by no means implies that there is torsion in the Grothendieck group $G(\mathfrak{T})$. For, in all our examples, $X_1+P\simeq X_2+P$ for a suitable polyhedron P (indeed, a sphere). Thus the question of the existence of torsion in the Grothendieck group remains open.

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A minimal hyperdegree

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Two sets of natural numbers have the same hyperdegree if each is hyperarithmetic in the other. A non-hyperarithmetic set is said to have *minimal hyperdegree* if all sets of lower hyperdegree are hyperarithmetic. In this paper we construct a set that has minimal hyperdegree, and we study a certain class of models of the hyperarithmetic comprehension axiom. We draw upon ideas occurring in Spector's construction of a minimal degree of unsolvability [9] and in Feferman's application of forcing to analysis [2]. Our argument mixes Cohen's forcing method [1] with classical truth considerations; however, the use of forcing is not essential to the construction of a set of minimal hyperdegree. Instead of forcing with finite conditions in the style of Feferman [2], we force with infinite, hyperarithmetic conditions. As one might expect, forcing with infinite conditions is much closer to truth than forcing with finite conditions. A set generic with respect to our notion of forcing must necessarily have minimal hyperdegree.

All of our forcing is with respect to a second order language L(S), which is virtually isomorphic to Feferman's language L*(S) ([2], p. 335). L(S) is the language of first order number theory augmented by the constant symbol **S**, some second order variables, and the membership symbol ϵ. Let O_1 be a π_1^1 subset of O [4], the set of all notations for recursive ordinals, such that each recursive ordinal has precisely one notation in O_1 [3]; if b is the unique notation in O_1 for the recursive ordinal β, we write $|b|=\beta$. In addition, the relation $|b|<|c|$ is the restriction of some recursively enumerable relation to O_1. For each $b\in O_1$, L(S) has set variables $X^b, Y^b, Z^b, \ldots$; L(S) also has set variables $X, Y, Z, \ldots$, number variables $x, y, z, \ldots$, a numeral $\overline{n}$ for each natural number n, and symbols for equality (=), successor (′), addition (+) and multiplication (·).

For each $b\in O_1$, the variable X^b is said to be *ranked*; the variable X is said to be *unranked*. A formula $\mathfrak{F}$ of L(S) is called *ranked* if every set

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variable occurring in $\mathfrak{F}$ is ranked. A formula of $\mathrm{L}(\mathbf{S})$ is called *existential* if it is ranked or if it is of the form $(\mathrm{E}X)\mathfrak{F}$ with X the only unranked variable occurring in $\mathfrak{F}$. The *ordinal rank* of a ranked formula $\mathfrak{F}$ is the least ordinal α such that $\alpha \geqslant |b|$ for every free variable X^b of $\mathfrak{F}$ and such that $\alpha > |b|$ for every bound variable X^b of $\mathfrak{F}$. A formula $\mathfrak{F}$ is called *arithmetical* if no bound set variables occur in $\mathfrak{F}$.

Let X be an arbitrary set of natural numbers. Following Feferman [2], for each $b \in O_1$, we inductively define a structure $\mathcal{M}_b(X) = \mathcal{M}_b$ and truth in $\bigcup\{\mathcal{M}_a|\ |a| < |b|\}$:

(i) A sentence $\mathfrak{F}$ of ordinal rank $\leqslant |b|$ is true in $\bigcup\{\mathcal{M}_a|\ |a| < |b|\}$ if it is true when $\mathbf{S}$ is interpreted as X, the number variables of $\mathfrak{F}$ are restricted to ω, and each second order variable X^a of $\mathfrak{F}$ is restricted to $\mathcal{M}_a$.

(ii) For each formula $\mathfrak{G}(x)$ (with only x free) of ordinal rank $\leqslant |b|$, let $\hat{x}\mathfrak{G}(x) = \{n|\ \mathfrak{G}(\overline{n})$ is true in $\bigcup\{\mathcal{M}_a|\ |a| < |b|\}\}$; then $\mathcal{M}_b$ consists of all such sets $\hat{x}\mathfrak{G}(x)$.

We define $\mathcal{M}(X) = \bigcup\{\mathcal{M}_b(X)|\ b \in O_1\}$. A sentence $\mathfrak{F}$ of $\mathrm{L}(\mathbf{S})$ is true in $\mathcal{M}(X)$ ($\models_{\mathcal{M}(X)} \mathfrak{F}$) if it is true when each unranked variable of $\mathfrak{F}$ is restricted to $\mathcal{M}(X)$ and the remaining symbols of $\mathfrak{F}$ are interpreted according to (i) above.

