Some consequences of the normality of the space of models *

by

Hidemitsu Sayeki (Montreal, Quebec)

The main purpose of the present paper is to give a new, and rather simple proof of Craig's Interpolation Theorem in first order predicate calculus [4]. We develop a bit further our knowledge of topological properties of the space of models, and we show that the Interpolation Theorem is a consequence of the Hausdorff compactness of the space of models.

It was Tarski's paper which interpreted for the first time Gödel's completeness theorem as stating the compactness of a certain topological space [11]. Afterwards E. W. Beth proved by a topological method the theorem of Gödel and that of Löwenheim-Skolem [1]. Among subsequent works in the topological treatment of the space of models we are indebted above all to Taimanov's paper in which a characterization of elementary classes was given [10].

In the following discussions we assume the Compactness Theorem of Gödel–Henkin–Malcev:

THEOREM 0. *Every consistent set of sentences has a model,*

in other words,

THEOREM 0′. *If every finite subset of Σ has a model, then Σ has a model.*

We shall not make use of AC nor GHC, but we assume elementary facts concerning general topology, for which we can refer to [6] and [9]. Our notations are due to A. Tarski [11], [12] and J. Keisler [5].

1. Preliminaries. We assume that ordinal numbers have been defined so that each ordinal number coincides with the set of all smaller ordinal numbers. Cardinal number will be understood as the corresponding initial ordinal number.

Throughout this paper set-theoretical notations ϵ, $\subseteq$, $\cup$, $\cap$ will be used as usual. CK, $\mathrm{Card}\,K$ will mean the *complement* and the *cardinality* of a set K, respectively.

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* This paper is dedicated to Professor Motokichi Kondo on his 60th anniversary. This paper was announced on the 4th of April, 1966 at the meeting of Association for Symbolic Logic in New York.

By a *type* we mean a function μ such that the domain of μ is an ordinal number ϱ and its range $\subseteq \omega$. By a *language* $L(\mu)$ of a type μ, we shall mean the first order predicate calculus with denumerable number of individual variables and $\mu(\lambda)$-placed predicate symbol P_λ $(\lambda < \varrho)$ besides identity. We shall use the symbols $\vee$, $\neg$, $\exists$ in the usual way. We assume that the notions of formula, free variable and sentence are known. $\vdash$ will mean deductive inference in the language and $\mathrm{Cn}(\Sigma)$ will mean the *set of all deductive inferences* of a set Σ of sentences. By $S(\mu)$ we shall mean the *class of all sentences* in $L(\mu)$. A system $\mathfrak{M} = \langle M, R_\lambda\rangle_{\lambda<\varrho}$ is said to be an *interpretation* of a language $L(\mu)$, or a *structure of type* μ, if M is a non-empty set and $R_\lambda \in 2^{M^{\mu(\lambda)}}$ for $\lambda < \varrho$. We shall denote by $\widetilde{M}(\mu)$ the *class of all structures of type* μ, i.e. $\widetilde{M}(\mu)$ is the class of all interpretations of the language $L(\mu)$. We assume that the definition of truth in an interpretation is also known. An interpretation $\mathfrak{M} \in \widetilde{M}(\mu)$ is said to be a *model of a sentence* $X \in S(\mu)$, in symbol: $\models_{\mathfrak{M}} X$, if and only if X is true in $\mathfrak{M}$. Let $\Sigma \subseteq S(\mu)$. We shall mean by Σ^* the class of all interpretations $\mathfrak{M}$ in $\widetilde{M}(\mu)$ such that $\models_{\mathfrak{M}} X$ for every X in Σ. If $\Sigma = \{X\}$, we write X^* instead of $\{X\}^*$. In other words, $\Sigma^* = \bigcap\{X^*\colon X \in \Sigma\}$. Dually, if $K \subseteq \widetilde{M}(\mu)$ we shall mean by K^* the class of all sentences X in $S(\mu)$ such that $\models_{\mathfrak{M}} X$ for every $\mathfrak{M}$ in K. If $K = \{\mathfrak{M}\}$ we write $\mathfrak{M}^*$ for $\{\mathfrak{M}\}^*$. We note that $K \subseteq (K^*)^*$, $\Sigma \subseteq (\Sigma^*)^*$ and $X \vdash Y$ if and only if $X^* \subseteq Y^*$.

