Topologies with T_1-complements

by

E. F. Steiner and A. K. Steiner (Albuquerque, New Mexico)

The family of all topologies definable on an arbitrary set X forms a complete lattice Σ under the partial ordering: $\tau_1 \leqslant \tau_2$ if and only if $\tau_1 \subseteq \tau_2$. The lattice operations $\wedge$ and $\vee$ are defined as: $\tau_1 \wedge \tau_2 = \tau_1 \cap \tau_2$ and $\tau_1 \vee \tau_2$ is the topology generated by the base $\mathfrak{B} = \{B\colon B = U_1 \cap U_2,\ U_1 \in \tau_1 \text{ and } U_2 \in \tau_2\}$. The greatest element, 1, is the discrete topology and the least element, 0, is the trivial topology. The lattice Σ has been recently studied ([2], [3], [4]) and has been shown to be complemented [5].

The family of all T_1-topologies definable on X forms a complete sublattice Λ of Σ, with greatest element 1, and least element, the co-finite topology $\mathfrak{C} = \{U\colon U = \emptyset \text{ or } X - U \text{ is finite}\}$. However, an example has been given [6] to show that Λ is not a complemented lattice, unless X is a finite set.

The question arises as to which T_1-topologies on X do have T_1-complements and whether it is possible to characterize these topologies. Several large classes of T_1-topologies which have T_1-complements have already been investigated ([1], [6]), but no characterization has been given. It is not even known if the set of real numbers with the usual topology has a T_1-complement.

It is the purpose of this paper to present more classes of T_1-topologies which have T_1-complements and to give some topological properties of a T_1-complement for the real numbers, if one exists.

It is shown that if a Hausdorff space (X, τ) satisfying the first axiom of countability has a T_1-complement τ', then τ' is countably compact on co-finite subsets of X. It is also shown that if a dense subspace (Y, τ_1) of (X, τ) has a T_1-complement which is compact on co-finite subsets of Y, then τ has a T_1-complement which is compact on co-finite subsets of X. (However, the converse of this theorem fails to be true.) Thus, if the rational numbers have a T_1-complement, so do the real numbers.

If $\tau \in \Lambda$, then τ' is a T_1-*complement* for τ if $\tau' \in \Lambda$, and $\tau \vee \tau' = 1$ and $\tau \wedge \tau' = \mathfrak{C}$.

Define $\mathfrak{I}(X)$ to be the family of co-finite subsets of X; that is, $\mathfrak{I}(X) = \{U \subseteq X\colon X - U \text{ is finite}\}$.

The symbol $|X|$ will be used to denote the cardinality of the set X.

THEOREM 1. *The order topology on any well-ordered set X has a T_1-complement.*

Proof. Let τ be the order topology on a well-ordered set X. Let τ' be the T_1-topology generated by unions of sets of the form:

(i) U, where $U \in \mathfrak{J}(X)$,

(ii) $\{x\}$, where x is a limit ordinal in the well-order of X.

Let $x \in X$. If x is a limit ordinal, then $\{x\} \in \tau'$ and if not, $\{x\} \in \tau$. Thus, $\tau \vee \tau' = 1$. Clearly, if $U \in \tau \wedge \tau'$ and $U \neq \emptyset$, then $U \in \mathfrak{J}(X)$ and $\tau \wedge \tau' = \mathfrak{C}$.

THEOREM 2. *Let (X, τ) be a Hausdorff space satisfying the first axiom of countability. If τ has a T_1-complement τ', then τ' must be countably compact on co-finite subsets of X.*

Proof. Since the discrete topology has a T_1-complement, compact on co-finite subsets, we can assume that $\tau \neq 1$.

Suppose that there is a T_1-topology τ' such that $\tau \vee \tau' = 1$ and $\tau \wedge \tau' = \mathfrak{C}$. Let E be a co-finite subset of X, that is $E \in \mathfrak{J}(X)$, and assume that there is a countable open covering $\mathcal{G} = \{U_n\colon U_n \in \tau',\ n = 1, 2, \ldots\}$ of E such that no finite subcollection of $\mathcal{G}$ covers E.

Well-order the elements of E as $\{x_\alpha\colon \alpha < \mu\}$, where μ is the smallest ordinal of cardinality $|E|$.

Let y_1 be an element of U_1 and define $S_1 = U_1 - \{y_1\}$. Since τ' is a T_1-topology, $S_1 \in \tau'$. Denote by y_2 the first $x_\alpha \in E$ such that $x_\alpha \notin S_1$ and $y_1 < x_\alpha$. Such an x_α exists since no finite number of U_n covers E. Define $S_2 = S_1 \cup (U_2 - \{y_1, y_2\})$. Then $S_2 \in \tau'$ and $y_1, y_2 \notin S_2$. If S_k has been defined for all $k < n$, denote by y_n the first $x_\alpha \in E$ such that $x_\alpha \notin S_{n-1}$ and $y_{n-1} < x_\alpha$. Define $S_n = S_{n-1} \cup \{U_n - \{y_1, y_2, \ldots, y_n\}\}$. Then

$$S_n = \bigcup_{i=1}^{n} \{U_i - \{y_1, y_2, \ldots, y_n\}\} \in \tau' .$$

For each $n = 1, 2, \ldots$, $S_n \in \tau'$, so $\bigcup_{n=1}^{\infty} S_n \in \tau'$ and $E - \bigcup_{n=1}^{\infty} S_n = Y = \{y_n\colon n = 1, 2, \ldots\}$. Thus $X - \bigcup_{n=1}^{\infty} S_n = Y \cup K$, where K is a finite subset of $X - E$.

