Proof. Let k be a fixed integer different from 0. We have

(1) $$E \triangle t^k E = \{n \in N\colon \cos n \cdot \cos(n-k) < 0\}.$$

Moreover, the sequence $a_n = \cos n \cdot \cos(n-k)$ is almost periodic (in the sense of H. Bohr) and for some n takes a negative value. The last properties of $\{a_n\}$ and (1) easily imply that E is a s.a. set.

Let us observe finally that a reasoning analogous to the proof of Theorem gives the following:

PROPOSITION. *Let X be a totally disconnected, compact, Hausdorff space and let t be a homeomorphism of X onto itself. Let $\mathcal{A}$ be the Boolean algebra of all clopen subsets of X. The the following conditions are equivalent:*

(i) *The set of all positive iterations of t is equicontinuous.*

(ii) *The same for negative iterations.*

(iii) *For every clopen set $E \subset X$ there is a $k \neq 0$ such that $t^k E = E$.*

(iv) *The algebra $\mathcal{A}$ is a union of a family of t-invariant and finite subalgebras of $\mathcal{A}$.*

Reference

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Metrically generated probabilistic metric spaces *

by

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1. Introduction. A. Špaček [10] has introduced the concept of a random metric consisting of a set S together with a probability measure μ on the set of all mappings of $S \times S$ into the reals and such that $\mu(M) = 1$, M being the set of all ordinary metrics for S. In [5], Menger, Schweizer and Sklar clarified the relationship between this concept and that of a probabilistic (statistical) metric ([3], [4], [6], [7]) and showed that the condition $\mu(M) = 1$ is extremely restrictive.

In this paper we continue the study of this relationship by investigating the probabilistic metric spaces which are generated (in the sense of definition 1) by a random metric and show that they are indeed of a very special type (Theorems 2 and 4). In part 3, we obtain a representation theorem giving sufficient conditions for a given probabilistic metric space to be generated in this way. We conclude by showing that our representation theorem is a best possible result in this direction.

For explicit definitions the reader is referred either to the paper [7] by Schweizer and Sklar or the paper [11] by Thorp. Also, following a previously established convention, we shall abbreviate "probabilistic metric space" to "PM space".

2. Metrically generated PM spaces. Let S be a set, let $\mathfrak{D}$ be a collection of ordinary metrics for S, and let μ be a measure for $\mathfrak{D}$ (i.e., a non-negative, countably additive set function defined on a σ-algebra of subsets of $\mathfrak{D}$, called *μ-measurable sets*) such that

(A) *for any pair p, q of points in S and any real number x, the set $\{d \in \mathfrak{D};\ d(p,q) < x\}$ is μ-measurable, and*

(B) $\mu(\mathfrak{D}) = 1$.

From the μ-measurability of the sets in (A), it follows that for each pair p, q of points in S, $d(p,q)$ is a numerically valued random variable on $\mathfrak{D}$ whose distribution function F_{pq} is given by

(1) $$F_{pq}(x) = \mu\{d \in \mathfrak{D};\ d(p,q) < x\}.$$

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Let F be the mapping from $S\times S$ into the set of one-dimensional distribution functions defined by $F(p,q)=F_{pq}$. Then we have

THEOREM 1. *If S, $\mathfrak{D}$, μ satisfy* (A) *and* (B) *and F is defined by* (1), *then (S,F) is a* PM *space.*

Proof. It is immediate that:

(I) $F_{pq}(x)=1$ for all $x>0$ if and only if $p=q$;

(II) $F_{pq}(0)=0$;

(III) $F_{pq}=F_{pq}$; and

(IV) if $F_{pq}(x)=1$ and $F_{qr}(y)=1$, then $F_{pr}(x+y)=1$.

Motivated by Theorem 1, we make the following:

DEFINITION 1. A PM space (S,F) is *metrically generated* if and only if there exists $\mathfrak{D}$, μ such that

(1a) $S,\mathfrak{D},\mu$ satisfy (A) and (B), and

(1b) $F_{pq}(x)=\mu\{d\in\mathfrak{D};\ d(p,q)<x\}$, for all p,q in S and all real x.