Let $K \subseteq \widetilde{M}(\mu)$. K is said to be an *elementary class*, or to be *finitely axiomatizable*, in symbol: $K \in \mathrm{EC}$, if and only if there is a sentence X in $S(\mu)$ such that $K = X^*$. K is said to be an *elementary class in wider sense*, or to be *axiomatizable*, in symbol: $K \in \mathrm{EC}_\Delta$, if and only if there is a set Σ of sentences in $S(\mu)$ such that $K = \Sigma^*$. Every axiomatizable class is an intersection of finitely axiomatizable classes.

2. Space of models. Two interpretations $\mathfrak{M}$, $\mathfrak{N}$ in $\widetilde{M}(\mu)$ are said to be *elementarily equivalent* in $L(\mu)$, in symbol: $\mathfrak{M} \approx \mathfrak{N}$, if and only if for any sentence X in $S(\mu)$, $\models_{\mathfrak{M}} X$ implies $\models_{\mathfrak{N}} X$ and *vice versa*. A class K of interpretations in $\widetilde{M}(\mu)$ is called *elementarily closed*, in symbol: $K \in \mathrm{ACL}$, if and only if $\mathfrak{M} \in K$ and $\mathfrak{M} \approx \mathfrak{N}$ imply $\mathfrak{N} \in K$. Note that $K \in \mathrm{EC}_\Delta$ implies $K \in \mathrm{ACL}$. We shall denote by $M(\mu)$ the family of equivalence classes of $\widetilde{M}(\mu)$ with respect to elementary equivalence relation. Natural mapping of $\widetilde{M}(\mu)$ onto $M(\mu)$ will be denoted by σ_μ (or by σ, in the case free from ambiguity). We introduce a topology into $M(\mu)$ taking $\{\sigma X^*\colon X \in S(\mu)\}$ as a basis for open sets. In fact, every interpretation in $\widetilde{M}(\mu)$ is a model of some sentences. If $\sigma\mathfrak{M} \in \sigma X^* \cap \sigma Y^*$, then $\sigma\mathfrak{M} \in \sigma(X \wedge Y)^* \subseteq \sigma X^* \cap \sigma Y^*$. And if $\sigma\mathfrak{M}$, $\sigma\mathfrak{N} \in \sigma X^*$ and $\sigma\mathfrak{M} \neq \sigma\mathfrak{N}$, then there is a $Y \in S(\mu)$ such that $\mathfrak{N} \in Y^*$, $\mathfrak{M} \in (\neg Y)^*$, whence $\sigma\mathfrak{N} \in \sigma(X \wedge Y)^* \subseteq \sigma X^*$.

THEOREM 2.1 (Taimanov [10]). *Let $K \subseteq \widetilde{M}(\mu)$. $K \in \mathrm{EC}_\Delta$ if and only if $K = \sigma^{-1}(F)$ for some closed set F in $M(\mu)$.*

Proof. Let us assume $K = \Sigma^*$, $\Sigma \subseteq S(\mu)$. It is sufficient to show that $\sigma\Sigma^*$ is closed since then K is elementarily closed. If $\sigma\mathfrak{M} \in M(\mu) - \sigma\Sigma^*$, then we can find a sentence X in Σ such that $\sigma\mathfrak{M} \notin \sigma X^*$, whence $\sigma\mathfrak{M} \in \sigma(\neg X)^*$. And $\sigma(\neg X)^* \cap \sigma\Sigma^* = 0$ implies $\sigma(\neg X)^* \subseteq M(\mu) - \sigma\Sigma^*$.

