1$, that is, define a decomposition $\mathfrak{D}$ of Z whose members are:

$$\text{the singletons } \{z\} = \{(a, t, k)\} \text{ if } a \neq \omega_1,$$
$$\text{the sets } E_t = \{(a, t, k)\colon a = \omega_1, \ 0 \leqslant k \leqslant \omega_0\}$$

and provide the family $\mathfrak{D}$ of equivalence classes with the quotient topology thereby obtaining the quotient space X in which a subset U is open if and only if $\pi^{-1}[U]$ is open in Z, where $\pi\colon Z \twoheadrightarrow X$ is the projection of Z onto X.

The topological space X will be referred to as the 'book-space' and the subspace

$$B = \bigcup_{t \in I} E_t$$

as the 'spine' of the book-space.

Note that there exists a homeomorphism $f\colon \mathfrak{I} \rightarrowtail\!\!\!\rightarrow B$ between the unit interval I and the spine B defined by

$$f(t) = \pi\big((\omega_1, t, k)\big)\,, \qquad \text{any } k \in N\,.$$

Next, let $r_0 = 0$, $r_1 = 1$, $r_2, r_3, \ldots$ be an enumeration of the rational numbers in $I = [0, 1]$. Let

$$A_k = \{(a, r_k, k)\colon\ a \in P\,,\ a < \omega_1\}\,.$$

Then

$$A_k \subseteq X\,, \qquad k \in N\,.$$

Let $C =$ that subset of X for which $a = \omega_1$ and $t =$ an irrational number $0 < t < 1$.

$$A = \Big(\bigcup_{k=0}^{\infty} A_k\Big) \cup C\,.$$

Clearly, $A \subset X$.

Finally, let

$$H = \text{that subset of } X \text{ for which } t = 0\,,$$
$$K = \text{that subset of } X \text{ for which } t = 1\,.$$

Then H, K are disjoint closed subsets of X. We assert that

(1) *X is normal;*

(2) *X is not hereditarily normal;*

(3) $\dim A = 0$;

(4) *if $G = X \setminus K$, then G is open and the closed set $H \subseteq G$; for every open set W such that*

$$H \subseteq W \subseteq \overline{W} \subseteq G\,, \qquad b(W) \cap A \neq \emptyset\,.$$

We shall establish (4) in the equivalent symmetric form:

(4′) *for any two open subsets U, V of X such that $H \subseteq U$, $K \subseteq V$, $U \cap V = \emptyset$*

$$\big(X \setminus (U \cup V)\big) \cap A \neq \emptyset\,.$$

Proof. (1) Since Z is compact, the quotient space X is compact. It suffices to show that X is Hausdorff or, equivalently, to observe that the decomposition $\mathfrak{D}$ is upper semi-continuous. (Let Z be a space and let $\mathfrak{C} = \{C_a\}$ be a collection of disjoint compact sets whose union is Z. The collection $\mathfrak{C}$ is said to be *upper semi-continuous* provided that for each C_a of $\mathfrak{C}$ and each open set U containing C_a there is an open set V with $C_a \subseteq V \subseteq U$ such that every element C_b of $\mathfrak{C}$ that intersects V lies in U. (See Hocking and Young [4].)

The collection $\mathfrak{D}$ clearly satisfies the requirements for being upper semi-continuous. Hence X is a compact Hausdorff space and is therefore normal.

(2) The space X contains homeomorphs of $P \times I$ in which $P \times N$, the 'Tychonov plank', can be imbedded. Since the 'Tychonov plank' is not hereditarily normal (see Kelley [5]), the space X is not hereditarily normal.

(3) Let $A = \bigcup\limits_{i=1}^{m} A \cap U_i$, where $U_1, U_2, \ldots, U_m$ are open in X. Then, since C is zero-dimensional, there exist J_i $(i = 1, 2, \ldots, m)$ open in I, J_i disjoint, such that

$$f(J_i) \subset U_i\,, \qquad \bigcup f(J_i) \supset C\,.$$

Since I has a countable base and the number of 'leaves' of the book-space X is countable, we can choose β_i so large that

$$V_i = \pi[\{(a, t, k)\colon\ \beta_2 < a \leqslant \omega_1\,,\ t \in J_i\,,\ 0 \leqslant k \leqslant \omega_0\}] \subset U_i\,.$$

Now $\bigcup V_i$ is open and $A \setminus \bigcup V_i$, the part of A not covered by the sets V_i, is the union of a family of disjoint zero-dimensional open subsets of A. Hence $\dim A = 0$.

(4′) Let U, V be any two open subsets of X such that $H \subseteq U$, $K \subseteq V$ and $U \cap V = \emptyset$. Put $Q = X \setminus (U \cup V)$. It remains to show that $Q \cap A \neq \emptyset$. Since B is connected, $Q \cap B \neq \emptyset$. If $Q \cap B$ has any point with irrational t-coordinate, then $Q \cap A \neq \emptyset$; if it does not, there is some $t_0 \in I$ such that $f(t_0)$ is an accumulation point of both U and V. Then t_0 is rational and all (a, t_0, k) with $\omega_1 > a >$ some ξ are accumulation points of both U and V; hence all such points are in Q. Hence $A \cap Q \neq \emptyset$.

The above construction thus shows that we cannot relax the condition 'hereditarily normal' to 'normal' in the above lemma.

In conclusion I wish to express my appreciation to Professor C. H. Dowker for drawing my attention to this problem and for valuable discussions.

References

[1] K. Morita, *Dimension theory for metric spaces*, Math. Annalen 128 (1954), pp. 350-362.

[2] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton 1952.

[3] J. Nagata, *Modern dimension theory*, North Holland 1965.

[4] J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley 1961.

[5] J. L. Kelley, *General topology*, Van Nostrand 1955.

UNIVERSITY COLLEGE, Swansea

Reçu par la Rédaction le 31. 3. 1967