A representation theorem for idempotent medial algebras

by

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1. An algebra $\mathfrak{M} = (X; f(x_1, \dots, x_n))$ is called a *medial algebra* if it satisfies the following two conditions:

$$(1)\quad f\big(f(x_{11}, \dots, x_{1n}), \dots, f(x_{n1}, \dots, x_{nn})\big) = f\big(f(x_{i_1j_1}, \dots, x_{i_1j_n}), \dots, f(x_{i_nj_1}, \dots, x_{i_nj_n})\big)$$

holds for every permutation $\{(i_1, j_1), (i_1, j_2), \dots, \dots, (i_n, j_{n-1}), (i_n, j_n)\}$ of the set $\{(1,1), (1,2), \dots, (n, n-1), (n, n)\}$ such that $(i_r, j_r) = (r, r)$ holds for $r = 1, 2, \dots, n$.

$$(2)\quad f\big(f(x_1, \dots, x_n), y_2, \dots, y_n\big) = f\big(x_1, f(x_2, y_2, x_3, \dots, x_n), y_3, \dots, y_n\big) = \dots = f\big(x_1, y_2, \dots, y_{n-1}, f(x_2, \dots, x_n, y_n)\big) .$$

Condition (1) in the case $n = 2$ coincides with the condition of mediality, considered by S. K. Stein in [7] for quasigroups. Condition (2) is a generalization of associativity to which it reduces in the case of $n = 2$.

In this note we shall give a constructive description of all medial algebras which are idempotent, i.e. which satisfy in addition to (1) and (2) the condition

$$(3)\quad f(x, x, \dots, x) = x .$$

We shall need the notion of the sum of a direct system of algebras, as defined in [4], [5]: if $\mathcal{A} = \langle I, \langle \mathfrak{A}_i\rangle_{i\in I}, \langle \varphi_{ij}\rangle_{i\leq j, i,j\in I}\rangle$ is a direct system with l.u.b. of similar algebras without fundamental nullary operations, then $S(\mathcal{A})$, the sum of the system $\mathcal{A}$, is an algebra whose carrier is equal to the sum of carriers of $\mathfrak{A}_i$'s (the algebras $\mathfrak{A}_i$ should be treated as disjoint) and the fundamental operations F_t are defined by

$$F_t(a_1, \dots, a_n) = F_t\big(\varphi_{i_1 i}(a_1), \dots, \varphi_{i_n i}(a_n)\big)$$

where $i = \text{l.u.b.}(i_1, \dots, i_n)$ and a_r belongs to the carrier of $\mathfrak{A}_{i_r}$.

We recall two results from [5] which we shall use in sequel:

(i) (See [4], theorem I.) *If $\mathcal{A}$ is not a trivial direct system of algebras* (i.e. if it consists of at least two algebras), *then in the algebra $S(\mathcal{A})$ all* *regular equations satisfied in all algebras of $\mathcal{A}$ are satisfied, whereas no other equation is satisfied in $S(\mathcal{A})$.*

(An equation $f = g$, where f and g are terms of an algebra, is called *regular* if the set of free variables occurring in f coincides with the set of free variables occurring in g.)

(ii) (See [4], theorem III.) *Let $\mathfrak{A}$ be an algebra belonging to an equational class K whose defining equations are all regular. Let $g(x, y)$ be a term of $\mathfrak{A}$ and let K^* be the equational class defined by the equations of the class K to which the equation $g(x, y) = x$ has been added. Then the term $g(x, y)$ induces a P-function for $\mathfrak{A}$ if and only if $\mathfrak{A}$ is representable as a sum of a direct system of algebras from the class K^*.* (For the definition of a P-function see [4], [5].)

Our description of idempotent medial algebras will use the following algebras: n-dimensional diagonal algebras (see [2], [3] for properties and representation theorem), i.e. algebras $(X; d(x_1, \ldots, x_n))$ such that $d(x, x, \ldots, x) = x$ and $d(d(x_{11}, \ldots, x_{1n}), \ldots, d(x_{n1}, \ldots, x_{nn})) = d(x_{11}, \ldots, x_{nn})$ and r_n-algebras (see [4], 2.4) which can be described in the following way: take an abelian group such that $(n-1)a = 0$ holds identically, and define $f(x_1, \ldots, x_n) = x_1 + \ldots + x_n$. The carrier of the group with f as the fundamental operation is a r_n-algebra. The class of r_n-algebras can also be described by using equations and this was done in [4], 2.4.

The results of this note were announced without proof in [5].