Conversely, suppose that $K = \sigma^{-1}(F)$ and F is closed in $M(\mu)$. Let $\sigma\mathfrak{M} \notin \sigma K = F$. Then there is a sentence X in $S(\mu)$ such that $\sigma\mathfrak{M} \in \sigma X^*$ and $\sigma X^* \cap \sigma K = 0$. Let

$$\Sigma = \{\neg X;\ X \in S(\mu),\ \sigma\mathfrak{M} \in \sigma X^*,\ \sigma X^* \cap \sigma K = 0\}\,.$$

Then

$$\begin{aligned}\sigma\Sigma^* &= \bigcap\{\sigma(\neg X)^*;\ X \in S(\mu),\ \sigma\mathfrak{M} \in \sigma X^*,\ \sigma X^* \cap \sigma K = 0\}\\ &= \bigcap\{\sigma X^*;\ X \in S(\mu),\ \sigma\mathfrak{M} \notin \sigma X^*,\ \sigma K \subseteq \sigma X^*\} = \sigma K\,,\end{aligned}$$

since σK is closed. So we have $K = \Sigma^*$, for $K = \sigma^{-1}(F)$.

COROLLARY 2.2. *Assume $K \subseteq \widetilde{M}(\mu)$. $K \in \mathrm{EC}$ if and only if $K = \sigma^{-1}(F)$, F is a clopen (i.e., open and closed simultaneously) subset of $M(\mu)$.*

THEOREM 2.3. *$M(\mu)$ is a compact Hausdorff topological space.*

Proof. Let $\sigma\mathfrak{M}$ and $\sigma\mathfrak{N}$ be two distinct elements of $M(\mu)$. Then there is a sentence X such that $\models_{\mathfrak{M}} X$ and not $\models_{\mathfrak{N}} X$. Hence $\sigma\mathfrak{M} \in \sigma X^*$, $\sigma\mathfrak{N} \in \sigma(\neg X)^*$ and $\sigma X^* \cap \sigma(\neg X)^* = 0$.

Let $\mathcal{F} = \{F_i;\ i \in I\}$ be a family of closed sets in $M(\mu)$ which has the finite intersection property. By Theorem 2.1 we can assume that $F_i = \Sigma_i^*$, $\Sigma_i \subseteq S(\mu)$ for each $i \in I$. Let $\Sigma = \bigcup\{\Sigma_i;\ i \in I\}$. Then for every finite set Σ' of Σ we can find a finite subset I' of indices of I such that $\bigcup\{\Sigma_i;\ i \in I'\}$ includes Σ'. By the finite intersection property $\bigcap\{F_i;\ i \in I'\}$ is non-void, i.e., $\bigcup\{\Sigma_i;\ i \in I'\}$ has a model which is a model of Σ'. Hence, by Theorem 0′, Σ has a model, i.e., $\mathcal{F}$ has a non-void intersection.

COROLLARY 2.4. *$M(\mu)$ is a normal space.*

Proof. Due to the fact that every compact Hausdorff space is normal.

3. Continuous mappings on $M(\mu)$. Suppose that μ' is a type which includes a type μ. Let the domain of μ' be an ordinal number ϱ'. Every formula defined in $L(\mu)$ is also a formula of $L(\mu')$. By the *μ-reduct* of an interpretation $\langle M, R_\lambda\rangle_{\lambda<\varrho'}$ of $L(\mu')$ we shall mean the interpretation $\langle M, R_\lambda\rangle_{\lambda<\varrho}$ of $L(\mu)$. Let $\mathfrak{M}$, $\mathfrak{N}$ be two interpretations of $L(\mu')$. If $\mathfrak{M}$, $\mathfrak{N}$ are elementarily equivalent in $L(\mu')$, then the μ-reducts of $\mathfrak{M}$ and $\mathfrak{N}$ are also elementarily equivalent in $L(\mu)$. A mapping $\pi_{\mu'\to\mu}$ defined on $M(\mu')$ is called the *μ-reduction* of $M(\mu')$ if and only if it assigns to every element $\sigma\mathfrak{M}$ of $M(\mu')$ the element of $M(\mu)$ whose representative is the μ-reduct of $\mathfrak{M}$. $A, B \subseteq M(\mu')$ and $A \subseteq B$ imply $\pi_{\mu'\to\mu}(A) \subseteq \pi_{\mu'\to\mu}(B)$. If $\mu'' \supseteq \mu' \supseteq \mu$, then $\pi_{\mu''\to\mu} = \pi_{\mu'\to\mu}\cdot\pi_{\mu''\to\mu'}$. When no confusion seems possible we may forget to mention the domain of μ-reduction and denote it by π_μ.

