Let τ' be the T_1-topology on Y generated by sets of the form:

(i) U, where $U \in \mathfrak{I}(Y)$,

(ii) V, where $V \in \mathfrak{I}(E_2)$,

(iii) W, where $W \in \mathfrak{I}(E_3)$,

(iv) $\{x\} \cup V \cup W$, where $x \in E_4$, $V \in \mathfrak{I}(E_2)$, and $W \in \mathfrak{I}(E_3)$.

It can easily be seen that $\tau \vee \tau' = 1$ and $\tau \wedge \tau' = \mathfrak{C}$. Moreover, since any co-finite subset of Y must contain points of E_1 and since any open set in τ' containing points of E_1 must itself be co-finite, τ' is compact on co-finite subsets of Y. However, if $X = E_1 \cup E_4$, then X is a dense subset of Y and $\tau_X = \tau|X$ has the following properties: $E_1, E_4 \in \tau_X$, $\tau_X|E_1 = 1$, and $\tau_X|E_4 = \mathfrak{C}$. Therefore τ_X has no T_1-complement (for proof, see [6]).

Since the real numbers with the usual topology $\mathfrak{R}$ satisfy the hypotheses of Theorems 2 and 4, if $\mathfrak{R}$ has a T_1-complement $\mathfrak{R}'$, then $\mathfrak{R}'$ cannot have a countable base and must be countably compact on all co-finite sets of real numbers.

Similarly, if $\mathfrak{R}_Q$ denotes the usual topology on the rational numbers, $\mathfrak{R}'_Q$ is a T_1-complement for $\mathfrak{R}_Q$ only if $\mathfrak{R}'_Q$ is compact on all co-finite subsets of rational numbers and $\mathfrak{R}'_Q$ is not Hausdorff.

Therefore, by Theorem 6, if $\mathfrak{R}_Q$ has a T_1-complement then $\mathfrak{R}$ has a T_1-complement which is compact on co-finite subsets of real numbers, and which is not Hausdorff. However, if $\mathfrak{R}_Q$ has no T_1-complement, no conclusion can be drawn about the existence of a T_1-complement for $\mathfrak{R}$.

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An extension of a theorem of Gaifman-Hales-Solovay

by

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Solovay [1] has found a remarkably simple proof of a theorem of Gaifman [2] and Hales [3]: *There are countably generated complete Boolean algebras of arbitrarily high cardinality.* Here we show that Solovay's methods can be extended to prove the stronger theorem: *Every Boolean algebra can be completely embedded in a countably generated complete Boolean algebra.* (A *complete embedding* of a Boolean algebra B into a complete Boolean algebra B' is a monomorphism of B into B' preserving all suprema that happen to exist in B. If B is itself a complete Boolean algebra, this means that *all* suprema are preserved [1].)

Like Solovay's [1], the present work was suggested by Cohen's notion of *forcing* [5]. We will follow Solovay, however, in making the present proof independent of any knowledge of Cohen's work, leaving the connection with Cohen to be divined by the *cognoscenti* [2].

Let S be any discrete topological space, and let S^ω be the product space of S taken countably many times as a factor. An element of S^ω can be represented as a function f from the positive integers into S. If we are given a finite sequence $\sigma = \langle s_1, \ldots, s_n \rangle$ of elements of S, the set of all functions $f \in S^\omega$ with $f(i) = s_i$ $(i = 1, \ldots, n)$ is an open set of the product space S^ω; call it $\mathfrak{O}(\sigma)$. The sets of form $\mathfrak{O}(\sigma)$ form a basis for the product topology of S^ω. We explicitly include the empty finite sequence; the corresponding basic open set is the entire space S^ω.

Solovay [1] has shown that the regular open algebra of S^ω, where S is any discrete space, is a countably generated complete Boolean algebra.

LEMMA 1. *Let S be a discrete space, and let B' be the regular open algebra of S^ω. For each element $s \in S$ and positive integer n, let $\varrho(n, s)$ be the*

(1) We use the notations of Halmos [4] for finite and infinite Boolean operations, except that the complement of b is denoted by $-b$. For all unexplained terminology, the reader should consult [1], [2], [3], or [4].

In the terminology of [4], p. 34, Ex. 6, Theorem 1 of the present paper would read: *Every Boolean algebra is isomorphic to a regular subalgebra of a countably generated complete Boolean algebra.* (So in particular, every complete Boolean algebra is isomorphic to a complete subalgebra of a countably generated complete Boolean algebra).

(2) The connection between Boolean algebras and forcing will be developed in a forthcoming paper by Scott and Solovay. I have not yet seen this paper, but I have benefited from hearing both authors expound some of their ideas.

set of all $f \in S^\omega$ such that $f(n) = s$. Then for each fixed $s \in S$, the supremum of the $\varrho(n, s)$ for all positive integers n in the complete Boolean algebra B' is 1.