2. The following theorem gives a complete description of idempotent medial algebras:

THEOREM. *An algebra $\mathfrak{A} = (X; f(x_1, \ldots, x_n))$ is an idempotent medial algebra if and only if A is the sum of a direct system of algebras, each of them being a product of an n-dimensional diagonal algebra and an r_n-algebra.*

Proof. The "if" part of the theorem follows immediately from the fact that diagonal algebras and r_n-algebras are medial and idempotent and that the class of medial algebras can be defined by using regular equations only, whence an application of (i) leads us to the desired result.

To prove the "only if" part, let us assume that $\mathfrak{A}$ is an idempotent medial algebra, and that $f(x_1, \ldots, x_n)$ is the fundamental operation of $\mathfrak{A}$. We shall need some lemmas.

LEMMA 1. *In $\mathfrak{A}$ the following equalities are true:*

$$f(f(f(x_1, \ldots, x_n), y_2, \ldots, y_n), z_2, \ldots, z_n) = f(f(f(x_1, y_2, \ldots, y_n), x_2, \ldots, x_n), z_2, \ldots, z_n),$$

and

$$f(z_1, \ldots, z_{k-1}, f(y_1, \ldots, y_{k-1}, f(x_1, \ldots, x_n), y_{k+1}, \ldots, y_n), z_{k+1}, \ldots, z_n)$$
$$= f(z_1, \ldots, z_{k-1}, f(x_1, \ldots, x_{k-1}, f(y_1, \ldots, y_{k-1}, x_k, y_{k+1}, \ldots, y_n), x_{k+1}, \ldots, x_n), z_{k+1}, \ldots, z_n) \quad (1 < k \leqslant n).$$

Proof. We prove only the first equality, the proof of the second being similar. We have

$$\begin{aligned} f(f(f(x_1, \ldots, x_n), y_2, \ldots, y_n), z_2, \ldots, z_n) &= f(f(x_1, \ldots, x_n), f(y_2, z_2, y_3, \ldots, y_n), z_3, \ldots, z_n) \\ &= f(f(x_1, y_2, \ldots, y_n), f(x_2, z_2, x_3, \ldots, x_n), z_3, \ldots, z_n) \\ &= f(f(f(x_1, y_2, \ldots, y_n), x_2, \ldots, x_n), z_2, \ldots, z_n). \end{aligned}$$

as asserted.

Now we introduce a binary operation by the formula $xy = f(x, y, \ldots, y)$.

LEMMA 2. *The following equalities are true:*

$$(xy)z = x(yz), \quad xx = x, \quad xyuv = xuyv.$$

Proof. We shall prove this lemma, and also the next few lemmas, only for the case of $n = 3$, the proof for the general case being identical in principle but involving complicated notation. In the case of $n = 3$ we shall write for shortness $[xyz]$ instead of $f(x, y, z)$.

We have

$$(xy)z = [[xyy]zz] = [[xyy]z[zzz]] = [x[yzy][zzz]] = [x[yzz][yzz]] = x(yz),$$

which proves the first equality. The second is evident, and the third is a consequence of lemma 1.

Now let us define another binary operation by means of the formula $x \circ y = f(f(x, y, \ldots, y), x, \ldots, x)$.

LEMMA 3. *The operation $x \circ y$ defines a P-function in $\mathfrak{A}$.*

Proof. According to the definition of P-functions we must check the following equalities:

$$(x \circ y) \circ z = x \circ (y \circ z), \tag{4}$$

$$x \circ x = x, \tag{5}$$

$$x \circ (y \circ z) = x \circ (z \circ y), \tag{6}$$

$$f(x_1, \ldots, x_n) \circ y = f(x_1 \circ y, \ldots, x_n \circ y), \tag{7}$$

$$y \circ f(x_1, \ldots, x_n) = y \circ f(y \circ x_1, \ldots, y \circ x_n), \tag{8}$$

$$f(x_1, \ldots, x_n) \circ x_k = f(x_1, \ldots, x_n) \quad (k = 1, 2, \ldots, n), \tag{9}$$

$$y \circ f(y, \ldots, y) = y. \tag{10}$$

We consider only the case of $n=3$. Equalities (5) and (10) are evident. Since clearly $x\circ y = xyx$, lemma 2 implies (6). Now $(x\circ y)\circ z = xyxzxyx = xyzx = xyzyx = x\circ(y\circ z)$ giving (4).