3.1. *Let $\mu' \supseteq \mu$ and let $K' \subseteq M(\mu')$. If $K = \pi_\mu(K')$, then $K^* = (K')^* \cap \cap S(\mu)$.*

Proof. Note that if $\mathfrak{M}' \in M(\mu')$, then $\mathfrak{M} = \pi_\mu(\mathfrak{M}')$ implies $\mathfrak{M}^* = (\mathfrak{M}')^* \cap S(\mu)$. Hence

$$K^* = \bigcap \{\mathfrak{M}^*;\ \mathfrak{M} = \pi_\mu(\mathfrak{M}'),\ \mathfrak{M}' \in K'\}$$
$$= \bigcap \{(\mathfrak{M}')^* \cap S(\mu);\ \mathfrak{M}' \in K'\} = (K')^* \cap S(\mu)\,.$$

THEOREM 3.2. *Let $\mu' \supseteq \mu$. Then the μ-reduction of $M(\mu')$ is a continuous mapping of $M(\mu')$ onto $M(\mu)$.*

Proof. Assume that X is a sentence in $S(\mu)$. Then X is true in an interpretation $\mathfrak{M}$ in $\widetilde{M}(\mu')$ if and only if X is true in the μ-reduct of $\mathfrak{M}$. Hence according to Theorem 2.1 the inverse image of a closed subset of $M(\mu)$ under the μ-reduction is closed.

3.3. *Suppose $\mu' \supseteq \mu$. The inverse image of every element of $M(\mu)$ under the μ-reduction is a closed compact subset of $M(\mu')$.*

Proof. By Theorem 2.3 every point in $M(\mu)$ is closed. Hence the theorem follows from Theorem 3.2 and the fact that every closed set contained in a compact space is compact.

THEOREM 3.4. *Let $\mu' \supseteq \mu$. Then the μ-reduction of $M(\mu')$ is a closed mapping (i.e., the image of each closed set is closed).*

Proof. The theorem follows from Theorems 2.1, 2.3 and 3.2 and the facts that (1) in a compact space every closed set is compact, (2) compactness of a subset is invariant under continuous mapping: $X \to Y$, if X is a compact space and Y is a Hausdorff space and (3) in a Hausdorff space every compact set is closed.

3.5. *Let $\mu' \supseteq \mu$ and let $\Sigma' \subseteq S(\mu')$. Then*

$$\pi_\mu \sigma(\Sigma')^* = \bigcap \{\sigma X^*;\ \Sigma' \vdash X \text{ in } L(\mu'),\ X \in S(\mu)\}\,.$$

Proof. According to Theorem 3.4, $\pi_\mu \sigma(\Sigma')^*$ is known to be closed in $M(\mu)$. Hence the theorem follows from the fact that every closed set coincides with its closure, i.e., the intersection of all closed sets which include the set.

Let $K \subseteq \widetilde{M}(\mu)$. K is said to be a *pseudo-elementary class (in wider sense)*, in symbol: $K \in \mathrm{PC}$ (PC_Δ), if and only if there is a type $\mu' \supseteq \mu$ such that K is the class of all μ-reducts of interpretations in an elementary class (in wider sense) in $M(\mu')$. If $K, K' \in \mathrm{PC}$ (PC_Δ), then $K \cup K'$, $K \cap K' \in \mathrm{PC}$ (PC_Δ), but $\mathrm{C}K$ is not necessarily in PC (PC_Δ).

COROLLARY 3.6. *Let $K \subseteq \widetilde{M}(\mu)$. Then $K \in \mathrm{PC}_\Delta$ if and only if σK is closed in $M(\mu)$.*

COROLLARY 3.7 (Büchi–Craig [4]). *$K \in \mathrm{PC}_\Delta$ and $K \in \mathrm{ACL}$ imply $K \in \mathrm{EC}_\Delta$.*

Let $K \subseteq \widetilde{M}(\mu)$. K will be called a *quasi-elementary class (in wider sense)*, in symbol: $K \in \mathrm{QC}$ (QC_Δ), if and only if there is a type $\mu' \supseteq \mu$ such that $\pi_\mu^{-1}(\sigma K) \in \mathrm{EC}$ (EC_Δ). If $K, K' \in \mathrm{QC}$, then $\mathrm{C}K$, $K \cup K'$, $K \cap \cap K' \in \mathrm{QC}$.