Proof. For fixed s, $\varrho(n, s)$ is open and closed in S^ω, so $\varrho(n, s)$ is a regular open set. The supremum $\bigvee_n \varrho(n, s) = \operatorname{int}\operatorname{cl}\big(\bigcup_n \varrho(n, s)\big)$. To show that this is 1, it is necessary and sufficient to prove that the closure of $\bigcup_n \varrho(n, s)$ is S^ω.

Let $f \in S^\omega$; we show that every basic open set containing f intersects $\bigcup_n \varrho(n, s)$. Let σ be a finite sequence of elements of S and $f \in \mathfrak{O}(\sigma)$. If σ is the empty sequence, $\mathfrak{O}(\sigma) = S^\omega$ and of course intersects $\bigcup_n \varrho(n, s)$. Otherwise, since $f \in \mathfrak{O}(\sigma)$, $\sigma = \langle f(1), \ldots, f(m)\rangle$ for some m. Define $g \in S^\omega$ by saying $g(n) = f(n)$ if $n \neq m+1$, and $g(m+1) = s$. Then $g \in \mathfrak{O}(\sigma)$ and $g \in \varrho(m+1, s)$. Hence $\mathfrak{O}(\sigma)$ intersects $\bigcup_n \varrho(n, s)$. This is the desired result, since every basic open set has the form $\mathfrak{O}(\sigma)$ for some σ. Q.E.D.

THEOREM 1. *Every Boolean algebra can be completely embedded in a countably generated complete Boolean algebra.*

Proof. Let B be a Boolean algebra. We assume a fixed well-ordering of B to be given. Let S be the set of all non-empty subsets s of B which have a supremum in B. Every finite non-empty subset of B is automatically in S; if B is a complete Boolean algebra, S contains all non-empty subsets of B. We consider S to be a topological space with the discrete topology. Let B' be the regular open algebra of S^ω; B' is a countably generated complete Boolean algebra.

To each finite sequence σ of elements of S, we correspond a non-zero element $\tau(\sigma)$ of B. We define $\tau(\sigma)$ by induction on the length of the sequence σ. If σ is the empty sequence, set $\tau(\sigma) = 1$. If σ has length $n+1$ and τ has been defined for sequences of length n, let s be the last $(n+1\,th)$ term of σ. Let σ' come from σ by dropping the last term s; σ' has length n, so $\tau(\sigma') \neq 0$ is defined. Now $s = \{b_i \mid i \in I\}$ is a subset of B with a supremum in B; we set $\tau(\sigma) = \tau(\sigma') \wedge b'$, where b' is the least element b_i of s (in the given well-ordering of B) such that $\tau(\sigma') \wedge b_i \neq 0$, if such an element exists; otherwise, $b' = c = -\bigvee_{i \in I} b_i$. We need to show that $\tau(\sigma) \neq 0$. Since $\bigvee_{i \in I} b_i \vee c = 1$,

$$\tau(\sigma') = \tau(\sigma') \wedge \big(\bigvee_{i \in I} b_i \vee c\big) = \bigvee_{i \in I} \big(\tau(\sigma') \wedge b_i\big) \vee \big(\tau(\sigma') \wedge c\big).$$

Hence since $\tau(\sigma') \neq 0$, either $\tau(\sigma') \wedge b_i \neq 0$ for some $i \in I$ or $\tau(\sigma') \wedge c \neq 0$. This shows that $\tau(\sigma) \neq 0$.

It is clear that if σ is a finite sequence of elements of S and σ' comes from σ by omitting the last term of σ, then $\tau(\sigma) \leqslant \tau(\sigma')$. It follows that if σ_1 and σ_2 are finite sequences of elements of S and σ_1 is an initial segment of σ_2, then $\tau(\sigma_2) \leqslant \tau(\sigma_1)$.

Let b be any element of B. Define $\varphi(b)$ to be the supremum in B' of those basic open sets $\mathfrak{O}(\sigma)$ of S^ω such that $\tau(\sigma) \leqslant b$. φ maps B into B'; we need to show that φ is a complete monomorphism. We first show that φ is a complete homomorphism; i.e., it preserves complementation and all suprema that exist in B.