Let U be a ranked or unranked set variable of $\mathrm{L}(\mathbf{S})$, and let t denote a number-theoretic term. We write $\mathfrak{F}(\hat{x}\mathfrak{G}(x))$ for the result of replacing each occurrence of $t \in U$ in $\mathfrak{F}(U)$ by $\mathfrak{G}(t)$.

If $a_0, a_1, \ldots, a_{m-1}$ is a finite sequence of natural numbers, then it is effectively represented by the sequence number $\pi\{p_i^{1+a_i}|\ i < m\}$, where p_i is the ith largest prime. Sequence numbers provide a means of referring to finite, initial segments of characteristic functions of sets of natural numbers as if they were natural numbers. We use $p, q, r, \ldots$ to denote finite, initial segments of characteristic functions. We write $p > r$ if $p \neq r$ and p is extended by r. Let P be a non-empty set of finite, initial segments; we say that P *defines* (or *is*) a *perfect, closed set* if

$$(p)_{p\in P}(\mathrm{E}q)_{q\in P}(\mathrm{E}r)_{r\in P}[q \neq r \,\&\, q \not> r \,\&\, r \not> q \,\&\, p > q \,\&\, p > r]\,.$$

We write $X \in P$ if infinitely many initial segments of the characteristic function of X belong to P. (If P defines a perfect, closed set, then $\{X|\ X \in P\}$ is a perfect, closed set in the standard sense.) Let $P, Q, R, \ldots$ denote hyperarithmetic, perfect, closed (h.p.c.) sets; i.e., the set of sequence numbers of the initial segments in P is hyperarithmetic. We write $P \geqslant Q$ if $(X)(X \in Q \to X \in P)$. It is routine to assign indices to h.p.c. sets so that the following relation in x and Y is π_1^1: x is the index of a h.p.c. set P and $Y \in P$. For this reason it makes sense to say that the relation "P is a h.p.c. set and $Y \in P$" is π_1^1. Similarly, it makes sense to say that the set of all formulas of $\mathrm{L}(\mathbf{S})$ is π_1^1.

The forcing relation $P \Vdash_h \mathfrak{F}$, where P is a h.p.c. set and $\mathfrak{F}$ is a sentence of $\mathrm{L}(\mathbf{S})$, is defined by means of five closure conditions:

(i) $P \Vdash_h \mathfrak{F}$ if $\mathfrak{F}$ is ranked and $(X)(X \in P \to \models_{\mathcal{M}(X)} \mathfrak{F})$.

(ii) $P \Vdash_h (\mathrm{E}x)\mathfrak{F}(x)$ if $\mathfrak{F}(x)$ is unranked and $P \Vdash_h \mathfrak{F}(\overline{n})$ for some n.

(iii) $P \Vdash_h (\mathrm{E}X^b)\mathfrak{F}(X^b)$ if $\mathfrak{F}(X^b)$ is unranked and $P \Vdash_h \mathfrak{F}(\hat{x}\mathfrak{G}(x))$ for some $\mathfrak{G}(x)$ of ordinal rank $\leqslant |b|$.

(iv) $P \Vdash_h (\mathrm{E}X)\mathfrak{F}(X)$ if $(\mathrm{E}b)_{b\in O_1}[P \Vdash (\mathrm{E}X^b)\mathfrak{F}(X^b)]$.

(v) $P \Vdash_h \mathfrak{F}_1 \,\&\, \mathfrak{F}_2$ if $\mathfrak{F}_1 \,\&\, \mathfrak{F}_2$ is unranked, $P \Vdash_h \mathfrak{F}_1$, and $P \Vdash_h \mathfrak{F}_2$.

(vi) $P \Vdash_h \sim\mathfrak{F}$ if $\mathfrak{F}$ is unranked and $(Q)_{P\geqslant Q}[\sim Q \Vdash_h \mathfrak{F}]$.

We defined $\Vdash_h$ in terms of closure properties rather than transfinite induction in order to simplify the statements of our proofs (cf. Feferman [2], p. 336). A set S is said to be *generic* if for each sentence $\mathfrak{F}$ there is a P such that $S \in P$ and $[P \Vdash_h \mathfrak{F}$ or $P \Vdash_h \sim\mathfrak{F}]$. It is not immediate that generic sets exist, because the definition of $\Vdash_h$ contains some peculiarities, particularly the contrast between conditions (i) and (vi). It is precisely these peculiarities which make possible a quick proof of the all-important Lemma 1.

LEMMA 1. *The relation $P \Vdash_h \mathfrak{F}$, restricted to ranked $\mathfrak{F}$, is π_1^1.*

Proof. It is readily verified that the relation $\mathfrak{F}$ is true in $\mathcal{M}(X)$, restricted to ranked $\mathfrak{F}$, is π_1^1 in $\mathfrak{F}$ and X; one need only note that this relation can be defined by Σ_1^1 closure conditions (cf. Feferman [2], p. 337), The relation $X \in P$ is arithmetical *on* the π_1^1 set of all hyperarithmetic. perfect, closed P's. (This last point is elaborated in the comments immediately preceding Lemma 6.)