3.8. $\mathrm{QC} \subseteq \mathrm{PC} \subseteq \mathrm{QC}_\Delta = \mathrm{PC}_\Delta$.

Proof. $\mathrm{QC} \subseteq \mathrm{PC}$, $\mathrm{PC} \subseteq \mathrm{PC}_\Delta$, $\mathrm{QC}_\Delta \subseteq \mathrm{PC}_\Delta$ follow from the definitions. Assume that $\sigma K \subseteq M(\mu)$ and $K \in \mathrm{PC}_\Delta$. Then Theorem 3.4 implies that σK is closed in $M(\mu)$. Hence $\pi_\mu^{-1}(\sigma K)$ is closed in $M(\mu')$, where π^μ is the μ-reduction of $M(\mu')$ for $\mu' \supseteq \mu$. From here it follows that $K \in \mathrm{QC}_\Delta$.

3.9. *$K \in \mathrm{QC}$ and $K \in \mathrm{ACL}$ imply $K \in \mathrm{EC}$.*

Proof. Assume $K \subseteq \widetilde{M}(\mu)$ and $K \in \mathrm{QC}$. Let μ' be the type $\supseteq \mu$ such that π_μ is the μ-reduction of $M(\mu')$ and $K' = \pi_\mu^{-1}(K)$ is clopen in $M(\mu')$. Then both $K = \pi_\mu(K')$ and $\mathrm{C}K = \pi_\mu(\mathrm{C}K')$ are closed, whence K is clopen in $M(\mu)$.

4. Definability. The μ-reduction π_μ of $M(\mu')$ induces an equivalence relation $\pi_\mu^{-1}\pi_\mu$ on $M(\mu')$, i.e. $(\mathfrak{M}, \mathfrak{N}) \in \pi_\mu^{-1}\pi_\mu$ if and only if $\pi_\mu(\mathfrak{M}) = \pi_\mu(\mathfrak{N})$. And the equivalence induces a decomposition $\mathfrak{D}$ of $M(\mu')$. Due to Theorem 3.4 the projection (quotient map) of $M(\mu')$ onto $\mathfrak{D}$ is closed and $M(\mu)$ is homeomorphic to the quotient space.

THEOREM 4.2. *The decomposition $\mathfrak{D}$ is upper semi-continuous (i.e., for each D in $\mathfrak{D}$ and each open set U containing D there is an open set V such that $D \subseteq V \subseteq U$ and V is the union of members of $\mathfrak{D}$).*

Proof. The theorem follows from Theorem 3.4 and the fact that a decomposition $\mathfrak{D}$ of a topological space X is upper semi-continuous if and only if the projection of X onto $\mathfrak{D}$ is closed [6].

Let $\mu' \supseteq \mu$. A class K of interpretations in $M(\mu')$ is called *definable in $L(\mu)$* if and only if $\pi_\mu^{-1}\pi_\mu(K) = K$.

4.2. *Let $\mu' \supseteq \mu$, $K, K' \subseteq M(\mu')$ and let K be definable in $L(\mu)$. Then $K \cap K' = 0$ implies $\pi_\mu(K) \cap \pi_\mu(K') = 0$.*

Proof. Assume $\mathfrak{M} \in \pi_\mu(K) \cap \pi_\mu(K')$. Then $\pi_\mu^{-1}(\mathfrak{M}) \cap K' \neq 0$ and $\pi_\mu^{-1}(\mathfrak{M}) \subseteq \pi_\mu^{-1}\pi_\mu(K) = K$, whence $K \cap K' \neq 0$.

A sentence X in $S(\mu')$ is said to be *definable in $L(\mu)$* if and only if σX^* is definable in $L(\mu)$. If X and Y are definable in $L(\mu)$, then $X \wedge Y$, $X \vee Y$ are definable in $L(\mu)$. And then $\pi_\mu^{-1}\pi_\mu(\mathrm{C}(\sigma X^*)) = \mathrm{C}(\sigma X^*)$, so $\neg X$ is also definable in $L(\mu)$.

