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Reçu par la Rédaction le 8. 2. 1967

A unique factorization theorem for countable products of circles

by

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To say that X is a *factor space* of Y means that Y is homeomorphic with $X \times Z$ for some space Z. One can often say something about X if something is known about Y. For example, it is not hard to show that if Y is a compact, connected, locally connected metric space, then so is X. As a further example, it can be shown that every one-dimensional factor space of the Hilbert cube is a tree (non-degenerate locally connected metric continuum containing no simple closed curve). R.D. Anderson has proved that in fact every tree is a factor space of the Hilbert cube [1]. In this note we consider the analogous problem of determining the one-dimensional factor spaces of countable products of circles. It is found that the circle is the only one. The author wishes to thank R. D. Anderson and A. Lelek for their suggestions.

The notion of an inessential space [2] will prove useful. An *inessential space* is a space X such that there is a homotopy $H\colon [0,1]\times X \to X$ with the property that $H(1,x) = x$ for each $x \in X$ and $H(0,X) \neq X$; in words, X can be continuously deformed to a proper subset of itself with a homotopy starting at the identity. A space is *essential* if no such homotopy exists. It follows from Lemma 1 of [3] that a countable product of circles is essential.

LEMMA 1. *Let X be a tree and let p be an endpoint of X. Then there is a homotopy $H\colon [0,1]\times X \to X$ such that $H(1,x) = x$ for $x \in X$, $H(0,X) = p$ and $H(t,p) = p$ for each $t \in [0,1]$. Consequently X is inessential.*

Proof. This is a corollary of Theorem A of [4], which states that X can be made into a semilattice with identity and zero p.

LEMMA 2. *Every factor space of an essential space is essential.*

* The results in this paper are contained in the author's dissertation written under the direction of Professor R. J. Koch.

Proof. Let X be an inessential space. Let Y be any space. Given a homotopy $H\colon [0,1]\times X\to X$ with the required property, define

$$K\colon [0,1]\times X\times Y\to X\times Y \text{ by } K(t,x,y)=\big(H(t,x),y\big).$$

Then K demonstrates the inessentiality of $X\times Y$. Hence the product of an inessential space with any space is inessential.

COROLLARY 3. *Let X be a one-dimensional factor space of a countable product of circles. Then X must contain at least one circle.*

Proof. If X contained no circle, then X would be a tree. This contradicts the fact that X is essential.

LEMMA 4. *Let X be a one-dimensional factor space of a countable product of circles. Then X contains at most one circle.*

Proof. Suppose that X contains more than one circle. We will show that the fundamental group of X, $\Pi_1(X)$, is then non-abelian. But $X\times Y$ is homeomorphic with a product of circles which has an abelian fundamental group since it is an arcwise connected topological group. This together with the fact that $\Pi_1(X\times Y)$ is isomorphic with $\Pi_1(X)\otimes\Pi_1(Y)$ implies that $\Pi_1(X)$ is abelian, a contradiction. The facts about fundamental groups used here may be found in [5].

To establish the claim that if X contains more than one circle then $\Pi_1(X)$ is non-abelian, suppose that A and B are distinct circles in X. There are two cases to consider:

Case 1. A and B are disjoint. Since X is arcwise connected, there is an arc C in X such that $A\cap C=a$ and $B\cap C=b$ are the end points of C. For $x\in B\setminus b$, let $U_x=\{y\in X\colon d(x,y)<\frac{1}{2}d(x,A\cup C)\}$, and let $U=\bigcup\{U_x\colon x\in B\setminus b\}$. Then U^*, the closure of U, contains B and $U^*\cap (A\cup C)=b$. Note also that $B\cup(U^*\setminus U)$ is a closed subset of U^* with $B\cap(U^*\setminus U)=b$. Hence, by theorem VI 4, page 83, of [5], there is a map $g\colon U^*\to B$ such that $g(x)=x$ for $x\in B$ and $g(x)=b$ for $x\in U^*\setminus U$. Now for each $x\in A\setminus a$, let $V_x=\{y\in X\colon d(x,y)<\frac{1}{2}d(x,U^*\cup C)\}$ and let $V=\bigcup\{V_x\colon x\in A\setminus a\}$. Again by Theorem VI 4 of [5], there is a map $h\colon V^*\to A$ such that $h(x)=x$ for $x\in A$ and $h(x)=a$ for $x\in V^*\setminus V$. Let $K=X\setminus(U\cup V)$. Then $C\cup(U^*\setminus U)\cup(V^*\setminus V)$ is a closed subset of K with $V^*\setminus V\cap U^*\setminus U=\emptyset$, the empty set. By the Tietze extension theorem there is a map $k\colon K\to C$ such that $k(x)=x$ for $x\in C$, $k(x)=b$ for $x\in U^*\setminus U$ and $k(x)=a$ for $x\in V^*\setminus V$. Hence the map $f\colon X\to A\cup C\cup B$ given by $f=g\cup h\cup k$ is a retraction of X onto $A\cup C\cup B$. Therefore $\Pi_1(A\cup C\cup B)$ is isomorphic with a subgroup of $\Pi_1(X)$. Since $\Pi_1(A\cup C\cup B)$ is non-abelian, we conclude that $\Pi_1(X)$ is non-abelian.