To prove (7) consider the following chain of equalities, in which lemma 1 is used:

$$\begin{aligned}[xyz]\circ u &= \big[[[xyz]uu][xyz][xyz]\big] = \big[[[xuu]yz][xyz][xyz]\big]\\ &= \big[[[xuu]xx][yyy][zzz]\big] = \big[[[xuu]xx]yz\big] = \big[[[xxx]uu]yz\big]\\ &= [[xuu]yz] = \big[[x[uuu][uuu]]yz\big] = \big[[[xuu]u[uuu]]yz\big]\\ &= \big[[xuu][uy[uuu]]z\big] = \big[[xuu][[u][uyu]u]z\big] = [[xuu][uyu][uuz]]\\ &= \big[[[xxx]uu][u[yyy]u][uu[zzz]]\big] = \big[[[xuu]xx][y[uyu]y][zz[uuz]]\big]\\ &= \big[[xuu][u[uyu]u]z\big] = [(x\circ u)(y\circ u)(z\circ u)]\,.\end{aligned}$$

We have also

$$\begin{aligned}[xyz]\circ x &= \big[[[xyz]xx][xyz][xyz]\big]\\ &= [xxx]yz][xyz][xyz]\big] = [[xyz][xyz][xyz]] = [xyz];\end{aligned}$$

moreover,

$$\begin{aligned}[xyz]\circ y &= \big[[[xyz]yy][xyz][xyz]\big] = \big[[[xyz]xz][yyy][xyz]\big]\\ &= \big[[[xyz]xz]y[xyz]\big] = [[xyz][xyz][xyz]] = [xyz]\end{aligned}$$

and similarly

$$[xyz]\circ z = [xyz]\,,$$

thus proving (9).

Finally, to prove (8) observe that

$$\begin{aligned}u\circ[xyz] &= \big[[u[xyz][xyz]]uu\big] = \big[[[uuu][xyz][xyz]][uuu][uuu]\big]\\ &= \big[[[uxx][uyy][uzz]][uuu][uuu]\big] = \big[[[uxx]uu][[uyy]uu][[uzz]uu]\big]\\ &= [(u\circ x)(u\circ y)(u\circ z)]\,.\end{aligned}$$

The lemma is thus proved.

This lemma together with (ii) implies that our algebra $\mathfrak{A}$ is the sum of a direct system of algebras which satisfy the conditions (1), (2) and (3) and additionally the condition $x\circ y = x$, i.e.

(11) $$f\big(f(x, y, \ldots, y), x, \ldots, x\big) = x\,.$$

Let $\mathfrak{P} = \big(Y; f(x_1, \ldots, x_n)\big)$ be an algebra satisfying (1), (2), (3) and (11). Our theorem will be proved if we can show that $\mathfrak{P}$ is the product of an n-dimensional diagonal algebra and an r_n-algebra.

Observe at first that the operation $xy = f(x, y, \ldots, y)$ is diagonal, i.e. satisfies formulas (4), (5) and $xyz = xz$. Indeed, lemma 1 implies that xy is associative, and (11) shows that $xyx = x$, whence the diagonality follows from a result given in [1], p. 108. Now let us define in $\mathfrak{P}$ a binary operation by the formula $x*y = f\big(f(x, \ldots, x, y), x, \ldots, x\big)$.

LEMMA 4. *The operation $*$ is diagonal in $\mathfrak{P}$, i.e. it satisfies* (4), (5) (*where $\circ$ should be replaced by $*$*) *and the formula $x*y*z = x*z$.*

Proof. Equality (5) is evident. Let us consider the case of $n=3$ only, since the proof for arbitrary n can be carried out along the same lines. We have

$$\begin{aligned}(x*y)*z &= \big[[[[xxy]xx][[xxy]xx]z][[xxy]xx][[xxy]xx]\big]\\ &= \big[[[[xxy][xxy]x][xxx]z][[xxy]x[xxy]][xxx]\big]\\ &= \big[[[[[xxy][xxx]x]xz][xxy][xxy]]xx\big] = \big[[[xyy]xx]xz]xx\big]\\ &= \big[[[xxx]xz]xx\big] = [[xxz]xx] = x*z.\end{aligned}$$

Moreover,

$$\begin{aligned}x*(y*z) &= \big[[xx[[yyz]yy]]xx\big] = \big[[[xy[xyz]]xy]xx\big]\\ &= \big[[x[yyy]z][yxx]x\big] = [[xyz][yxx]x]\\ &= [[xxz][yxy]x] = \big[[[xxz]yy]xx\big] = [[xxz]xx] = x*z\,.\end{aligned}$$

The lemma is thus proved.

