A representation theorem for idempotent medial algebras
by
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1. An algebra $\mathfrak{A} = (X; f(x_1, \ldots, x_n))$ is called a medial algebra if it satisfies the following two conditions:

   (1) $f(f(x_{i_1}, \ldots, x_{i_k}), \ldots, f(x_{i_m}, \ldots, x_{i_n}))$

   $= f(f(x_{i_k}, \ldots, x_{i_{k+1}}), \ldots, f(x_{i_n}, \ldots, x_{i_1}))$

   holds for every permutation $((i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n))$ of the set $(1, 1), (1, 2), \ldots, (n, n - 1), (n, n)$ such that $(i_r, j_r) = (r, r)$ holds for $r = 1, 2, \ldots, n$.

   (2) $f(f(x_1, \ldots, x_n), y_1, \ldots, y_n) = f(x_1, f(x_2, y_2, \ldots, y_n), \ldots, f(x_n, y_1, \ldots, y_{n-1})), y_n) = f(x_1, y_1, \ldots, f(x_n, y_{n-1}, f(x_1, \ldots, x_n, y_n))$

   Condition (1) in the case $n = 2$ coincides with the condition of mediality, considered by S. K. Stein in [7] for quasigroups. Condition (2) is a generalization of associativity to which it reduces in the case of $n = 2$.

   In this note we shall give a constructive description of all medial algebras which are idempotent, i.e. which satisfy in addition to (1) and (2) the condition

   (3) $f(x, x, \ldots, x) = x$.

   We shall need the notion of the sum of a direct system of algebras, as defined in [4], [5]: if $\mathcal{A} = \langle I, \langle A_I \rangle_{I \in \mathfrak{I}}, \langle p_i \rangle_{i \in \mathfrak{I}} \rangle$ is a direct system with l.u.b. of similar algebras without fundamental nullary operations, then $S(\mathcal{A})$, the sum of the system $\mathcal{A}$, is an algebra whose carrier is equal to the sum of carriers of $\mathcal{A}_I$'s (the algebras $\mathcal{A}_I$ should be treated as disjoint) and the fundamental operations $F_i$ are defined by

   $F_i(a_1, \ldots, a_n) = F_i(p_1(a_1), \ldots, p_n(a_n))$

   where $i = \text{l.u.b.}(i_1, \ldots, i_n)$ and $a_i$ belongs to the carrier of $\mathcal{A}_{i}$.

   We recall two results from [5] which we shall use in sequel:

   (i) (See [4], theorem I) If $\mathcal{A}$ is not a trivial direct system of algebras (i.e. if it consists of at least two algebras), then in the algebra $S(\mathcal{A})$ all
regular equations satisfied in all algebras of \( A \) are satisfied, whereas no other equation is satisfied in \( S(4) \).

(An equation \( f = g \), where \( f \) and \( g \) are terms of an algebra, is called regular if the set of free variables occurring in \( f \) coincides with the set of free variables occurring in \( g \).)

(ii) (See [4], theorem III.) Let \( \mathbb{A} \) be an algebra belonging to an equational class \( K \) whose defining equations are all regular. Let \( g(x, y) \) be a term of \( \mathbb{A} \) and let \( K^* \) be the equational class defined by the equations of the class \( K \) to which the equation \( g(x, y) = x \) has been added. Then the term \( g(x, y) \) induces a P-function for \( \mathbb{A} \) if and only if \( \mathbb{A} \) is representable as a sum of a direct system of algebras from the class \( K^* \). (For the definition of a P-function see [4], [5].)

Our description of idempotent algebras will use the following algebras: \( n \)-dimensional diagonal algebras (see [2], [3] for properties and representation theorem), i.e., algebras \( \{ X; d(x_1, \ldots, x_n) \} \) such that \( d(x_1, x_\ldots, x_n) = x \) and \( d(d(x_1, \ldots, x_n), \ldots, d(x_1, \ldots, x_n)) = d(x_1, \ldots, x_n) \) and \( r_n \)-algebras (see [4], 2.4) which can be described in the following way: take an abelian group such that \( (n-1)a = 0 \) holds identically, and define \( f(a_1, \ldots, a_n) = a_1 + \cdots + a_n \). The carrier of the group with \( f \) as the fundamental operation is a \( r_n \)-algebra. The class of \( r_n \)-algebras can also be described by using equations and this was done in [4], 2.4.

The results of this note were announced without proof in [5].