Kreisel's Lemma ([7], p. 307) states: if $P(x, y)$ is π_1^1 and $(x)(\mathrm{E}y)P(x, y)$, then there exists a hyperarithmetic function f such that $(x)P(x, f(x))$. Kreisel's argument also establishes a slightly stronger fact needed below: if $P(x, y)$ is π_1^1, then there exists a partial π_1^1 function f such that $(x)[(\mathrm{E}y)P(x, y) \to f(x)$ is defined $\&\, P(x, f(x))]$. (A partial function is π_1^1 if its graph is π_1^1).

The purpose of Lemma 2 (the "sequential" lemma) is to standardize a construction which we use repeatedly, and which is characteristic of forcing with closed, perfect sets.

LEMMA 2. *Let $\{\mathfrak{F}_m\}$ be a hyperarithmetic sequence of existential sentences If $(m)(Q)_{P\geqslant Q}(\mathrm{E}R)_{Q\geqslant R}[R \Vdash_h \mathfrak{F}_m]$, then $(\mathrm{E}Q)_{P\geqslant Q}(m)[Q \Vdash_h \mathfrak{F}_m]$.*

Proof, Suppose $(m)(Q)_{P\geqslant Q}(\mathrm{E}R)_{Q\geqslant R}[R \Vdash_h \mathfrak{F}_m]$. By Lemma 1 and condition (iv) of the definition of $\Vdash_h$, the following relation is π_1^1 in m, Q, and R: $P \geqslant Q \geqslant R \,\&\, R \Vdash_h \mathfrak{F}_m$. The argument of Kreisel's Lemma ([7], p. 307) shows that R can be regarded as a partial π_1^1 function f of m and Q: $R = f(m, Q) \to P \geqslant Q \geqslant R \,\&\, R \Vdash_h \mathfrak{F}_m$. Let us say that Q_1 and Q_2 are basic, disjoint h.p.c. subsets of Q if there exist $q_1, q_2 \epsilon Q$ such that $q_1 \neq q_2$, $q_1 \not\geqslant q_2$, $q_2 \not\geqslant q_1$, $Q_1 = \{p|\ p \epsilon Q \ \&\ q_1 \geqslant p\}$, and $Q_2 = \{p|\ p \epsilon Q \ \&\ q_2 \geqslant p\}$. By iterating the partial π_1^1 function f, we can define a hyperarithmetic, partial function Q_i^m with the following properties: $Q_0^0 = P$; for each $m \geqslant 0$ and $i < 2^m$, Q_i^{m+1} and $Q_{i+2^m}^{m+1}$ are basic, disjoint, h.p.c. subsets of Q_i^m, $Q_i^{m+1} \Vdash_h \mathfrak{F}_m$, and $Q_{i+2^m}^{m+1} \Vdash_h \mathfrak{F}_m$. We define Q by

$$p \epsilon Q \leftrightarrow (m)(\mathrm{E}_1 i)_{i<2^m}(p \epsilon Q_i^m)\ .$$

Then Q is a h.p.c. set, $P \geqslant Q$, and

$$X \epsilon Q \leftrightarrow (m)(\mathrm{E}_1 i)_{i<2^m}(X \epsilon Q_i^m)\ .$$

Fix m and $X \epsilon Q$. There exists an i such that $X \epsilon Q_i^{m+1}$ and $Q_i^{m+1} \Vdash_h \mathfrak{F}_m$. Since $\mathfrak{F}_m$ is existential, $\models_{\mathcal{M}(X)} \mathfrak{F}_m$. But then $Q \Vdash_h \mathfrak{F}_m$.

Lemma 3. $(\mathfrak{F})(P)(\mathrm{E}Q)_{P\geqslant Q}[Q \Vdash_h \mathfrak{F} \vee Q \Vdash_h \sim \mathfrak{F}]$.

Proof. Condition (vi) of the definition of $\Vdash_h$ makes it safe to assume that $\mathfrak{F}$ is ranked. Conditions (i) and (ii) of the definition of $\Vdash_h$ make it safe to assume that $\mathfrak{F}$ is in prenex normal form. By the full ordinal rank of $\mathfrak{F}$, we mean a function f defined on O_1 such that for each $b \epsilon O_1$, $f(b)$ equals the number of occurrences of b in $\mathfrak{F}$. We say

$$f < g \quad \text{if} \quad (\mathrm{E}b)_{b \epsilon O_1}\big[f(b) < g(b)\ \&\ (c)_{|c|>|b|}\big(f(c) = g(c)\big)\big]\ .$$