Now we define two relations in $\mathfrak{P}$. We put xR_1y if and only if $x*y = x$, and xR_2y if and only if $x*y = y$.

LEMMA 5. *The relations R_1 and R_2 are both congruences in $\mathfrak{P}$.*

Proof. We shall prove this for R_1 and, as before, restrict ourselves to $n=3$. Lemma 4 implies that R_1 is an equivalence. If $x_kR_1y_k$ for $k = 1, 2, 3$, then

$$\begin{aligned}&\big[[[x_1x_2x_3][x_1x_2x_3][y_1y_2y_3]][x_1x_2x_3][x_1x_2x_3]\big]\\ &\quad = \big[[[x_1x_1y_1][x_2x_2y_2][x_3x_3y_3]][x_1x_2x_3][x_1x_2x_3]\big]\\ &\quad = \big[[[x_1x_1y_1]x_1x_1][[x_2x_2y_2]x_2x_2][[x_3x_3y_3]x_3x_3]\big] = [x_1x_2x_3]\,,\end{aligned}$$

which shows that

$$[x_1x_2x_3]R_1[y_1y_2y_3]\,.$$

LEMMA 6. *The algebra $\mathfrak{P}/R_1$ is an r_n-algebra.*

Proof. Observe first that $[x_1x_2x_3]R_1[x_2x_1x_3]$. Indeed, we have

$$\big[[[x_1x_2x_3][x_1x_2x_3][x_2x_1x_3]][x_1x_2x_3][x_1x_2x_3]\big]$$
$$= \big[[[x_1x_2x_3][x_1x_2x_3][x_1x_2x_3]][x_1x_2x_3][x_1x_2x_3]\big] = [x_1x_2x_3]$$

as needed. Moreover, $[x_1x_2x_3]R_1[x_3x_2x_1]$ because

$$\big[[[x_1x_2x_3][x_1x_2x_3][x_3x_2x_1]][x_1x_2x_3][x_1x_2x_3]\big]$$
$$= \big[[[x_1x_2x_3][x_3x_2x_1][x_1x_2x_3]][x_2x_3x_2][x_1x_2x_3]\big]$$
$$= \big[[[x_1x_2x_3][x_1x_2x_3][x_1x_2x_3]][x_1x_2x_3][x_1x_2x_3]\big] = [x_1x_2x_3] .$$

Finally, $xR_1[xyy]$ because

$$\big[[xx[xyy]]xx\big] = \big[[x[xyy]x]xx\big] = \big[[x[yyx]x]xx\big]$$
$$= \big[[xx[yyx]]xx\big] = \big[[[xyy]xx]xx\big] = [[xxx]xx] = x .$$

It follows that in the algebra $\mathfrak{P}/R_1$ the fundamental operation $f(x_1, x_2, x_3)$ is symmetric and satisfies $f(x, y, y) = f(x) = x$, and conditions (1) and (2) which hold in $\mathfrak{P}$ imply that in $\mathfrak{P}/R_1$ the following equality is true:

$$f\big(f(x_1, x_2, x_3), x_4, x_5\big) = f\big(x_1, f(x_2, x_3, x_4), x_5\big)$$
$$= f\big(x_1, x_2, f(x_3, x_4, x)\big) .$$

As shown in [5], 2.4, these conditions imply that $\mathfrak{P}/R_1$ is an r_n-algebra.

The proof for the general case is similar.

LEMMA 7. *The algebra $\mathfrak{P}/R_2$ is a diagonal algebra.*

Proof. In view of (2) and (3), by using theorem II of [6] it is enough to show that $[[x_1x_2x_3]x_1x_1]R_2x_1$ holds in $\mathfrak{P}$. But we have

$$\big[[x_1x_1[[x_1x_2x_3]x_1x_1]]x_1x_1\big] = \big[[x_1x_1[x_1x_1[x_2x_3x_1]]]x_1x_1\big]$$
$$= \big[[[x_1x_1x_1]x_1[x_2x_3x_1]]x_1x_1\big] = \big[[x_1x_1[x_2x_3x_1]]\,x_1x_1$$
$$= \big[[[x_1x_2x_3]x_1x_1]x_1x_1\big] = \big[[x_1x_2x_3]x_1x_1\big] ,$$

as needed.