Let b be any element of B; to show that $\varphi(-b)$ is the complement of $\varphi(b)$ in B', we need to show $\varphi(b) \wedge \varphi(-b) = 0$ and $\varphi(b) \vee \varphi(-b) = 1$. Since

$$\Phi(b) \wedge \Phi(-b) = \Big(\bigvee_{\tau(\sigma_1) \leqslant b} \mathfrak{O}(\sigma_1)\Big) \wedge \Big(\bigvee_{\tau(\sigma_2) \leqslant -b} \mathfrak{O}(\sigma)\Big)$$
$$= \bigvee_{\tau(\sigma_1) \leqslant b,\, \tau(\sigma_2) \leqslant -b} \mathfrak{O}(\sigma_1) \wedge \mathfrak{O}(\sigma_2),$$

the assertion $\varphi(b) \wedge \varphi(-b) = 0$ amounts to saying that if σ_1 and σ_2 are finite sequences such that $\tau(\sigma_1) \leqslant b$ and $\tau(\sigma_1) \leqslant -b$, then $\mathfrak{O}(\sigma_1)$ and $\mathfrak{O}(\sigma_2)$ are disjoint. Now $\mathfrak{O}(\sigma_1)$ and $\mathfrak{O}(\sigma_2)$ will be disjoint, unless either σ_1 is an initial segment of σ_2 or σ_2 is an initial segment of σ_1. Suppose that σ_1 is an initial segment of σ_2. Then $\tau(\sigma_2) \leqslant \tau(\sigma_1) \leqslant b$, and $\tau(\sigma_2) \leqslant -b$; hence $\tau(\sigma_2) = 0$, contrary to what has been proved above. So σ_1 is not an initial segment of σ_2. Similarly σ_2 is not an initial segment of σ_1, so $\mathfrak{O}(\sigma_1)$ and $\mathfrak{O}(\sigma_2)$ are disjoint, and $\varphi(b) \wedge \varphi(-b) = 0$.

To show $\varphi(b) \vee \varphi(-b) = 1$, we need only show that for each n, $\varrho(n, \{b\}) \leqslant \varphi(b) \vee \varphi(-b)$; for then, by Lemma 1, $1 = \bigvee_n \varrho(n, \{b\}) \leqslant \varphi(b) \vee \varphi(-b)$. Let $f \in \varrho(n, \{b\})$, let σ be the finite sequence $\langle f(1), \ldots, f(n)\rangle$, and let σ' be $\langle f(1), \ldots, f(n-1)\rangle$ (the empty sequence, if $n = 1$). Since $f \in \varrho(n, \{b\})$, $f(n) = \{b\}$. So either $\tau(\sigma) = \tau(\sigma') \wedge b$ or $\tau(\sigma) = \tau(\sigma') \wedge (-b)$. If $\tau(\sigma) = \tau(\sigma') \wedge b$, then $\mathfrak{O}(\sigma) \leqslant \varphi(b)$, since $\tau(\sigma) \leqslant b$; hence, since $f \in \mathfrak{O}(\sigma)$, $f \in \varphi(b)$. Similarly, if $\tau(\sigma') = \tau(\sigma) \wedge (-b)$, then $f \in \varphi(-b)$. So $\varrho(n, \{b\}) \subseteq \varphi(b) \cup \varphi(-b)$; hence $\varrho(n, \{b\}) \leqslant \varphi(b) \vee \varphi(-b)$, as desired.

We now show that, for any set $s = \{i \mid b_i \in I\} \subseteq B$,

$$\bigvee_{i \in I} \varphi(b_i) = \varphi\big(\bigvee_{i \in I} b_i\big)$$

whenever $\bigvee_{i \in I} b_i$ exists. If s is empty, this is trivial, so we assume s non-empty and $\bigvee_{i \in I} b_i$ exists; i.e., $s \in S$. To show that $\bigvee_{i \in I} \varphi(b_i) \leqslant \varphi\big(\bigvee_{i \in I} b_i\big)$, we need only show that for each $i \in I$, $\varphi(b_i) \leqslant \varphi\big(\bigvee_{i \in I} b_i\big)$. But, using the definition of φ, this amounts to the assertion that if $\tau(\sigma) \leqslant b_i$, then also $\tau(\sigma) \leqslant \bigvee_{i \in I} b_i$, which is trivial. So we need only show that $\varphi\big(\bigvee_{i \in I} b_i\big) \leqslant \bigvee_{i \in I} \varphi(b_i)$. To show this, it suffices to show for each n, that

$$\varrho(n, s) \wedge \varphi\big(\bigvee_{i \in I} b_i\big) \leqslant \bigvee_{i \in I} \varphi(b_i);$$

for then, using Lemma 1,

$$\varphi(\bigvee_{i\in I} b_i) = 1 \wedge \varphi(\bigvee_{i\in I} b_i) = \big(\bigvee_n \varrho(n, s)\big) \wedge \varphi(\bigvee_{i\in I} b_i)$$

$$= \bigvee_n \big(\varrho(n, s) \wedge \varphi(\bigvee_{i\in I} b_i)\big) \leqslant \bigvee_{i\in I} \varphi(b_i)\,.$$