THEOREM 2. *If (S,F) is a metrically generated* PM *space then:*

(i) *(S,F) is a Menger space under the t-norm T_m, defined by*

$$T_m(a,b)=\mathrm{Max}(a+b-1,0);$$

(ii) *Every distribution function F_{pq} $(p\neq q)$ is continuous at zero.*

Proof. To verify (i) we have to show that the Menger triangle inequality under T_m, i.e., the inequality

$$F_{pr}(x+y)\geqslant T_m\big(F_{pq}(x),F_{qr}(y)\big),$$

is satisfied. To this end, let p,q,r in S and $x,y\geqslant 0$ be given. Define

$$(*)\qquad \begin{aligned} E_1&=\{d;\ d(p,q)<x\},\\ E_2&=\{d;\ d(q,r)<y\},\\ E_3&=\{d;\ d(p,r)<x+y\}. \end{aligned}$$

Since $d(p,r)\leqslant d(p,q)+d(q,r)$, it follows that $E_1\cap E_2\subset E_3$ and

$$\begin{aligned} F_{pr}(x+y)&=\mu(E_3)\geqslant\mu(E_1\cap E_2)\\ &=\mu(E_1)+\mu(E_2)-\mu(E_1\cup E_2)\\ &=F_{pq}(x)+F_{qr}(y)-\mu(E_1\cup E_2)\\ &\geqslant F_{pq}(x)+F_{qr}(y)-1. \end{aligned}$$

Since $F_{pr}(x+y)$ is always $\geqslant 0$, the desired inequality follows.

To verify (ii), let $\{x_n\}$ be a decreasing sequence of positive real numbers whose limit is zero. Let

$$E_n=\{d;\ d(p,q)<x_n\};\qquad n=1,2,\ldots$$

Then $\lim E_n=\emptyset$ and, since $\mu(E_1)<\infty$,

$$\lim F_{pq}(x_n)=\lim\mu(E_n)=\mu(\emptyset)=0=F_{pq}(0),$$

which, together with the fact that $F_{pq}(x)=0$ for $x\leqslant 0$, shows that F_{pq} is continuous at zero.

Theorem 2 is a best possible result. This is shown by

THEOREM 3. *If T is a t-norm stronger than T_m, then there exists a metrically generated* PM *space (S,F) which is not a Menger space under T. Moreover, the space may be chosen so that the distance distribution functions F_{pq} $(p\neq q)$ are continuous.*

Proof. Since T is stronger than T_m, for some a,b from the open interval $(0,1)$ we have $T(a,b)>T_m(a,b)$. We consider two cases.

Case 1. $a+b\leqslant 1$. Let (S,F) be the PM space which is metrically generated by $S,\mathfrak{D},\mu$ where:

$$S=\{p,q,r\},\qquad \text{a three element set};$$
$$\mathfrak{D}=\{d_t;\ 0<t<1\},$$

where the d_t are metrics for S given by

$$d_t(p,q)=t+3,\quad d_t(q,r)=t+5,\quad d_t(p,r)=t+8,\quad \text{for}\quad 0<t\leqslant\tfrac12,$$

and

$$d_t(p,q)=t+5,\quad d_t(q,r)=t+3,\quad d_t(p,r)=t+8,\quad \text{for}\quad \tfrac12<t<1;$$

let g be the function defined on $(0,1)$ by

$$g(t)=\begin{cases} 8at, & 0<t\leqslant\tfrac12,\\ 8(1-a)(t-\tfrac12), & \tfrac12<t<1 \end{cases}$$

and let μ be the measure for $\mathfrak{D}$ defined by

$$\mu(D)=\int g(t)\,dt$$

where the integral is taken in the Lebesgue sense over the set of all t in $(0,1)$ for which $d_t\in D$, for any set $D\subset\mathfrak{D}$ for which this integral exists, i.e., for which $\{t;\ d_t\in D\}$ is Lebesgue measurable. Routine calculations show that the distance distribution functions F_{pq},F_{qr},F_{pr} are continuous everywhere and that

$$\begin{aligned} F_{pr}(4+4)=0=T_m(a,b)&<T(a,b)\leqslant T(a,1-a)\\ &=T\big(F_{pq}(4),F_{qr}(4)\big), \end{aligned}$$

whence (S,F) is not a Menger space under T.