4.3. *Let $\mu' \supseteq \mu$ and $X \in S(\mu')$. X is definable in $L(\mu)$ if and only if there is a sentence Y in $S(\mu)$ such that X and Y are deductively equivalent (i.e., $X \vdash Y$ and $Y \vdash X$) in $L(\mu')$.*

Proof. If X is definable in $L(\mu)$, then $\pi_\mu(\sigma X^*)$ is clopen in $M(\mu)$, since $\pi_\mu(\sigma X^*)$ and $\pi_\mu \sigma(\neg X)^*$ are disjoint closed subsets whose union is $M(\mu)$. Hence there is a sentence Y in $S(\mu)$ such that $\pi_\mu(\sigma X^*) = \sigma Y^*$. Hence $\sigma X^* = \pi_\mu^{-1}\pi_\mu(\sigma X^*) = \pi_\mu^{-1}(\sigma Y^*) = \sigma Y^*$. The converse is obvious.

Assume that $\mu' \supseteq \mu$ and the language $L(\mu')$ can be obtained from $L(\mu)$ by adjoining a sequence α (possibly empty) of new predicates. For convenience, we often represent the type μ' by the notation $\mu+\alpha$.

A predicate P is said to be *definable in* $L(\mu)$ if and only if every sentence in $L(\mu+\{P\})$ is definable in $L(\mu)$. A predicate P is definable in $L(\mu)$ if and only if $M(\mu+\{P\})$ is a homeomorph of $M(\mu)$.

Due to E. Beth [2] a predicate is said to be *implicitly definable in* $L(\mu)$ if and only if the μ-reduction of $M(\mu+\{P\})$ is a one-to-one mapping.

THEOREM 4.4. *A predicate P is definable in $L(\mu)$ if and only if P is implicitly definable in $L(\mu)$.*

Proof. Due to the fact that every continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism.

5. Interpolation Theorem.

THEOREM 5.1. *Let $\mu' \supseteq \mu$. If K', K'' are two disjoint closed sets of $M(\mu')$ and if $\pi_\mu K' \cap \pi_\mu K'' = 0$, then there is a clopen set K of $M(\mu')$ such that $K' \subseteq K$, $K \cap K'' = 0$ and K is definable in $L(\mu)$.*

Proof. According to Theorem 3.4, $\pi_\mu K'$ and $\pi_\mu K''$ are disjoint closed subsets of $M(\mu)$. Hence by normality and compactness of the space $M(\mu)$ there is a clopen neighbourhood σX^* such that $\pi_\mu K' \subseteq \sigma X^* \subseteq \mathrm{C}(\pi_\mu K'')$, $X \in S(\mu)$. Then $K = \pi_\mu^{-1}(\sigma X^*)$ is the required clopen set.

Assume that α, β are two disjoint sequences of predicates and that $\mu' = \mu+\alpha+\beta$. Two sets Σ', Σ'' of sentences in $S(\mu')$ are said to be *prime modulo* μ if and only if every sentence in Σ' is definable in $L(\mu+\alpha)$, and every sentence in Σ'' is definable in $L(\mu+\beta)$.

LEMMA 5.2. *Let $\mu' \supseteq \mu$. Assume that $\Sigma', \Sigma'' \subseteq S(\mu')$ and that Σ' and Σ'' are prime modulo μ. If for some interpretation $\mathfrak{M}$ in $M(\mu)$, $\pi_\mu^{-1}(\mathfrak{M}) \cap \sigma\Sigma'^* \neq 0$ and $\pi_\mu^{-1}(\mathfrak{M}) \cap \sigma\Sigma''^* \neq 0$, then $\pi_\mu^{-1}(\mathfrak{M}) \cap \sigma\Sigma'^* \cap \sigma\Sigma''^* \neq 0$.*

