Translation properties of finite partitions of the positive integers

by

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1. Introduction. Let N be the set of natural numbers $\{1, 2, \ldots\}$, βN the Stone–Čech compactification of N, and $N' = \beta N - N$ (the set-theoretic difference). If $A \subset N$, and if $\bar{A}$ is its closure in βN, A' denotes $\bar{A} \cap N'$; A' is open and closed in N'. All open-closed subsets of N' are obtained in this way, and they form a basis for the topology of N', which is a compact Hausdorff space. We shall call A an *antecedent* of A'. If A and B are each antecedents of A', then $A-B$ and $B-A$ are each finite sets. Let t: $N \to N$ be defined by $t(n) = n+1$; then t, by continuous extension to $\bar{N}$, induces a homeomorphism of N' onto N'. This homeomorphism will still be denoted by t. We denote by T the family $\{t^k: k = 0, 1, 2, \ldots\}$, where t^0 indicates the identity mapping. If $A \subset N$ and $t^k \in T$, $t^k(A') = (t^k A)'$.

A subset $K \subset N'$ is called *t-invariant* if $tK = K$. T is then a family of homeomorphisms of K onto K. We shall call T *equicontinuous on* K if: Given any covering of K by open sets $\{W_\alpha: \alpha \in A\}$, there exists another covering $\{V_\beta: \beta \in B\}$ such that each translate $t^k V_\beta$ is contained in some set W_α, α depending on k and β.

THEOREM 1. *If K is a closed t-invariant subset of N', then T is not equicontinuous on K.*

The proof of Theorem 1 is given in [1] Sec. 3.2, and will not be reproduced here. It is based on a theorem of W. Rudin ([2], Theorem 6).

The purpose of the present paper is to re-interpret Theorem 1 in strictly set-theoretic terms about N.

2. Restatement of the theorem. Non-equicontinuity of T on K means there exists a certain open covering $\mathfrak{W} = \{W_\alpha\}$ of K such that if $\mathfrak{V} = \{V_\beta\}$ is any other such covering of K, $t^p V_\beta$ will lie in no member of $\mathfrak{W}$, provided p and β are suitably chosen. Since the open-closed sets are a basis for the topology of K, $\mathfrak{W}$ may be replaced by a finer covering $\mathfrak{U}$, whose members are open-closed, and $t^p V_\beta$ will lie in no member of $\mathfrak{U}$. Since K is compact, a finite subfamily of $\mathfrak{U}$ will still cover K, and have the same property. If $\{U_1, U_2, \dots, U_n\}$ is this subfamily, we may put $W_1 = U_1$, $W_2 = U_2 - U_1, \dots, W_n = U_n - \bigcup_1^{n-1} U_j$. Then $\{W_i\colon i = 1, 2, \dots, n\}$ is a *partition* of K into open-closed subsets, and this partition may be used in place of the original covering $\mathfrak{W}$ to give the same conclusion.

Furthermore, $W_i = N'_i \cap K$ for a suitable choice of N'_i open-closed in N'. If we put $M'_1 = N'_1 \cup \{N' - \bigcup_1^n N'_i\}$, $M'_2 = N'_2 - N'_1, \dots, M'_n = N'_n - \bigcup_1^{n-1} N'_j$, we arrive at a partition of N' into open-closed sets $\{M'_i\}$ such that $\{M'_i \cap K\colon i = 1, 2, \dots, n\}$ is as good as the original covering $\mathfrak{W}$. And if the covering $\mathfrak{V}$ of the theorem also happens to be derived from a partition of N' into open-closed subsets, the conclusion still follows. Thus we have proved

THEOREM 2. *Let K be a closed t-invariant subset of N'. Then there exists a partition $\mathfrak{M} = \{M'_i\colon i = 1, 2, \dots, n\}$ of N' into open-closed subsets such that if $\mathfrak{V} = \{V'_i\colon i = 1, 2, \dots, k\}$ is another such partition of N', there exist values of p and j such that $t^p(V'_j \cap K)$ is contained in no member of $\mathfrak{M}$.*

3. t-ideals and t-filters.

If $\mathfrak{A}$ is a family of subsets of N, we shall call $\mathfrak{A}$ a *t-ideal* if it satisfies the following conditions:

(a) $A \in \mathfrak{A}$, $B \in \mathfrak{A}$ *implies* $A \cup B \in \mathfrak{A}$.

(b) $A \in \mathfrak{A}$, $B \subset A$ *implies* $B \in \mathfrak{A}$.

(c) $N \notin \mathfrak{A}$.

(d) $A \in \mathfrak{A}$, $t^k \in T$ *implies* $t^k A \in \mathfrak{A}$, *and conversely*, $t^k A \in \mathfrak{A}$ *implies* $A \in \mathfrak{A}$.