Case 2. $a+b>1$. Let (S,F) be the PM space which is metrically generated by $S,\mathfrak{D},\mu$ where:

$$S=\{p,q,r\},\qquad \text{a three element set};$$
$$\mathfrak{D}=\{d_t;\ 0<t<1\},$$

where the d_t are metrics for S given by

$$\begin{array}{llll} d_t(p,q)=t+1, & d_t(q,r)=t+5, & d_t(p,r)=t+6, & \text{for}\quad 0<t\leqslant\frac{1}{3};\\ d_t(p,q)=t+2, & d_t(q,r)=t+2, & d_t(p,r)=t+4, & \text{for}\quad \frac{1}{3}<t\leqslant\frac{2}{3};\\ d_t(p,q)=t+5, & d_t(q,r)=t+1, & d_t(p,r)=t+6, & \text{for}\quad \frac{2}{3}<t<1. \end{array}$$

Let g be defined on $(0,1)$ by

$$g(t)=\begin{cases} 18(1-b)t, & 0<t\leqslant\frac{1}{3},\\ 18(a+b-1)(t-\frac{1}{3}), & \frac{1}{3}<t\leqslant\frac{2}{3},\\ 18(1-a)(t-\frac{2}{3}), & \frac{2}{3}<t<1, \end{cases}$$

and let μ be the measure for $\mathfrak{D}$ defined by

$$\mu(D)=\int g(t)\,dt$$

where the integral is taken as in the previous case.

As in case 1, the distance distribution functions are continuous and

$$F_{pr}(3+3)=a+b-1=T_m(a,b)<T(a,b)=T\big(F_{pq}(3),F_{qr}(3)\big),$$

so that (S,F) is not a Menger space under the t-norm T.

Schweizer, Sklar and Thorp ([8], Theorem 2) have shown that if (S,F) is a Menger space under a continuous t-norm, then the ε,λ-topology for S is metrizable. Since T_m is continuous, this, together with Theorem 2 above, yields

THEOREM 4. *If (S,F) is a metrically generated* PM *space, then the ε,λ-topology for S is metrizable.*

If, for some $k>0$, the kth moments of the random variables $d(p,q)$ are finite, then a metric for S can be explicitly exhibited in terms of these moments.

LEMMA. *Let (S,F) be a* PM *space, metrically generated by $(S,\mathfrak{D},\mu)$. Let $k>0$ be given and suppose that for any p, q in S, the k-th moment*

$$\int_{\mathfrak{D}} d^k(p,q)\,d\mu$$

of $d(p,q)$ is finite. Then the function δ_k from $S\times S$ into the non-negative reals defined by

$$\delta_k(p,q)=\begin{cases} \int_{\mathfrak{D}} d^k(p,q)\,d\mu & \text{if}\quad 0<k\leqslant 1,\\ \Big[\int_{\mathfrak{D}} d^k(p,q)\,d\mu\Big]^{1/k} & \text{if}\quad k>1, \end{cases}$$

is a metric for S.

Proof. For $0<k\leqslant 1$, d^k is a metric for S and the ordinary triangle inequality is satisfied. For $k>1$, the conclusion of the lemma is an immediate consequence of Minkowski's inequality.

THEOREM 5. *If the* PM *space (S,F) is metrically generated by $(S,\mathfrak{D},\mu)$ and if, for some $k>0$, the k-th moments of the random variables $d(p,q)$ are finite, then the ε,λ-topology for S is weaker than the δ_k-metric topology for S, in the sense that each ε,λ-open set is a δ_k-open set.*

Proof. By Theorem 7.2 of [7], the ε,λ-neighborhoods

$$N_p(\varepsilon,\lambda)=\{q\in S;\ F_{pq}(\varepsilon)>1-\lambda\},\qquad \varepsilon,\lambda>0,$$

form a base for the ε,λ-topology. Thus it suffices to show that any ε,λ-neighborhood of a point p in S contains an open sphere of the form

$$S_p(r)=\{q\in S;\ \delta_k(p,q)<r\}.$$

We shall show that for any $\varepsilon>0$, $\lambda>0$ and any $p\in S$:

(i) $$S_p(\varepsilon^k\lambda)\subset N_p(\varepsilon,\lambda)\quad\text{if}\quad 0<k\leqslant 1;$$

and

(ii) $$S_p(\varepsilon\lambda^{1/k})\subset N_p(\varepsilon,\lambda)\quad\text{if}\quad k>1.$$

Suppose $0<k\leqslant 1$ and that (i) does not hold. Then there exist $\varepsilon>0$, $\lambda>0$ and p, q in S such that

$$\delta_k(p,q)<\varepsilon^k\lambda\quad\text{and}\quad F_{pq}(\varepsilon)\leqslant 1-\lambda.$$

Let $E=\{d;\ d(p,q)\geqslant\varepsilon\}$. Then $\mu(E)\geqslant\lambda$ and

$$\delta_k(p,q)\geqslant\int_E d^k(p,q)\,d\mu\geqslant\int_E \varepsilon^k d\mu = \varepsilon^k\mu(E)\geqslant\varepsilon^k\lambda.$$

This contradiction completes the proof of (i).

If $k>1$ and (ii) does not hold, then there exist $\varepsilon>0$, $\lambda>0$ and p, q in S such that

$$\delta_k(p,q)<\epsilon\,\lambda^{1/k}\quad\text{and}\quad F_{pq}(\varepsilon)\leqslant 1-\lambda.$$

Let $E=\{d;\ d(p,q)\geqslant\varepsilon\}$. Then $\mu(E)\geqslant\lambda$ and

$$\delta_k(p,q)\geqslant\Big[\int_E d^k(p,q)\,d\mu\Big]^{1/k}\geqslant\Big[\int_E \varepsilon^k d\mu\Big]^{1/k} = \varepsilon\mu^{1/k}(E)\geqslant\varepsilon\lambda^{1/k}.$$

This contradiction completes the proof of (ii).

THEOREM 6. *If the* PM *space (S,F) is metrically generated by $S,\mathfrak{D},\mu$ and if for some $k>0$, the $(2k)$-th moments of the random variables $d(p,q)$ are finite and uniformly bounded, then the ε,λ-topology and the δ_k-metric topology for S are equivalent.*

Proof. By hypothesis, there is an $M>0$ such that, for all p, q in S,

$$\int_{\mathfrak{D}} d^{2k}(p,q)\,d\mu<M.$$

It follows from this inequality that the kth moments are finite. Thus, in view of Theorem 5, it suffices to show that for any $p \in S$ and any $\varepsilon > 0$, there exists $\lambda > 0$ such that

$$N_p(\lambda, \lambda) \subset S_p(\varepsilon) .$$

We again consider the two cases separately.

Case (i). $0 < k \leqslant 1$.

Let $p \in S$ and let $\varepsilon > 0$. Choose $\lambda > 0$ so small that

$$(M\lambda)^{1/2} + \lambda^k < \varepsilon ,$$

and suppose that $q \in N_p(\lambda, \lambda)$. Then $F_{pq}(\lambda) > 1 - \lambda$. Let $E = \{d;\ d(p,q) \geqslant \lambda\}$ and let χ_E be the characteristic function of E. Since $\mu(E) < \lambda$, we have

$$\begin{aligned}\delta_k(p,q) &= \int_E d^k(p,q)\,d\mu + \int_{\mathfrak{D}-E} d^k(p,q)\,d\mu \\ &= \int_E d^k(p,q)\,\chi_E\,d\mu + \int_{\mathfrak{D}-E} d^k(p,q)\,d\mu \\ &\leqslant \Big[\int_E d^{2k}(p,q)\,d\mu\Big]^{1/2}\Big[\int_E \chi_E^2\,d\mu\Big]^{1/2} + \lambda^k \int_{\mathfrak{D}-E} d\mu \\ &\leqslant M^{1/2}\mu^{1/2}(E) + \lambda^k < (M\lambda)^{1/2} + \lambda^k < \varepsilon .\end{aligned}$$

Thus, $q \in S_p(\varepsilon)$ and $N_p(\lambda, \lambda) \subset S_p(\varepsilon)$.

Case (ii). $k > 1$.