2. The following theorem gives a complete description of idempotent medial algebras:

**Theorem.** An algebra \( \mathbb{A} = \{ X; f(x_1, \ldots, x_n) \} \) is an idempotent medial algebra if and only if \( \mathbb{A} \) is the sum of a direct system of algebras, each of them being a product of an \( n \)-dimensional diagonal algebra and an \( r_n \)-algebra.

**Proof.** The “if” part of the theorem follows immediately from the fact that diagonal algebras and \( r_n \)-algebras are medial and idempotent and that the class of medial algebras can be defined by using regular equations only, whence an application of (i) leads us to the desired result.

To prove the “only if” part, let us assume that \( \mathbb{A} \) is an idempotent medial algebra, and that \( f(a_1, \ldots, a_n) \) is the fundamental operation of \( \mathbb{A} \). We shall need some lemmas.

**Lemma 1.** In \( \mathbb{A} \) the following equalities are true:

\[
(f(x_1, \ldots, x_n), y, \ldots, y, z_1, \ldots, z_n) = f(f(x_1, y, \ldots, y, z_1, \ldots, z_n), z_1, \ldots, z_n),
\]

and

\[
(f(x_1, \ldots, x_n), y_1, \ldots, y_k, f(x_1, \ldots, x_n), y_{k+1}, \ldots, y_n, z_1, \ldots, z_n) = f(f(x_1, \ldots, x_n), y_1, \ldots, y_k, f(y_1, \ldots, y_k, f(x_1, \ldots, x_n), y_{k+1}, \ldots, y_n, z_1, \ldots, z_n), z_1, \ldots, z_n) (1 < k \leq n).
\]

**Proof.** We prove only the first equality, the proof of the second being similar. We have

\[
f(f(x_1, \ldots, x_n), y_1, \ldots, y_n, z_1, \ldots, z_n) = f(f(x_1, y_1, \ldots, y_n, z_1, \ldots, z_n), y_1, \ldots, y_n, z_1, \ldots, z_n) = f(f(x_1, y_1, \ldots, y_n, z_1, \ldots, z_n), y_1, \ldots, y_n, z_1, \ldots, z_n)
\]

as asserted.

Now we introduce a binary operation by the formula \( xy = f(x, y, \ldots, y) \).

**Lemma 2.** The following equalities are true:

\[
(xy)z = x(yz), \quad xz = zx, \quad yz = zy.
\]

**Proof.** We shall prove this lemma, and also the next few lemmas, only for the case of \( n = 3 \), the proof for the general case being identical in principle but involving complicated notation. In the case of \( n = 3 \) we shall write for shortness \( [xyz] \) instead of \( f(x, y, z) \).

We have

\[
(xy)z = [xyz]z = [xyz][z]z = [x(yz)]z = x([yz]z) = x(yz),
\]

which proves the first equality. The second is evident, and the third is a consequence of lemma 1.

Now let us define another binary operation by means of the formula

\[
\alpha \ast \beta = f(f(x, y, \ldots, y), x, \ldots, x).
\]

**Lemma 3.** The operation \( \alpha \ast \beta \) defines a P-function in \( \mathbb{A} \).

**Proof.** According to the definition of P-functions we must check the following equalities:

\[
\begin{align*}
(\alpha \ast \beta) \ast \gamma &= \alpha \ast (\beta \ast \gamma), \\
\alpha \ast \alpha &= \alpha, \\
\alpha \ast (\beta \ast \gamma) &= \alpha \ast (\beta \ast \gamma), \\
f(\alpha_1, \ldots, \alpha_n) \ast \gamma &= f(\alpha_1, \ldots, \alpha_n) \ast \gamma, \\
\gamma \ast f(\alpha_1, \ldots, \alpha_n) &= \gamma \ast f(\alpha_1, \ldots, \alpha_n), \\
f(\alpha_1, \ldots, \alpha_n) \ast \gamma &= f(\alpha_1, \ldots, \alpha_n) \ast \gamma (k = 1, 2, \ldots, n), \\
\gamma \ast f(\alpha_1, \ldots, \alpha_n) &= f(\alpha_1, \ldots, \alpha_n) \ast \gamma.
\end{align*}
\]
We consider only the case of $n = 3$. Equalities (5) and (10) are evident. Since clearly $x + y = x$, lemma 2 implies (6). Now $(x + y) + z = x + y + z = x + y + z$ giving (4).

To prove (7) consider the following chain of equalities, in which lemma 1 is used:


and similarly


We also have


moreover,


and similarly


thus proving (9).

Finally, to prove (8) observe that

$$x + [xyz] = \ldots = \ldots$$

The lemma is thus proved.