Properties (a)-(c) define $\mathfrak{A}$ as a proper ideal in the Boolean algebra of all subsets of N; (d) is an extra requirement for present purposes. The simplest and smallest non-zero t-ideal is the class $\mathfrak{F}$ of all finite subsets of N.

Corresponding to each t-ideal $\mathfrak{A}$ is its dual, the *t-filter* $\mathfrak{A}^c = \{N - A\colon A \in \mathfrak{A}\}$. $\mathfrak{A}^c$ is a filter and is translation-invariant, i.e. $\mathfrak{A}^c$ also satisfies (d).

If we put $K = \bigcap \{B'\colon B \in \mathfrak{A}^c\}$ and $Z = \bigcup \{A'\colon A \in \mathfrak{A}\}$, then it is easily verified that K is closed in N', Z is open in N', $K \cap Z = \emptyset$, $K \cup Z = N'$, and that K and Z are each t-invariant subsets of N'. We shall call K the *support* of $\mathfrak{A}$ (or of $\mathfrak{A}^c$) and Z the *null set* of $\mathfrak{A}$ (or of $\mathfrak{A}^c$).

This terminology derives from a popular measure-theoretic interpretation of certain t-ideals. If m is a *Banach mean*, i.e. a finitely additive, positive, t-invariant set-function on all the subsets of N, with $m(N) = 1$, then $\mathfrak{A} = \{A \subset N\colon m(A) = 0\}$ is a t-ideal. On the other hand, m induces a countable additive t-invariant probability measure on N', for which K turns out to be exactly the support set.

Dually, we may begin with any closed t-invariant non-empty subset $K \subset N'$, and define $\mathfrak{A}^c = \{B \subset N\colon B' \supset K\}$. Then the dual family $\mathfrak{A}$ is

a t-ideal and K is its support. It is, however, worth observing that since not every such K is the support of a Banach measure (N' itself, for example), not all t-ideals are obtained from Banach means.

4. The non-equicontinuity theorem on N.

Let $\mathfrak{A}$ be any t-ideal, and let $\mathfrak{M} = \{M_i\}$ and $\mathfrak{V} = \{V_i\}$ each be a finite family of pairwise disjoint subsets of N. We say that $\mathfrak{V}$ is an *$\mathfrak{A}$-refinement* of $\mathfrak{M}$ if each member of $\mathfrak{V}$ is contained, except for a set in $\mathfrak{A}$, in some member of $\mathfrak{M}$. Thus $\mathfrak{V}$ is an $\mathfrak{A}$-refinement of $\mathfrak{M}$ if, given $V_i \in \mathfrak{V}$, there exists a set $M_j \in \mathfrak{M}$ and a set $A \in \mathfrak{A}$ such that $V_i \subset M_j \cup A$. If $\mathfrak{V}$ is as above, and $t^k \in T$, $t^k\mathfrak{V}$ will denote the (pairwise disjoint) family $\{t^k V_i\colon V_i \in \mathfrak{V}\}$.

THEOREM 3. *Let $\mathfrak{A}$ be any t-ideal on N. Then there exists a finite partition $\mathfrak{M}$ of N such that if $\mathfrak{V}$ is any other finite partition, some translate of $\mathfrak{V}$, $t^k\mathfrak{V}$, is not an $\mathfrak{A}$-refinement of $\mathfrak{M}$.*

Proof. Let K be the support of $\mathfrak{A}$, and let $\mathfrak{M}'$ be the partition of N' which by Theorem 2 corresponds to K, say $\mathfrak{M}' = \{M'_i\colon i = 1, 2, \dots, n\}$. Let $\mathfrak{M} = \{M_i\colon i = 1, 2, \dots, n\}$ be a set of antecedents to the sets M'_i, so chosen as to constitute a partition of N'; then $\mathfrak{M}$ has the required property. Indeed, suppose that $\mathfrak{V} = \{V_i\colon i = 1, 2, \dots, k\}$ is any partition of N. Then $\{V'_i\colon i = 1, 2, \dots, k\}$ partitions N', and by Theorem 2 there exist p, j such that $t^p(V'_j \cap K)$ is contained in no member of $\mathfrak{M}'$. Then it will follow that $t^p V_j \subset M_r \cup A$ for some $A \in \mathfrak{A}$ and some $M_r \in \mathfrak{M}$ is impossible, by the following reasoning:

If $t^p V_j \subset M_r \cup A$, then

$$t^p(V'_j) \subset M'_r \cup A', \quad \text{and} \quad t^p(V'_j) \cap K \subset (M'_r \cup A') \cap K .$$

But A' is in the null set Z, i.e. $A' \cap K = \emptyset$, so we have

$$t^p(V'_j) \cap K \subset M'_r \cap K .$$

Since $t^p K = K$,

$$t^p(V'_j) \cap K = t^p(V'_j) \cap t^p K = t^p(V'_j \cap K) ,$$

and we end with

$$t^p(V'_j \cap K) \subset M'_r \cap K .$$

This last set is a subset of a member o1 $\mathfrak{M}'$, which contradicts Theorem 2.