By the *arithmetical rank* of $\mathfrak{F}$, we mean the number m of occurrences of number-theoretic quantifiers in $\mathfrak{F}$. The rank of $\mathfrak{F}$ is (f, m). We say $(f, m) < (g, n)$ if $f < g$ or if $f = g$ and $m < n$. We prove the lemma for ranked $\mathfrak{F}$ by induction on the rank of $\mathfrak{F}$. Suppose that $\mathfrak{F}$ is $(\mathrm{E}X^b)\mathfrak{F}(X^b)$. Let $\{\mathfrak{G}_i(x)\}$ be a hyperarithmetic enumeration of all formulas (with just x free) of ordinal rank $\leqslant |b|$. For each i, $\mathfrak{F}\big(\hat{x}\mathfrak{G}_i(x)\big)$ has lower full ordinal rank than $(\mathrm{E}X^b)\mathfrak{F}(X^b)$. Our inductive hypothesis is: $(i)(P)(\mathrm{E}Q)_{P\geqslant Q}\big[Q \Vdash_h \mathfrak{F}\big(\hat{x}\mathfrak{G}_i(x)\big)$ or $Q \Vdash_h \sim \mathfrak{F}\big(\hat{x}\mathfrak{G}_i(x)\big)\big]$. Fix P. If $(\mathrm{E}i)(\mathrm{E}Q)_{P\geqslant Q}\big[Q \Vdash_h \mathfrak{F}\big(x\mathfrak{G}_i(x)\big)\big]$, then all is well; suppose not. Then it follows from the inductive hypothesis that $(i)(Q)_{P\geqslant Q}(\mathrm{E}R)_{Q\geqslant R}\big[R \Vdash_h \sim \mathfrak{F}\big(\hat{x}\mathfrak{G}_i(x)\big)\big]$. By Lemma 2, there is a Q such that $P \geqslant Q$ and $(i)\big[Q \Vdash_h \sim \mathfrak{F}\big(\hat{x}\mathfrak{G}_i(x)\big)\big]$; but then $Q \Vdash_h \sim (\mathrm{E}X^b)\mathfrak{F}(X^b)$.

If $\mathfrak{F}$ is of the form $(\mathrm{E}x)\mathfrak{F}(x)$, we note that $\mathfrak{F}(\overline{n})$ has lower arithmetical rank than $(\mathrm{E}x)\mathfrak{F}(x)$ and then proceed as above. If $\mathfrak{F}$ has no quantifiers, then there is a $q \epsilon P$ such that the desired $Q = \{p|\ p \epsilon P \ \&\ q \geqslant p\}$.

The existence of generic sets follows from Lemma 3. A standard transfinite induction shows: if S is generic, then $(\mathfrak{F})[\models_{\mathcal{M}(X)} \mathfrak{F} \leftrightarrow \leftrightarrow (\mathrm{E}P)(S \epsilon P \ \&\ P \Vdash_h \mathfrak{F})]$.

Lemma 4. *If S is genetric, $\mathfrak{F}(X)$ is arithmetical, and $\models_{\mathcal{M}(S)} (\mathrm{E}X)\mathfrak{F}(X)$, then $(\mathrm{E}X)[\mathfrak{F}(X)\ \&\ X$ is hyperarithmetic $\&\ X \epsilon \mathcal{M}(S)]$.*

Proof. Since S is generic, there must be a P such that $S \epsilon P$ and $P \Vdash_h (\mathrm{E}X^b)\mathfrak{F}(X^b)$ for some $b \epsilon O_1$. Since P is a hyperarithmetic, perfect, closed set, there must exist a hyperarithmetic $H \epsilon P$. It follows from the definition of $\Vdash_h$ for ranked sentences that $(\mathrm{E}X)[X \epsilon \mathcal{M}_b(H)\ \&\ \mathfrak{F}(X)]$. A transfinite induction on O_1 in the style of Kleene ([4], p. 35) shows that every member of $\mathcal{M}_b(X)$ is hyperarithmetic in X for every X; consequently, $\mathfrak{F}(K)$ holds for some hyperarithmetic K. Following Kleene ([4], p. 35), we can find a ranked formula $\mathfrak{G}(x)$ such that $K = \{n|\ \models_{\mathcal{M}(X)} \mathfrak{G}(\overline{n})\}$ holds for every X.