Proof. Let $\alpha = \{P_a\}_{a<\alpha}$, $\beta = \{P_b\}_{b<\beta}$ be disjoint sequences of predicates such that $\mu' = \mu+\alpha+\beta$ and Σ', Σ'' be definable in $L(\mu+\alpha)$, $L(\mu+\beta)$, respectively. Let $\mathfrak{M} = \langle M, R_\lambda\rangle_{\lambda<\varrho}$ and

$$\mathfrak{M}' = \langle M, R_\lambda, R'_a, R'_b\rangle_{\lambda<\varrho, a<\alpha, b<\beta} \in \pi_\mu^{-1}(\mathfrak{M}) \cap \sigma\Sigma'^*,$$

$$\mathfrak{M}'' = \langle M, R_\lambda, R''_a, R''_b\rangle_{\lambda<\varrho, a<\alpha, b<\beta} \in \pi_\mu^{-1}(\mathfrak{M}) \cap \sigma\Sigma''^*.$$

Then we can verify that $\langle M, R_\lambda, R'_a, R''_b\rangle_{\lambda<\varrho, a<\alpha, b<\beta} \in \pi_\mu^{-1}(\mathfrak{M}) \cap \sigma\Sigma'^* \cap \sigma\Sigma''^*$.

LEMMA 5.3. *Let $\mu' \supseteq \mu$. Assume that $\Sigma', \Sigma'' \subseteq S(\mu')$ and that Σ' and Σ'' are prime modulo μ. If $\sigma\Sigma'^* \cap \sigma\Sigma''^* = 0$ in $\widetilde{M}(\mu')$, then $\pi_\mu(\sigma\Sigma'^*) \cap \pi_\mu(\sigma\Sigma''^*) = 0$.*

Proof. Direct consequence of Lemma 5.2.

The following is a generalization of Craig's Theorem:

THEOREM 5.4. *Let $\mu' \supseteq \mu$. Assume that $\Sigma', \Sigma'' \subseteq S(\mu')$ and that Σ' and Σ'' are prime modulo μ. If $\Sigma' \cup \Sigma''$ is inconsistent in $L(\mu')$, then there is a sentence X in $S(\mu)$ such that $X \in \mathrm{Cn}(\Sigma')$ and $\Sigma'' \cup \{X\}$ is inconsistent in $L(\mu')$.*

Proof. Due to Theorem 5.1 and Lemma 5.3.

6. A remark on the Interpolation Theorem.

It has been shown by K. Scütte that the Craig's Interpolation Theorem holds in intuitionistic logic, [8] (1). However our Theorem 5.4, a generalized form of the Interpolation Theorem seems to fail in intuitionistic logic.

We shall distinguish intuitionistic notions from classical ones, by adding the symbol $'$ as a superscript.

In order to preserve the characterization of axiomatizable classes we introduce a somewhat stronger topology into the class $M'(\mu)$ of all interpretations of $L'(\mu)$, the intuitionistic predicate calculus of type μ. Let $\{\mathfrak{N};\ \text{not} \models_{\mathfrak{N}} X,\ X \in S'(\mu)\}$ be a basis for the open sets. Then $M'(\mu)$ is a T_1-space but it is not a Hausdorff space. In fact, let $\mathfrak{M}, \mathfrak{N}$ be two distinct interpretations of $L'(\mu)$ such that all sentences true in $\mathfrak{M}$ are also true in $\mathfrak{N}$; then for some sentence $X \in S'(\mu)$, $\mathfrak{N} \in X^*$ and $\mathfrak{M} \notin X^*$. Hence both of $\mathfrak{M}$ and $\mathfrak{N}$ have neighbourhoods $\{\mathfrak{A};\ \text{not} \models_{\mathfrak{A}} \neg X\}$, $\{\mathfrak{B};\ \text{not} \models_{\mathfrak{B}} X\}$ which are disjoint from $\mathfrak{N}, \mathfrak{M}$, respectively, though they may meet each other. It is not difficult to verify that $K \subset \widetilde{M}'(\mu)$ is axiomatizable if and only if $K = \sigma'^{-1}(F)$ for some closed set F in $M'(\mu)$ in this topology. Therefore 5.4 represent so-called Tietze's (or T_4-) Separation Axiom, and it is easy to see that every T_1-space with T_4-axiom should be a Hausdorff space.

7. Appendix. Point set theoretical properties of $M(\mu)$.

In this section we supplement some point set-theoretical properties of $M(\mu)$.

A set Σ of sentences in $S(\mu)$ is said to be *complete* if and only if for every sentence X in $S(\mu)$, either $X \in \mathrm{Cn}(\Sigma)$ or $\neg X \in \mathrm{Cn}(\Sigma)$. It is trivial that if K is inconsistent or if K is empty, then K is complete.