Let $p \in S$ and let $\varepsilon > 0$. Choose $\lambda > 0$ so that

$$(M\lambda)^{1/2} + \lambda^k < \varepsilon^k$$

and suppose that $q \in N_p(\lambda, \lambda)$. Then $F_{pq}(\lambda) > 1 - \lambda$. Let E and χ_E be as defined above. Since $\mu(E) < \lambda$, we have

$$\delta_k^k(p,q) = \int_{\mathfrak{D}} d^k(p,q)\,d\mu < (M\lambda)^{1/2} + \lambda^k < \varepsilon^k .$$

Thus $\delta_k(p,q) < \varepsilon$ and $q \in S_p(\varepsilon)$.

This shows that $N_p(\lambda, \lambda) \subset S_p(\varepsilon)$ and completes the proof.

3. Metric generation of PM spaces. The previous section was devoted primarily to the derivation of various properties of metrically generated PM spaces. Many of its results may be viewed as necessary conditions for a PM space to be metrically generated. In this section we consider the converse question—that is to say, we seek sufficient conditions for a PM space to be metrically generated. We begin with the following necessary and sufficient condition for a simple space (see [7]) to be metrically generated.

THEOREM 7. *Let (S, F) be a simple space generated by the metric space (S, d) and the distribution function G, so that for any $p \neq q$ in S and any $x \geqslant 0$, $F_{pq}(x) = G\big(x/d(p,q)\big)$. Then (S, F) is metrically generated if and only if G is continuous at zero.*

Proof. The necessity of the condition follows from Theorem 2. To show sufficiency, let $\mathfrak{D} = \{d_y;\ y > 0\}$ where the d_y are metrics for S defined by

$$d_y(p,q) = y \cdot d(p,q) \quad \text{for all } p, q \text{ in } S \text{ and all } y > 0 .$$

Let σ be the Lebesgue–Stieltjes measure for the open half line $y > 0$ determined by the distribution function G and let μ be the measure for $\mathfrak{D}$ defined by

$$\mu(D) = \sigma\{y > 0;\ d_y \in D\}$$

for all sets $D \subset \mathfrak{D}$ for which $\{y > 0;\ d_y \in D\}$ is σ-measurable. It is routine to show that $S, \mathfrak{D}, \mu$ satisfy (A) and (B). Furthermore, for all $p \neq q$ in S and for all real x we have

$$\begin{aligned}F_{pq}(x) = G\big(x/d(p,q)\big) &= \sigma\{y > 0;\ y < x/d(p,q)\} \\ &= \sigma\{y > 0;\ y \cdot d(p,q) < x\} \\ &= \mu\{d_y \in \mathfrak{D};\ d_y(p,q) < x\} .\end{aligned}$$

Thus (S, F) is metrically generated by $S, \mathfrak{D}, \mu$.

In order to present the main results concerning metric generation of PM spaces, we need the concept of the quasi-inverse of a distribution function, as introduced by Sklar in [9].

DEFINITION 2. Let F be a distribution function. The *quasi-inverse* F^* of F is the function defined on the open interval $0 < t < 1$ by

$$F^*(t) = \sup\{x;\ F(x) < t\} .$$

We also list the following conditions which a given PM space (S, F) may or may not satisfy:

(C) *For each $0 < t < 1$, the function d_t defined on $S \times S$ by*

$$d_t(p,q) = F_{pq}^*(t)$$

is a metric for S.

(D) *(S, F) is metrically generated by $S, \mathfrak{D}, \mu$ where $\mathfrak{D}$ is linearly ordered* ([1], p. 14) *by the relation:*

$$d_1 \leqslant d_2 \quad \textit{if and only if } d_1(p,q) \leqslant d_2(p,q) \quad \textit{for all } p, q \textit{ in } S .$$

(E) *(S, F) is a Menger space under the t-norm $T = \mathrm{Min}$, given by*

$$\mathrm{Min}(a,b) = \begin{cases} a, & \text{if } a \leqslant b , \\ b, & \text{if } a > b . \end{cases}$$

THEOREM 8. *If (S, F) is a PM space, then the following implications hold:*

$$(C) \Rightarrow (D) \Rightarrow (E) .$$

Proof. (i) Suppose that (S, F) satisfies (C). Let $\mathfrak{D} = \{d_t;\ 0 < t < 1\}$, where the d_t are the metrics for S given in (C), and let μ be the measure for $\mathfrak{D}$ defined by

$$\mu(D) = \nu\{0 < t < 1;\ d_t \in D\}$$

(where ν is the Lebesgue measure for $0 < t < 1$) for all sets D such that $\{t \in (0, 1);\ d_t \in D\}$ is Lebesgue measurable.