Lemma 5. *For each π_1^1 set A, there exists an existential formula $P(x)$ such that for all generic S, $(n)[n \epsilon A \leftrightarrow \models_{\mathcal{M}(S)} P(\overline{n})]$.*

Proof. By Gandy [3] there exists an arithmetical predicate $B(x, Y)$ such that for all n,

$$n \epsilon A \rightarrow (\mathrm{E}Y)[B(n, Y)\ \&\ Y \text{ is hyperarithmetical}]\ .$$

At the end of the proof of Lemma 4, it was noted that every hyperarithmetic set belongs to $\mathcal{M}(X)$ for every X. It follows from Lemma 4 that for all n,

$$n \epsilon A \leftrightarrow \models_{\mathcal{M}(S)} (\mathrm{E}Y)B(\overline{n}, Y)\ .$$

For the sake of Lemma 6, it is necessary to sharpen our previous observations concerning the assignment of indices to hyperarithmetic, closed sets. Kreisel ([7], p. 307) mentions an arithmetical formula $A(x, Y)$ with the following property: if $x \epsilon O$, then $(\mathrm{E}_1 Y)A(x, Y)$ and $(Y)[A(x, Y) \rightarrow \rightarrow Y$ is a hyperarithmetic set of the same Turing degree as $H_x]$. Since each hyperarithmetic set is recursive in H_x for some x, it is possible to assign indices to hyperarithmetic sets and to obtain an arithmetical formula $B(x, Y)$ such that the set of indices I is π_1^1 and such that: if $x \epsilon I$, then $(\mathrm{E}_1 Y)B(x, Y)$ and $(Y)[B(x, Y) \rightarrow (Y)_0 =$ the hyperarithmetic set whose index is $x]$, where $(Y)_0 = \{m|\ 2^m \epsilon Y\}$.

I and B can be modified to obtain a π_1^1 set I_1 of indices for hyperarithmetic, perfect, closed sets and an arithmetical formula $B_1(x, Y)$ with the following properties: if $x \epsilon I_1$, then $(\mathrm{E}_1 Y)B_1(x, Y)$ and $(Y)[B_1(x, Y) \rightarrow (Y)_0 =$ the h.p.c. set whose index is $x]$. Let $D(Y, Z)$ be an arithmetical formula which says: $(Y)_0$ contains infinitely many sequence numbers which represent initial segments of the characteristic function of Z. Then $x \epsilon I_1\ \&\ (Y)[B_1(x, Y) \rightarrow D(Y, Z)]$ says: x is the index of a h.p.c. P and $Z \epsilon P$.

Lemma 6. *Let A be a π_1^1 set (of indices) of hyperarithmetic, perfect, closed sets, and let S be generic. If $(P)(\mathrm{E}Q)_{P\geqslant Q}(Q \epsilon A)$, then $(\mathrm{E}Q)(Q \epsilon A\ \&\ \&\ S \epsilon Q)$.*

Proof. By Lemma 5 there is a formula $P(x)$ such that for all generic S, $(n)[n \in A \leftrightarrow \models_{\mathcal{M}(S)} P(\bar{n})]$. Let $\mathfrak{F}$ denote the following formula of $L(\mathbf{S})$:

$$(\mathrm{E}x)\big[P(x) \,\&\, (Y)\big(B_1(x, Y) \to D(Y, \mathbf{S})\big)\big] .$$

Then for all generic S, $(\mathrm{E}Q)[Q \in A \,\&\, S \in Q] \leftrightarrow \models_{\mathcal{M}(S)} \mathfrak{F}$. Now fix the generic set S. To prove the lemma, it is enough to find a P such that $P \Vdash_h \mathfrak{F}$ and $S \in P$. Suppose (for the sake of a contradiction) that no such P exists. Then there must be a P such that $P \Vdash_h \sim\mathfrak{F}$ and $S \in P$. There must also be a Q_1 such that $P \geqslant Q_1$ and $Q_1 \in A$. By Lemma 3, there exists a generic set $S' \in Q_1$. Since $P \Vdash_h \sim\mathfrak{F}$ and $P \geqslant Q_1$, it follows $Q_1 \Vdash_h \sim\mathfrak{F}$. Thus $\models_{\mathcal{M}(S')} \sim\mathfrak{F}$, and consequently, $\sim(\mathrm{E}Q)[Q \in A \,\&\, S' \in Q]$. But $Q_1 \in A$ and $S' \in Q_1$.

LEMMA 7. *Let $\mathfrak{F}(x, Y)$ be a formula of $L(\mathbf{S})$ whose only free variables are x and Y and whose only unranked variable is Y. If $P \Vdash_h (x)(\mathrm{E}Y)\mathfrak{F}(x, Y)$, then $(\mathrm{E}Q)_{P \geqslant Q}(\mathrm{E}b)_{b \in O_1}[Q \Vdash_h (x)(\mathrm{E}Y^b)\mathfrak{F}(x, Y^b)]$.*

Proof. Let $P \Vdash_h (x)(\mathrm{E}Y)\mathfrak{F}(x, Y)$. Then Lemma 3 and the definition of $\Vdash_h$ imply

$$(n)(Q)_{P \geqslant Q}(\mathrm{E}R)_{Q \geqslant R}[R \Vdash_h (\mathrm{E}Y)\mathfrak{F}(\bar{n}, Y)] .$$