THEOREM 7.1. *The following conditions are equivalent:*

(1) *Σ is a non-empty consistent and complete set of sentences in $S(\mu)$.*

(2) *$\sigma\Sigma^*$ consists of a single point in $M(\mu)$.*

(3) *$\mathfrak{M}^* = \mathrm{Cn}(\Sigma)$ for some interpretation $\mathfrak{M} \in \widetilde{M}(\mu)$.*

(1) For this information the author thanks Professor Gaishi Takeuti. The author is also indebted to Dr. Mitsuru Yasuhara who gave a proof of this fact independently.

Proof. (1)→(2). If Σ is complete, two interpretations of $L(\mu)$ in which Σ hold are elementarily equivalent, for otherwise, Σ has two models $\mathfrak{M}$, $\mathfrak{N}$ in $\widetilde{M}(\mu)$ and $\models_{\mathfrak{M}} X$, $\models_{\mathfrak{N}} \neg X$ for some X in $S(\mu)$. Then by Theorem 0 both $K \cup \{X\}$ and $K \cup \{\neg X\}$ are consistent, whence neither X nor $\neg X$ is a deductive inference of K.

The implication (2)→(3) is clear.

(3)→(1). Assume that $\mathfrak{M}^* = \mathrm{Cn}(\Sigma)$. Then for any sentence X in $S(\mu)$, either $\sigma\mathfrak{M}^* \epsilon \sigma X^*$ or $\sigma\mathfrak{M}^* \epsilon \sigma(\neg X)^*$, that is, either $X \epsilon \mathrm{Cn}(\Sigma)$ or $\neg X \epsilon \mathrm{Cn}(\Sigma)$.

Corollary 7.2 (A. Robinson [7]). *If Σ is a non-empty complete set of sentences and Σ', Σ'' are two consistent sets of sentences such that $\Sigma \subseteq \Sigma' \cap \Sigma''$, then $\Sigma' \cup \Sigma''$ is consistent.*

The following fundamental theorems are also easy to see as consequences of Theorem 7.1 and of the compactness of $M(\mu)$:

Theorem 7.3 (Lindenbaum). *Every non-empty consistent set of sentences has a complete and consistent extension.*

Theorem 7.4 (Tarski). *Every non-empty consistent set of sentences is the intersection of all its complete and consistent extensions.*

7.5. *Σ is a non-empty, consistent and complete finite set of sentences in $S(\mu)$ if and only if $\sigma\Sigma^*$ is an isolated point in $M(\mu)$.*

Proof. A point x in a topological space is isolated if and only if $\{x\}$ is open, whence the theorem follows from Corollary 2.2.

As an application of Vaught's Test for completeness, we know that the elementary theory of densely ordered system with first and last elements provides an example of non-empty consistent and complete finite system. Hence the space $M(\mu)$ is not dense itself if $L(\mu)$ has the binary relation $\leqslant$ besides identity.

Theorem 7.6. *Let Σ be a non-empty consistent finite set of sentences. If $\mathrm{Card}(\sigma\Sigma^*) < c$, then there is a finite extension of Σ which is consistent and complete.*

Proof. Assume that Σ has no finite extension which is consistent and complete. Then $\sigma\Sigma^*$ does not contain any isolated point of $M(\mu)$. Moreover, $\sigma\Sigma^*$ is a clopen set, whence is a perfect subset of $M(\mu)$. The theorem follows from the fact that every non-empty perfect and compact set contained in a normal space has cardinal $\geqslant c$ [9].

If we assume that the domain ϱ of the type μ is an ordinal $\subseteq \omega$, i.e. that the language $L(\mu)$ has denumerable number of predicates, then Theorem 7.6 can be strengthened as follows:

7.7. *Let Σ be a non-empty consistent finite set of sentences. If every finite consistent extension of Σ is not complete, then $\mathrm{Card}(\sigma\Sigma^*) = c$.*

Proof. The theorem follows from Theorem 7.6 and the fact that every topological space with a countable basis has cardinal $\leqslant c$.

References

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Reçu par la Rédaction le 27. 6. 1966