It is straightforward to show that (S, F) is metrically generated by S, $\mathfrak{D}$, μ. Further, $\mathfrak{D}$ is linearly ordered by $\leqslant$ since the quasi-inverses F^*_{pq} are non-decreasing functions of t on $(0, 1)$. Thus, condition (D) is satisfied.

(ii) Suppose that (S, F) satisfies condition (D). Let p, q, r be in S and let $x, y \geqslant 0$. Let E_1, E_2, E_3 be defined as in $(*)$ of the proof of Theorem 2. We first show that either $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$. To this end, suppose that $E_1 \not\subseteq E_2$ and that $d_1 \in E_1 - E_2$. Then, since $\mathfrak{D}$ is linearly ordered, we have

(a) for all $d \leqslant d_1$, $d \in E_1$,

(b) $d_1(q, r) \geqslant y$,

(c) for all $d \in E_2$, $d(q, r) < y < d_1(q, r)$ and $d \leqslant d_1$.

From (a) and (c) it follows that $E_2 \subseteq E_1$, whence either $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$. Finally, since $E_1 \cap E_2 \subseteq E_3$,

$$\begin{aligned} F_{pr}(x+y) = \mu(E_3) \geqslant \mu(E_1 \cap E_2) &= \operatorname{Min}\big(\mu(E_1), \mu(E_2)\big) \\ &= \operatorname{Min}\big(F_{pq}(x), F_{qr}(y)\big) . \end{aligned}$$

Thus, (S, F) is a Menger space under Min and the proof is complete.

THEOREM 9. *If (S, F) is a PM space whose distance distribution functions F_{pq} $(p \neq q)$ are continuous then* (C), (D), *and* (E) *are equivalent.*

Proof. In view of Theorem 8, it suffices to show that (E) implies (C). Suppose therefore that (S, F) satisfies condition (E). Let $0 < t < 1$. We shall show that the function d_t given in (C) is a metric for S. Clearly, for any $t \in (0, 1)$ and all p, q in S,

(a) $d_t(p, q) \geqslant 0$, and $d_t(p, q) = 0$ if and only if $p = q$,

(b) $d_t(p, q) = d_t(q, p)$.

If p, q, r are distinct points of S then, since F_{pq} and F_{qr} are continuous, there exist x, $y > 0$ such that

$$t = F_{pq}(x) = F_{qr}(y)$$

and

$$x = F^*_{pq}(t) , \quad y = F^*_{qr}(t) .$$

Using the assumption that (S, F) is a Menger space under the t-norm $T = \operatorname{Min}$, we have

$$F_{pr}\big(F^*_{pq}(t) + F^*_{qr}(t)\big) = F_{pr}(x+y) \geqslant \operatorname{Min}\big(F_{pq}(x), F_{qr}(y)\big) = \operatorname{Min}(t, t) = t .$$

It follows that $F^*_{pq}(t) + F^*_{qr}(t) \geqslant F^*_{pr}(t)$; i.e., that

(c) $d_t(p, r) \leqslant d_t(p, q) + d_t(q, r)$.

(We note that (c) is obviously satisfied if p, q, r are not all distinct.) Thus d_t is a metric for S and (S, F) satisfies condition (C).

The following theorem follows immediately from Theorem 9 and is the main result concerning metric generation of PM spaces.

THEOREM 10. *If (S, F) is a Menger space under the t-norm $T = $* Min *and if each distance distribution function F_{pq} $(p \neq q)$ is continuous, then (S, F) is metrically generated.*

Since Min is the strongest possible t-norm, one might conjecture that Theorem 10 admits a considerable improvement. This, however, is not the case. For our final result, we show that Theorem 10 is best possible in the sense that if the t-norm Min is replaced by any weaker continuous t-norm, then the resulting proposition is false.