COROLLARY. *There exists a finite partition $\mathfrak{M}$ of N such that if $\mathfrak{V}$ is another finite partition of N, some translate t^pV of one of the members V of $\mathfrak{V}$ intersects at least two members of $\mathfrak{M}$ in an infinite set.*

Proof. This is merely Theorem 3 for the case where $\mathfrak{A}$ is the ideal $\mathfrak{F}$ of all finite subsets of N. The support K, in this case, is N' itself.

5. Generalizations.

The translation mapping $t\colon n \to n+1$ is not the only mapping of N into N for which the entire discussion given above is true *verbatim*. Rudin's theorem, on which Theorem 1 is based, requires only that t be a 1:1 mapping with the property that the induced homeomorphism on N' have no periodic points on the support set K in question. It is not even necessary that this homeomorphism of K be onto. A sufficient condition for t to have no finite orbits (i.e. no periodic points) in N' is that t have no finite orbits in N, but this is far from necessary. [1] contains a discussion of this question.

References

[1] R. A. Raimi, *Homeomorphisms and invariant measures for $\beta N - N$*, Duke Math. Journ. 33 (1966), pp. 1-12.

[2] W. Rudin, *Averages of continuous functions on compact spaces*, ibid. 25 (1958), pp. 197-204.

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Remark on Raimi's theorem on translations

by

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The present note gives an alternative proof of Theorem 3 from Ralph A. Raimi's paper *Translation properties of finite partitions of the positive integers* [1] in a changed and refined form. All notations and terminology are preserved. Only, for the sake of simplicity, the set of natural numbers is replaced by the set N of all integers. The author claims that this modifications is completely irrelevant. To make this note self-contained all definitions of the notions used below are reproduced here.

Let t denote the transformation $t(x) = x+1$ in the set N. Let $\mathfrak{A}$ be an arbitrary proper ideal of subsets of N which is *t-invariant* (i.e. $E \in \mathfrak{A}$ iff $tE \in \mathfrak{A}$ for all $E \subset N$). A partition $\mathfrak{B}$ of the set N is an *$\mathfrak{A}$-refinement* of a partition $\mathfrak{M}$ iff each member of $\mathfrak{B}$ is contained mod $\mathfrak{A}$ in some member of $\mathfrak{M}$. A set $E \subset N$ is called *strongly aperiodic* (s.a.) iff for every integer $k \neq 0$ we have

$$\bigcup_{j=0}^{m} t^j(E \triangle t^k E) = N$$

for some m. ($X \triangle Y = X \cup Y \setminus X \cap Y$.)

THEOREM. *If a set $E \subset N$ is s.a., then for any finite partition $\mathfrak{B}$ of N there is a translate $t^k\mathfrak{B}$ which is not an $\mathfrak{A}$-refinement of the partition $\mathfrak{M} = \{E, N\setminus E\}$.*

Proof. Suppose to the contrary that all $t^k\mathfrak{B}$ ($k = 1, 2, \ldots$) for some finite $\mathfrak{B} = \{V_i\}_{i\in I}$ are $\mathfrak{A}$-refinements of $\mathfrak{M}$. Since the set I is finite, there is a partition $I = I_0 \cup (I\setminus I_0)$ and two translations t^p and t^q ($p \neq q$) such that

$$t^pV_i\setminus E \in \mathfrak{A} \quad \text{and} \quad t^qV_i\setminus E \in \mathfrak{A} \quad \text{for} \quad i \in I_0,$$
$$t^pV_i\setminus(N\setminus E) \in \mathfrak{A} \quad \text{and} \quad t^qV_i\setminus(N\setminus E) \in \mathfrak{A} \quad \text{for} \quad i \in I\setminus I_0.$$

Consequently, for $k = p-q$, the set E is t^k-invariant mod $\mathfrak{A}$ (i.e. $E \triangle t^kE \in \mathfrak{A}$). Hence in view of strong aperiodicity of E we get a contradiction with our assumption $N \notin \mathfrak{A}$.

The next lemma shows the existence of s.a. sets.

LEMMA. *The set $E = \{n \in N: \cos n > 0\}$ is s.a.*