This lemma together with (ii) implies that our algebra $\mathfrak{H}$ is the sum of a direct system of algebras which satisfy the conditions (1), (2) and (3) and additionally the condition $x + y = x$, i.e.

$$f(x, y, \ldots, z) = x.$$

Let $\mathfrak{H} = (X; f(x_1, \ldots, x_n))$ be an algebra satisfying (1), (2), (3) and (11). Our theorem will be proved if we can show that $\mathfrak{H}$ is the product of an $n$-dimensional diagonal algebra and an $x_n$-algebra.

Observe at first that the operation $xy = f(x, y, \ldots, y)$ is diagonal, i.e. satisfies formulas (4), (5) and $xy = x$. Indeed, lemma 1 implies that $xy$ is associative, and (11) shows that $xy = x$, whence the diagonality follows from a result given in (1), p. 108. Now let us define in $\mathfrak{H}$ a binary operation by the formula $2xy = f(x, \ldots, x, y, \ldots, y)$, where $u$ should be replaced by $e$, and the formula $2xy = 2x$. We have

$$(xy)z = [[[2xy]z][2xy]z][2xy]z = [[[2xy]z][2xy]z][2xy]z$$

and similarly

$$2(2xy)z = [[[2xy][2][2xy]z][2xy]z][2xy]z = [[[2xy][2][2xy]z][2xy]z][2xy]z$$

Moreover,$$

2x(2xy)z = [[[2x][2xy][2][2xy]z][2xy]z][2xy]z = [[[2x][2xy][2][2xy]z][2xy]z][2xy]z$$

The lemma is thus proved.

Now we define two relations in $\mathfrak{H}$. We put $xR_y$ if and only if $xy = x$, and $xR_y$ if and only if $2xy = 2x$. We shall prove this for $R_1$ and, as before, restrict ourselves to $n = 3$. Lemma 4 implies that $R_1$ is an equivalence. If $aR_ky_k$ for

$$k = 1, 2, 3,$$

then

$$([a_1, a_2, a_2, a_3][y_1, y_2, y_3][c_1, c_2, c_3][a_1, a_2, a_3]$$

and similarly

$$([a_1, a_2, a_2, a_3][y_1, y_2, y_3][c_1, c_2, c_3][a_1, a_2, a_3]$$

which shows that

$$[a_1, a_2, a_3]R_k[y_1, y_2, y_3].$$
Lemma 6. The algebra \( \mathcal{B}_n \) is an \( r_n \)-algebra.

Proof. Observe first that \([a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2] = [a_1, a_2]\) as needed. Moreover, \([a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2] \) because

\[
\begin{align*}
[[[a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2]] & = [[a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2]] \\
& = [[a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2]] \\
& = [[a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2]] \\
& = [[a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2]] \\
& = [a_1, a_2].
\end{align*}
\]

Finally, \( aB[a y y y] \) because

\[
\begin{align*}
[[[a y y y]]] & = [[[a y y y]]] \\
& = [[[a y y y]]] \\
& = [[[a y y y]]] \\
& = [[[a y y y]]] \\
& = [a y y y].
\end{align*}
\]

It follows that in the algebra \( \mathcal{B}_n \), the fundamental operation \( f(a_1, a_2, a_3) \) is symmetric and satisfies \( f(x, y, y) = f(x) = y \), and conditions (1) and (2) which hold in \( \mathcal{B}_n \), the following equality is true:

\[
\begin{align*}
f(f(a_1, a_2, a_3), a_4, a_5) & = f(a_1, f(a_2, a_3, a_4), a_5) \\
& = f(a_1, a_2, f(a_3, a_4, a_5)).
\end{align*}
\]

As shown in [3, 2.4], these conditions imply that \( \mathcal{B}_n \) is an \( r_n \)-algebra.

The proof for the general case is similar.

Lemma 7. The algebra \( \mathcal{A}_n \) is a diagonal algebra.

Proof. In view of (2) and (3) by using theorem II of [6] it is enough to show that \( [a_1, a_2]a_n [a_1, a_2]a_n \) holds in \( \mathcal{A}_n \). But we have

\[
\begin{align*}
[[[a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2]] & = [[[a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2]]] \\
& = [[[a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2]]] \\
& = [[[a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2]]] \\
& = [[[a_1, a_2][a_1, a_2][a_1, a_2][a_1, a_2]]] \\
& = [a_1, a_2].
\end{align*}
\]

as needed.