It follows from Lemma 2 that there is a Q such that $P \geqslant Q$ and $(n)[Q \Vdash_h (\mathrm{E}Y)\mathfrak{F}(\bar{n}, Y)]$. This means that $(n)(\mathrm{E}b)_{b \in O_1}[Q \Vdash_h (\mathrm{E}Y^b)\mathfrak{F}(\bar{n}, Y^b)]$. By Lemma 1 and Kreisel's Lemma ([7], p. 307), there is a hyperarithmetic function f such that $(n)[Q \Vdash_h (\mathrm{E}Y^{f(n)})\mathfrak{F}(\bar{n}, Y^{f(n)})]$. By Spector [8], there is a $b \in O_1$ such that $(n)(|f(n)| \leqslant |b|)$. But then $Q \Vdash_h (x)(\mathrm{E}Y^b)\mathfrak{F}(x, Y^b)$.

LEMMA 8. *Let $\mathfrak{G}(x)$ be a ranked formula with only x free. For each P, there exists a Q such that $P \geqslant Q$ and either* (i) *or* (ii) *holds:*

(i) $(X)\big(X \in Q \to X$ *is hyperarithmetic in* $\{n \mid \models_{\mathcal{M}(X)} \mathfrak{G}(\bar{n})\}\big)$;

(ii) $(X)\big(X \in Q \to \{n \mid \models_{\mathcal{M}(X)} \mathfrak{G}(\bar{n})\}$ *is hyperarithmetic*$\big)$.

Proof. We proceed in the spirit of the proof of Lemma 2.

Case 1. $(Q)_{P \geqslant Q}(\mathrm{E}Q_1)_{Q \geqslant Q_1}(\mathrm{E}Q_2)_{Q \geqslant Q_2}(\mathrm{E}n)[Q_1 \Vdash_h \mathfrak{G}(\bar{n}) \,\&\, Q_2 \Vdash_h \sim\mathfrak{G}(n)]$.

It follows from Lemma 1 and the argument of Kreisel's Lemma ([7], p. 307) that we can regard Q_1, Q_2, and n as partial π_1^1 functions of Q. By iterating these partial π_1^1 functions, we can define a hyperarithmetic, partial function Q_i^m with the following properties: $Q_0^0 = P$; for each $m \geqslant 0$ and $i < 2^m$, Q_i^{m+1} and $Q_{i+2^m}^{m+1}$ are basic, disjoint h.p.c. subsets of Q_i^m, and $(\mathrm{E}n)[Q_i^{m+1} \Vdash_h \mathfrak{G}(\bar{n}) \,\&\, Q_{i+2^m}^{m+1} \Vdash_h \sim\mathfrak{G}(\bar{n})]$. We define Q by:

$$p \in Q \leftrightarrow (m)(\mathrm{E}_1 i)_{i<2^m}(p \in Q_i^m) .$$

Then Q is a h.p.c. set, $P \geqslant Q$, and

$$X \in Q \leftrightarrow (m)(\mathrm{E}_1 i)_{i<2^m}(X \in Q_i^m) .$$

We say that t puts X in Q if $(m)(X \in Q_{t(m)}^m)$. Each $X \in Q$ is put in Q by a unique t, and is hyperarithmetic in that t. We claim that Q satisfies condition (i) of the lemma. Fix $X \in Q$. Let $\mathfrak{G} = \{n \mid \models_{\mathcal{M}(X)} \mathfrak{G}(\bar{n})\}$. To see that X is hyperarithmetic in $\mathfrak{G}$, it is enough to see that the unique t which puts X in Q is hyperarithmetic in $\mathfrak{G}$. Consider the following definition of t:

$$t(0) = 0 \,\&\, \big(t(m+1) = t(m) \vee t(m+1) = t(m) + 2^m\big) ;$$

$$t(m+1) = t(m) \leftrightarrow (\mathrm{E}n)[Q_{t(m)}^{m+1} \Vdash_h \mathfrak{G}(\bar{n}) \,\&\, Q_{t(m)+2^m}^{m+1} \Vdash_h \sim\mathfrak{G}(n) \,\&\, n \in \mathfrak{G}] .$$

The above equations define t, because $\mathfrak{G}(\bar{n})$ is a ranked formula, and consequently, $[X \in Q_i^{m+1} \,\&\, Q_i^{m+1} \Vdash_h \mathfrak{G}(\bar{n})] \to n \in \mathfrak{G}$. But then t is hyperarithmetic in $\mathfrak{G}$, since the relation $P \Vdash_h \mathfrak{F}$ is hyperarithemtic when P is restricted to be a member of $\{Q_i^m \mid i < 2^m \,\&\, m \geqslant 0\}$ and $\mathfrak{F}$ is restricted to be a member of $\{\mathfrak{G}(\bar{n}), \sim\mathfrak{G}(\bar{n}) \mid n \geqslant 0\}$.