THEOREM 11. *Let T be a continuous t-norm which is weaker than* Min. *Then there exists a PM space (S, F) such that*

(i) *(S, F) is a Menger space under T;*

(ii) *each F_{pq} $(p \neq q)$ is continuous;*

(iii) *(S, F) is not metrically generated.*

Proof. The hypothesis implies that there exists $0 < a < 1$ such that $T(a, a) < a$. Let (S, F) be the PM space defined by:

(1) $S = \{p, q, r\}$, a three element set;

(2) The distance distribution functions F_{pq}, F_{qr}, F_{pr} are

$$F_{pq}(x) = F_{qr}(x) = \begin{cases} 0 , & 0 \leqslant x \leqslant \frac{1}{4} , \\ a(4x-1) , & \frac{1}{4} < x \leqslant \frac{1}{2} , \\ a , & \frac{1}{2} < x \leqslant \frac{3}{4} , \\ (1-a)(4x-3)+a , & \frac{3}{4} < x \leqslant 1 , \\ 1 , & 1 < x ; \end{cases}$$

and

$$F_{pr}(x) = \sup_{0 \leqslant t \leqslant x} T\big(F_{pq}(x-t), F_{qr}(t)\big) \quad \text{for all } x \geqslant 0 .$$

It is easily verified that (S, F) satisfies (i) and (ii). Also,

$$F_{pr}(1) = T(a, a) \quad \text{and} \quad F_{pr}(1.5) = T(1, a) = a .$$

Suppose now that there exists $\mathfrak{D}$ and μ such that (S, F) is metrically generated by $S, \mathfrak{D}, \mu$. Then we would have

$$\begin{aligned}
1-a = 1-F_{pr}(1.5) &= 1-\mu\{d;\ d(p,r) < 1.5\} \\
&= \mu\{d;\ d(p,r) \geqslant 1.5\} \\
&\leqslant \mu\{d;\ d(p,r) \geqslant 1.5 \text{ and } d(p,q) < .5 \text{ and } d(q,r) < .5\} \\
&\quad + \mu\{d;\ d(p,r) \geqslant 1.5 \text{ and } d(p,q) < .5 \text{ and } .5 \leqslant d(q,r) < 1\} \\
&\quad + \mu\{d;\ d(p,r) \geqslant 1.5 \text{ and } .5 \leqslant d(p,q) < 1 \text{ and } d(q,r) < .5\} \\
&\quad + \mu\{d;\ d(p,r) \geqslant 1.5 \text{ and } .5 \leqslant d(p,q) < 1 \text{ and } .5 \leqslant d(q,r) < 1\} \\
&\quad + \mu\{d;\ d(p,q) \geqslant 1 \text{ or } d(q,r) \geqslant 1\},
\end{aligned}$$

since the μ-measurable sets on the right side of the inequality cover the set $\{d;\ d(p,r) \geqslant 1.5\}$. The ordinary triangle inequality implies that the first three terms on the right side are zero. The fifth term is zero since $F_{pq}(1) = F_{qr}(1) = 1$. Thus,

$$\begin{aligned}
1-a &\geqslant \mu\{d;\ d(p,r) \geqslant 1.5 \text{ and } .5 \leqslant d(p,q) < 1 \text{ and } .5 \leqslant d(q,r) < 1\} \\
&\leqslant \mu\{d;\ d(p,q) \geqslant .5 \text{ and } d(q,r) \geqslant 5\} \\
&= 1-\mu\{d;\ d(p,q) < .5 \text{ or } d(q,r) < .5\} \\
&= 1-[\mu\{d;\ d(p,q) < .5\} + \mu\{d;\ d(q,r) < .5\} \\
&\qquad\qquad - \mu\{d;\ d(p,q) < .5 \text{ and } d(q,r) < .5\}] \\
&= 1-F_{pq}(.5)-F_{qr}(.5)+\mu\{d;\ d(p,q) < .5 \text{ and } d(q,r) < .5\} \\
&\leqslant 1-a-a+\mu\{d;\ d(p,r) < 1\} \\
&= 1-2a+F_{pr}(1) = 1-2a+T(a,a).
\end{aligned}$$

Thus $1-a \leqslant 1-2a+T(a,a)$ and $a \leqslant T(a,a)$, which is contrary to the hypothesis. Therefore, (S, F) is not metrically generated and the proof is complete.

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