To finish the proof of the theorem it is sufficient to observe (see e.g. [3], p. 313) that in the algebra $(Y; *)$, in which the relations R_1 and R_2 are defined in the same way as it was done above, every class $\bmod R_1$ intersects every class $\bmod R_2$ at exactly one point, which shows that $\mathfrak{P}$ is the product of factor algebras $\mathfrak{P}/R_1$ and $\mathfrak{P}/R_2$, which by lemmas 6 and 7 are diagonal, or resp. are r_n-algebras. The theorem is thus proved.

COROLLARY 1. *For $n = 2$ every algebra of the direct system considered above is diagonal, since r_2-algebras can have only one element, which is easy to prove.*

Note that in this case this was proved by Yamada and Kimura in [8], theorem 6,8.

COROLLARY 2. *An idempotent medial algebra with the fundamental operation $f(x_1, \ldots, x_n)$ is the sum of a direct system of diagonal algebras if and only if the condition*

$$f\big(f(x_1, \ldots, x_n), x_2, \ldots, x_n\big) = f(x_1, \ldots, x_n)$$

is satisfied.

In fact, the necessity is obvious, since the condition is satisfied in diagonal algebras and (i) implies that it is also satisfied in their sum.

To prove the sufficiency it is enough to check that the congruence R_1 is trivial in our case, and this results from the following chains of equalities (again we write it down only for $n = 3$):

$$[[xxy]xx] = \big[[[xxy]xy]xx\big] = \big[[xxy][xxy]x\big] = \big[[xxx][yxy]x\big]$$
$$= \big[[[xxx]yy]xx\big] = x .$$

COROLLARY 3. *A medial idempotent algebra with the fundamental operation $f(x_1, \ldots, x_n)$ is the sum of a direct system of algebras, each of which is equal to the product of an r_n-algebra and a trivial algebra (i.e. one in which $f(x_1, \ldots, x_n) = x_1$ is the only fundamental operation) if and only if the equality*

$$f(x_1, \ldots, x_n) = f(x_1, x_{i_2}, \ldots, x_{i_n})$$

holds for every permutation $(i_2, \ldots, i_n)$ of the set $(2, 3, \ldots, n)$.

The necessity is obvious again and the sufficiency follows from the easy observation that our condition implies the equality $f(x_1, \ldots, x_n) = x_1$ in that factor of any algebra of the direct system given by our theorem which must be diagonal.

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On the Grothendieck group of compact polyhedra

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1. Introduction. In an earlier note [3] we constructed a set of examples of the following phenomenon: X_1 and X_2 are compact connected polyhedra with isomorphic homology and homotopy groups but of different homotopy types. The demonstration fell into two parts. First it was shown that it is possible to construct polyhedra X_1, X_2 of different stable homotopy types such that $X_1+S \simeq X_2+S$, where S is a suitable sphere and $+$ denotes the disjoint union with base points identified. Secondly it was shown that if $X_1+A \simeq X_2+A$ for a suitable compact connected polyhedron A, then the suspensions of X_1 and X_2 have isomorphic homotopy groups, $\pi_i \Sigma X_1 \cong \pi_i \Sigma X_2$.

In this paper we make a more systematic study of both parts of the argument and considerably strengthen the relevant statements. In section 2 we deal with the second part of the argument. We find it unnecessary to pass to the suspensions of X_1 and X_2 provided X_1, X_2, A are themselves already suspensions of connected polyhedra. This effects considerable improvement when it comes to finding examples. We also show that the homotopy groups kill the torsion in the Grothendieck group of suspensions of connected polyhedra. That is to say we may interpret the statement

$$\pi_i X_1 \cong \pi_i X_2 \quad \text{if} \quad X_1+A \simeq X_2+A$$

as saying that π_i may be regarded as being defined on those elements of the Grothendieck group $G(\Sigma\mathcal{T}^1)$ of homotopy classes of suspensions of compact connected polyhedra which are represented by polyhedra; call this subset $G^+(\Sigma\mathcal{T}^1)$. Then we actually prove the statement

$$\pi_i X_1 \cong \pi_i X_2 \quad \text{if} \quad tX_1+A \simeq tX_2+A \quad \text{for some integer } t>0;$$

that is, if X_1 and X_2 represent the same element of $G(\Sigma\mathcal{T}^1)$ modulo its torsion subgroup. Although this improvement is, at this stage, purely theoretical, it fits better into the general algebraic picture. For π_i maps $\Sigma\mathcal{T}^1$ to $\mathcal{A}b_0$, interpreted as the collection of isomorphism classes of finitely generated abelian groups. If we form the Grothendieck group of $\mathcal{A}b_0$