Let $f \in \varrho(n, s) \cap \varphi(\bigvee_{i\in I} b_i)$. Then $f(n) = s$. Let $\sigma = \langle f(1), \ldots, f(n)\rangle$, and let $\sigma' = \langle f(1), \ldots, f(n-1)\rangle$ (empty, if $n = 1$). Then either $\tau(\sigma) = \tau(\sigma') \wedge b_i$ for some $i \in I$ or $\tau(\sigma) = \tau(\sigma') \wedge (-\bigvee_{i\in I} b_i)$. Suppose $\tau(\sigma) = \tau(\sigma') \wedge (-\bigvee_{i\in I} b_i)$. Then

$$\mathfrak{O}(\sigma) \leqslant \varphi(-\bigvee_{i\in I} b_i) = -\varphi(\bigvee_{i\in I} b_i)\,.$$

Since $f \in \mathfrak{O}(\sigma), f \in -\varphi(\bigvee_{i\in I} b_i)$, which contradicts our hypothesis that $f \in \varphi(\bigvee_{i\in I} b_i)$. This contradiction shows that $\tau(\sigma) \neq \tau(\sigma') \wedge (-\bigvee_{i\in I} b_i)$; so for some $i \in I, \tau(\sigma) = \tau(\sigma') \wedge b_i$. Hence $\tau(\sigma) \leqslant b_i$, so $\mathfrak{O}(\sigma) \leqslant \varphi(b_i)$, hence $f \in \varphi(b_i)$, and hence $f \in \bigvee_{i\in I} \varphi(b_i)$. This shows that

$$\varrho(n, s) \wedge \varphi(\bigvee_{i\in I} b_i) \leqslant \bigvee_{i\in I} \varphi(b_i)\,,$$

as desired.

Finally, we need to show that φ is a monomorphism. Let $b \in B$, $b \neq 0$; we show $\varphi(b) \neq 0$. Let σ be the one-termed sequence $\langle\{b\}\rangle$. Then since $b \neq 0$, $\tau(\sigma) = 1 \wedge b = b$. Hence $\mathfrak{O}(\sigma) \leqslant \varphi(b)$. Since $\mathfrak{O}(\sigma)$ is non-empty, $\varphi(b) \neq 0$. Q.E.D.

Theorem 1 includes the result of Gaifman and Hales, since there are Boolean algebras of arbitrarily high cardinality; it also includes the known result that *every Boolean algebra can be completely embedded in a complete Boolean algebra*, which is usually proved by other methods (see [4], section 21).

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Core decompositions of continua

by

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Let S be the space consisting of the $\sin(1/x)$ curve, $0 < x \leqslant 1$, and its limit continuum C on the y-axis. Shrinking C to a point gives rise to a decomposition G of S. The decomposition G is monotone with an arc as hyperspace, and if H is any other monotone decomposition of S whose hyperspace is an arc, then G refines H. Thus we say that G is the *core* decomposition of S with respect to the property — "Is monotone with an arc as hyperspace". (See Definition 1.1.).

Let P be a property of decompositions and $\mathcal{S}$ a class of topological spaces. By a *method of core decomposition for* $\mathcal{S}$ *with respect to* P, we mean a method of decomposition which, when applied to any $S \in \mathcal{S}$, yields the core decomposition of S with respect to P. Thus Kuratowski, by his decomposition into "tranches" ([7], p. 248), has described a method of core decomposition for *the class* $\mathcal{S}$ of compact continua irreducible between two points and P the property — "Is monotone with locally connected hyperspace". W. A. Wilson's decomposition into "oscillatory sets" ([15], p. 381) is a method of core decomposition for P as above and $\mathcal{S}$ the class of compact, one-dimensional, m-cyclic, and separable continua.

The principle result of this work is a method of core decomposition for the class $\mathcal{S}$ of compact Hausdorff continua and P the property — "Is monotone with semi-locally-connected hyperspace".

If M and N are set functions on a set S, then M is an *expansion* of N if and only if $M(A) \supset N(A)$ for all $A \subset S$. Our method consists of three successive expansions of the useful set function T of [1], [2] and of Definition 1.2 below. We first expand T to its minimal closure T^* ([4], p. 61) of Definition 3.1 below and use T^* to obtain a modification of H. Hahn's prime parts of a continuum ([5], p. 225). But $\{T^*(x)\colon x \in S\}$ is not a decomposition for every compact continuum S. We next, by a chaining process, expand T^* to its recursive chain closure $M\mathrm{Ch}T$ of Definition 4.4 below. For compact continuum S, $\{M\mathrm{Ch}T(x)\colon x \in S\}$ is

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