Case 2. $(\mathrm{E}R)_{P \geqslant R}(Q_1)_{R \geqslant Q_1}(Q_2)_{R \geqslant Q_2}(n) \sim[Q_1 \Vdash_h \mathfrak{G}(\bar{n}) \,\&\, Q_2 \Vdash_h \sim\mathfrak{G}(\bar{n})]$.

By Lemma 3, $(n)(\mathrm{E}Q)_{R \geqslant Q}[Q \Vdash_h \mathfrak{G}(\bar{n}) \vee Q \Vdash_h \sim\mathfrak{G}(\bar{n})]$. For each n, let

$$F_n = \begin{cases} \mathfrak{G}(\bar{n}) & \text{if} \quad (\mathrm{E}Q)_{R \geqslant Q}[Q \Vdash_h \mathfrak{G}(\bar{n})] , \\ \sim\mathfrak{G}(\bar{n}) & \text{i-} \quad (\mathrm{E}Q)_{R \geqslant Q}[Q \Vdash_h \sim\mathfrak{G}(\bar{n})] . \end{cases}$$

The defining property of R guarantees that F_n is well-defined; in addition $\{F_n\}$ is a hyperarithmetic sequence of ranked formulas. The nature of R also guarantees that

$$(n)(Q_1)_{R \geqslant Q_1}(\mathrm{E}Q_2)_{Q_1 \geqslant Q_2}[Q_2 \Vdash_h F_n] .$$

It follows from Lemma 2 that $(\mathrm{E}Q)_{R \geqslant Q}(n)[Q \Vdash_h F_n]$. If $X \in Q$, then $\models_{\mathcal{M}(X)} \mathfrak{G}(\bar{n}) \leftrightarrow Q \Vdash_h \mathfrak{G}(\bar{n})$. Thus Q satisfies condition (ii) of the lemma, since the relation $Q \Vdash_h \mathfrak{G}(\bar{n})$ is hyperarithmetic in n.

LEMMA 9. *If S is generic, then the hyperarithmetic comprehension axiom holds in $\mathcal{M}(S)$.*

Proof. The argument of Feferman ([2], p. 339) makes clear that it is sufficient to show the following: let S be generic, and let $\mathfrak{F}(x, Y)$ be a formula of $L(\mathbf{S})$ whose only free variables are x and Y and whose only unranked variable is Y; if $\models_{\mathcal{M}(S)} (x)(\mathrm{E}Y)\mathfrak{F}(x, Y)$, then for some $b \in O_1$,

$$\models_{\mathcal{M}(S)} (x)(\mathrm{E}Y^b)\mathfrak{F}(x, Y^b) .$$

Let P be such that $S \in P$ and $P \Vdash_h (x)(\mathrm{E}Y)\mathfrak{F}(x, Y)$. Let

$$A_1 = \{Q \mid (X)(X \in Q \to X \notin P)\} ,$$

and let

$$A_2 = \{Q \mid P \geqslant Q \,\&\, (\mathrm{E}b)_{b \in O_1}[Q \Vdash_h (x)(\mathrm{E}Y^b)\mathfrak{F}(x, Y^b)]\} .$$

Let A be the set of indices of all members of $A_1 \cup A_2$. It follows from Lemma 1 that A is π_1^1. It is our intention to apply Lemma 6 to A. We must first show $(Q)(\mathrm{E}R)_{Q \geqslant R}(R \in A)$. Fix Q. The last sentence of the proof of Lemma 4 implies there exists a ranked sentence $\mathfrak{F}$ such that $(X)[X \in P \leftrightarrow$

$\leftrightarrow \models_{\mathcal{M}(X)}\mathfrak{F}]$. By Lemma 3, there exists a Q_1 such that $Q \geqslant Q_1$, and $Q_1 \Vdash_h \mathfrak{F}$ or $Q_1 \Vdash_h \sim\mathfrak{F}$. If $Q_1 \Vdash_h \mathfrak{F}$, then $P \geqslant Q_1$, and by Lemma 7, $(\mathrm{E}R)_{Q_1 \geqslant R}(R \in A_2)$. If $Q_1 \Vdash_h \sim\mathfrak{F}$, then $Q_1 \in A_1$.

By Lemma 6, there is a $Q_2 \in A$ such that $S \in Q_2$. Since $S \in P$, it must be that $Q_2 \in A_2$. But then $Q_2 \Vdash_h (x)(\mathrm{E}Y^b)\mathfrak{F}(x, Y^b)$ for some $b \in O_1$, and consequently, $\models_{\mathcal{M}(S)}(x)(\mathrm{E}Y^b)\mathfrak{F}(x, Y^b)$.