Case 2. $A\cap B$ is non-void. Let C be a component of $B\setminus A$. Then clearly C^* is an arc which meets A in its endpoints or $C^*=B$. If $C^*=B$ then $A\cup B$ is a figure eight, and a retraction of X onto $A\cup B$ can be constructed by a simplification of the procedure used in Case 1. As in Case 1, this implies that $\Pi_1(X)$ is non-abelian. If $A\cap C^*=\{p,q\}$ where $p\neq q$, then $A\cup C^*$ is a θ curve. Label the components of $A\setminus\{p,q\}$ D and E. There are two possibilities:

(i) C, D, and E do not all lie in the same component of $X\setminus\{p,q\}$. Then there is a component K of $X\setminus\{p,q\}$ containing exactly one of C, D, and E, say C. Let $K'=X\setminus K$. Note that K^* and K' are closed subsets of X with $K^*\cap K'=\{p,q\}$ and $K^*\cup K'=X$. Using Theorem VI 4 of [5] and the Tietze extention theorem, there are retractions $g\colon K'\to A$ and $h\colon K^*\to C^*$. The union of g and h is then a retraction of X onto $A\cup C^*$. We conclude that $\Pi_1(X)$ is non-abelian.

(ii) C, D, and E lie in the same component of $X\setminus\{p,q\}$. Let F be an arc in $X\setminus\{p,q\}$ which joins exactly two of C, D, and E, say C and D. Let r and s denote the endpoints of F, with $r\in C$, $s\in D$. Now if the subarc (p,s) of D lies in a component of $X\setminus\{p,s\}$ not containing q, then we can retract X onto a θ curve as in (i). Hence we may assume that there is an arc G joining (p,s) with $(A\cup C\cup F)\setminus[p,s]$. In fact, we may assume that G joins (p,s) with the subarc (r,q) of C; otherwise, it is seen that X contains two disjoint circles or a figure eight. By the same reasoning we may assume there is an arc H joining E with $F\setminus\{r,s\}$. Now in the figure $A\cup C\cup F\cup G\cup H$, there are two disjoint circles. Hence $\Pi_1(X)$ must be non-abelian. This concludes the proof of Lemma 4. The author is grateful to Dr. H. Patkowska for pointing out an error in the original version of Lemma 4.

THEOREM 5. *Let X be a one-dimensional factor space of a countable product of circles. Then X is a circle. Thus the factorization of a countable product of circles into one-dimensional factor spaces is unique.*

Proof. Let A denote the circle in X and assume that $X\setminus A$ is non-void. Let C be one of the components of $X\setminus A$. Then C^* is a tree which meets A in a point p. Take a homotopy $H\colon [0,1]\times C^*\to C^*$ such that $H(1,x)=x$ for $x\in C^*$, $H(t,p)=p$ for $t\in[0,1]$ and $H(0,C^*)=p$ (such a homotopy exists by Lemma 1). Extend H to $[0,1]\times X$ by defining $H(t,x)=x$ for all $x\in X\setminus C^*$ and all $t\in[0,1]$. Then H demonstrates the inessentiality of X. But X is essential. Therefore $X=A$, and the proof is complete.

It appears to the author that Theorem 5 should hold for arbitrary products of circles. In connection with this, it would be nice to know the answer to the following question: Is every one-dimensional factor space of a product of circles metric? Indeed, one can ask the same question about finite-dimensional factor spaces of products of metric spaces in general.

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Reçu par la Rédaction le 11. 2. 1967

A counter-example in dimension theory

by

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In proving results involving the *strong inductive dimension* 'Ind' of a topological space one frequently uses the following

LEMMA. *Let A be a subset of the hereditarily normal space X. If* $\dim A \leqslant 0$ *then for any pair of a closed set F and an open set G with $F \subseteq G$ there exists an open set V such that*

$$F \subseteq V \subseteq G, \quad b(V) \cap A = \emptyset$$

where $b(V) = \overline{V} \setminus V$ is the boundary of V (see, e.g. Morita [1], Hurewicz and Wallman [2], Nagata [3]).

That this is not true for every normal space is shown by the construction which follows.

Let ω_0 be the first infinite ordinal and let ω_1 be the first uncountable ordinal and provide each of the sets

$$N = \{k:\ k \text{ and ordinal},\ 0 \leqslant k \leqslant \omega_0\},$$
$$P = \{a:\ a \text{ an ordinal},\ 0 \leqslant a \leqslant \omega_1\}$$

with the order topology.

Let the set $I = \{t:\ t \text{ real},\ 0 \leqslant t \leqslant 1\}$ be provided with the usual topology. Form the topological product

$$Z = P \times I \times N .$$

Then since P, I, N are all compact Hausdorff spaces, so is their product Z. Finally, form the quotient space X by identifying all points of Z with the same t-coordinate for $a = \omega_1$, that is, define a decomposition $\mathfrak{D}$ of Z whose members are:

$$\text{the singletons } \{z\} = \{(a, t, k)\} \text{ if } a \neq \omega_1,$$
$$\text{the sets } E_t = \{(a, t, k):\ a = \omega_1,\ 0 \leqslant k \leqslant \omega_0\}$$

and provide the family $\mathfrak{D}$ of equivalence classes with the quotient topology thereby obtaining the quotient space X in which a subset U is open if and only if $\pi^{-1}[U]$ is open in Z, where $\pi\colon Z \twoheadrightarrow X$ is the projection of Z onto X.