To finish the proof of the theorem it is sufficient to observe (see e.g. [3, p. 313]) that in the algebra \( (Y, \ast) \), in which the relations \( B_n \) and \( B_n \) are defined in the same way as it was done above, every class mod \( B_n \)

intersects every class mod \( B_n \) at exactly one point, which shows that \( \mathcal{A}_n \)

is the product of factor algebras \( \mathcal{B}_n \) and \( \mathcal{A}_n \), which by lemmas 6 and 7 are diagonal, or resp. are \( r_n \)-algebras. The theorem is thus proved.

Corollary 1. For \( n = 2 \) every algebra of the direct system considered above is diagonal, since \( r_2 \)-algebras can have only one element, which is easy to prove.

Note that in this case this was proved by Yamada and Kimura in [8], theorem 6.8.

Corollary 2. An idempotent medial algebra with the fundamental operation \( f(a_1, ..., a_n) \) is the sum of a direct system of diagonal algebras if and only if the condition

\[
\begin{align*}
f(f(a_1, ..., a_2), a_3, ..., a_2) & = f(a_1, ..., a_2)
\end{align*}
\]

is satisfied.

In fact, the necessity is obvious, since the condition is satisfied in diagonal algebras and (4) implies that it is also satisfied in their sum.

To prove the sufficiency it is enough to check that the congruence \( B_n \) is trivial in our case, and this results from the following chains of equalities (again we write it down only for \( n = 3 \)):

\[
\begin{align*}
[[[a y y]]] & = [[[a y y]]] \\
& = [[[a y y]]] \\
& = [[[a y y]]] \\
& = [[[a y y]]] \\
& = [a y y].
\end{align*}
\]

Corollary 3. A medial idempotent algebra with the fundamental operation \( f(a_1, ..., a_n) \) is the sum of a direct system of algebras, each of which is equal to the product of an \( r_n \)-algebra and a trivial algebra (i.e. one in which \( f(a_1, ..., a_n) = a_n \) is the product of \( r_n \)-algebra and a trivial algebra) if and only if the equality

\[
\begin{align*}
f(a_1, ..., a_n) & = f(a_1, a_n, ..., a_n)
\end{align*}
\]

holds for every permutation \( (a_1, ..., a_n) \) of the set \( (2, 3, ..., n) \).

The necessity is obvious again and the sufficiency follows from the easy observation that our condition implies the equality \( f(a_1, ..., a_n) = a_k \) in that factor of any algebra of the direct system given by our theorem which must be diagonal.

References

On the Grothendieck group of compact polyhedra

by

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1. Introduction. In an earlier note [3] we constructed a set of examples of the following phenomenon: $X_i$ and $X_4$ are compact connected polyhedra with isomorphic homology and homotopy groups but of different homotopy types. The demonstration fell into two parts. First it was shown that it is possible to construct polyhedra $X_i$, $X_4$ of different stable homotopy types such that $X_i+S \simeq X_4+S$, where $S$ is a suitable sphere and $+$ denotes the disjoint union with base points identified. Secondly it was shown that if $X_i+A \simeq X_4+A$ for a suitable compact connected polyhedron $A$, then the suspensions of $X_i$ and $X_4$ have isomorphic homotopy groups, $\pi_t \Sigma X_i \simeq \pi_t \Sigma X_4$.

In this paper we make a more systematic study of both parts of the argument and considerably strengthen the relevant statements. In section 2 we deal with the second part of the argument. We find it unnecessary to pass to the suspensions of $X_i$ and $X_4$ provided $X_i$, $X_4$, $A$ are themselves already suspensions of connected polyhedra. This effects considerable improvement when it comes to finding examples. We also show that the homotopy groups kill the torsion in the Grothendieck group of suspensions of connected polyhedra. That is to say we may interpret the statement

$$\pi_t X_i \simeq \pi_t X_4 \text{ if } X_i+A \simeq X_4+A$$

as saying that $\pi_t$ may be regarded as being defined on those elements of the Grothendieck group $G(\Sigma^p)$ of homotopy classes of suspensions of compact connected polyhedra which are represented by polyhedra; call this subset $G_t(\Sigma^p)$. Then we actually prove the statement

$$\pi_t X_i \simeq \pi_t X_4 \text{ if } tX_i+A \simeq tX_4+A \text{ for some integer } t>0;$$

that is, if $X_i$ and $X_4$ represent the same element of $G(\Sigma^p)$ modulo its torsion subgroup. Although this improvement is, at this stage, purely theoretical, it fits better into the general algebraic picture. For $\pi_t$ maps $\Sigma^p$ to $\mathcal{M}_n$, interpreted as the collection of isomorphism classes of finitely generated abelian groups. If we form the Grothendieck group of $\mathcal{M}_n$,