LEMMA 10. *If S is generic, then S has minimal hyperdegree.*

Proof. Let S be generic. First we show that S is not hyperarithmetic. Suppose that S equals the hyperarithmetic set H. Following Kleene ([4], p. 35), we can find a ranked formula $\hat{x}\mathfrak{G}(x)$ such that $H = \{n \mid \models_{\mathcal{M}(X)}\mathfrak{G}(\bar{n})\}$ for all X. Since S is generic, there exists a P with the property that $P \Vdash_h \mathrm{S} = \hat{x}\mathfrak{G}(x)$. But then $(X)(X \in P \rightarrow X = H)$. This last is absurd, since P is uncountable.

Now suppose that K is hyperarithmetic in S. It follows from Lemma 9 and the argument of Kreisel ([6], p. 114) that $K \in \mathcal{M}(S)$. Let $\mathfrak{G}(x)$ be a ranked formula such that $n \in K \leftrightarrow \models_{\mathcal{M}(S)}\mathfrak{G}(\bar{n})$. Let

$$A_1 = \{Q \mid (X)(X \in Q \rightarrow X \text{ is hyperarithmetic in } \{n \mid \models_{\mathcal{M}(X)}\mathfrak{G}(\bar{n})\})\},$$

and let

$$A_2 = \{Q \mid (X)(X \in Q \rightarrow \{n \mid \models_{\mathcal{M}(X)}\mathfrak{G}(\bar{n})\} \text{ is hyperarithmetic})\}.$$

Let $A = A_1 \cup A_2$. Then the following observations imply that A is π_1^1: the relation "X is hyperarithmetic in Y" is π_1^1; the relation "$\models_{\mathcal{M}(X)}\mathfrak{G}(\bar{n})$" is hyperarithmetic (in X and n). By Lemma 8, $(P)(\mathrm{E}Q)_{P \geqslant Q}(Q \in A)$. By Lemma 6, $(\mathrm{E}Q)(Q \in A \ \& \ S \in Q)$. If $Q \in A_1$, then S is hyperarithmetic in K. If $Q \in A_2$, then K is hyperarithmetic.

THEOREM. *There exists a set of minimal hyperdegree less than the hyperdegree of O.*

Proof. The relation $P \Vdash_h \mathfrak{F}$, where P is a h.p.c. set and $\mathfrak{F}$ is an arbitrary sentence of $\mathrm{L}(\mathrm{S})$, is hyperarthmetic in O (cf. [2], p. 337). Let $\{\mathfrak{F}_m \mid m \geqslant 0\}$ be an enumeration hyperarithmetic in O of all sentences of $\mathrm{L}(\mathrm{S})$. By Lemma 3, we have $(m)(P)(\mathrm{E}Q)_{P \geqslant Q}[P \Vdash_h \mathfrak{F}_m \vee P \Vdash_h \sim \mathfrak{F}_m]$. The argument of Kreisel's Lemma ([7], p. 307), relativized to O, makes it possible to regard Q as a partial function of m and P whose graph is π_1^1 in O: $(m)(P)[P \geqslant f(m, P) \ \& \ (f(m, P) \Vdash_h \mathfrak{F}_m \vee f(m, P) \Vdash_h \sim \mathfrak{F}_m)]$. By iterating f, we can define a function P_m hyperarithmetic in O with the following properties: $P_m \geqslant P_{m+1} \ \& \ [P_m \Vdash_h \mathfrak{F}_m \vee P_m \Vdash_h \sim \mathfrak{F}_m]$. We define S by the formula:

$$n \in S \leftrightarrow (\mathrm{E}m)[P_m \Vdash_h \bar{n} \in \mathrm{S}].$$

Then S is generic and hyperarithmetic in O. (S is, in fact, the unique X such that $(m)(X \in P_m)$.) By Lemma 10, S has minimal hyperdegree. In [2], p. 340, it was observed that O is not of minimal hyperdegree.

We conclude with a word on the elimination of forcing from the construction of a set of minimal hyperdegree. The key lemmas underlying the proof of the theorem are Lemmas 7 and 8. Lemma 7 provides the means of insuring that ω_1^S, the least ordinal not recursive in S, equals ω_1, the least non-recursive ordinal. Lemma 8 provides the means of insuring that S has minimal hyperdegree, given that $\omega_1^S = \omega_1$. Both of the key lemmas are consequences of the nature of the forcing relation restricted to ranked sentences, where it coincides with the classical truth relation. These considerations, if pressed hard, lead to a construction of a set of minimal hyperdegree recursive in O without any recourse to the idea of forcing.

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