Since $\tau \wedge \tau' = \mathfrak{C}$, the only closed sets which can be common to τ and τ' are either finite subsets of X, or X. And since $K \cup Y$ is closed in τ', if $K \cup Y$ is closed in τ then $K \cup Y = X$. But if $K \cup Y = X$, then $S_n = \emptyset$ for each $n = 1, 2, \ldots$ This implies that each U_n is finite and that for each $x \in E$, $\{x\} \in \tau'$. However, since τ is Hausdorff and $\tau \neq 1$, there is a $w \in X$ such that w has no finite neighborhood in τ. Let $x \in E$ such that $x \neq w$. Then there are disjoint open sets V_1 and V_2 in τ such that $w \in V_1$ and $x \in V_2$. Since $X - E$ is finite, $V_2 \cap E \in \tau$ and $V_2 \cap E \in \tau'$. But, $V_1 \subset X - \{V_2 \cap E\}$ and V_1 is not finite. Hence, $V_2 \cap E \notin \mathfrak{C}$, and $\tau \wedge \tau' \neq \mathfrak{C}$.

Thus $K \cup Y$ is not closed in τ, and since K is finite, Y is not closed in τ. However, τ is first countable and Hausdorff and so there exists a subsequence $\{y_n'\}$ of $\{y_n\}$ which converges to a point $z \in X$, and the set $Y' = \{z, y_1', y_2', \ldots\}$ is closed in τ.

Define

$$R_n = \bigcup_{i=1}^{n} \{U_i - \{y_1', y_2', \ldots, y_n', z\}\} \quad \text{for each } n = 1, 2, \ldots .$$

Since τ' is a T_1-topology, $R_n \in \tau'$ for each n and $\bigcup_{n=1}^{\infty} R_n \in \tau'$. Hence $X - \bigcup_{n=1}^{\infty} R_n = Y' \cup K'$, where K' is a finite subset of $X - E$, is closed in τ and in τ' and is different from X. This contradicts the assumption that τ' is a T_1-complement for τ and therefore τ' must be countably compact on co-finite subsets of X.

COROLLARY. *Let (X, τ) be a countable Hausdorff space. If τ has a T_1-complement τ', then τ' must be compact on co-finite subsets of X.*

THEOREM 3. *If a countable Hausdorff space (X, τ) has a T_1-complement, τ', then τ' cannot be Hausdorff.*

Proof. Let us suppose that τ' is Hausdorff and let $x \in X$. By the corollary to Theorem 2, $X - \{x\}$ is compact in the topology τ'. Hence (τ' being Hausdorff), $X - \{x\}$ is a closed subset of (X, τ'). This implies that $\tau' = 1$ and since τ' is a T_1-complement for τ, it implies that $\tau = \mathfrak{C}$. This is a contradiction since τ is Hausdorff.

THEOREM 4. *Let (X, τ) be a T_1-space satisfying the second axiom of countability, where $|X| > \aleph_0$. If τ has a T_1-complement τ', then τ' cannot be second countable.*

Proof. If τ and τ' both have a countable base, then $\tau \vee \tau'$ has a countable base. If $|X| > \aleph_0$, the discrete topology on X does not have a countable base, so $\tau \vee \tau' \neq 1$.

THEOREM 5. *Let X be an open dense subset of a T_1-space (Y, τ). If $\tau|X = \tau_X$ has a T_1-complement τ_X', then τ has a T_1-complement.*

Proof. Let τ' be the T_1-topology generated by unions of sets of the form:

(i) $\{y\}$, *for* $y \in Y - X$,

(ii) $U \cup F$, where $U \in \tau_X'$ and $F \in \mathfrak{J}(Y - X)$.

Since τ_X' is a T_1-topology, τ' is T_1. Also, $\tau'|X = \tau_X'$.

If $y \in Y - X$, then $\{y\} \in \tau'$. If $y \in X$, there exist sets $U \in \tau_X$ and $V \in \tau_X'$ such that $\{x\} = U \cap V$. But $U \in \tau$ and $V^* = V \cup (Y - X) \in \tau'$ so $\{x\} = U \cap V^* \in \tau \vee \tau'$. Hence $\tau \vee \tau' = 1$.

Let $U \in \tau \wedge \tau'$, $U \neq \emptyset$. Then $U \cap X \in \tau_X \wedge \tau'_X$, so $U \cap X \in \mathfrak{I}(X)$. (Since X is dense, $U \cap X \neq \emptyset$.) But $U \in \tau'$ implies that $U \in \mathfrak{I}(Y)$. Hence $\tau \wedge \tau' = \mathfrak{C}$.

However, the converse of Theorem 5 is not true, as may be seen from the following example.

Let $Y = E_1 \cup E_2 \cup S$, where E_1, E_2, and S are mutually disjoint infinite subsets of Y. Let τ be the T_1-topology on Y generated by sets of the form:

(i) $\{x\}$, for $x \in E_1$,

(ii) U, where $U \in \mathfrak{I}(E_2)$,

(iii) $\{s\} \cup W$, where $s \in S$ and $W \in \mathfrak{I}(E_1 \cup E_2)$,

(iv) W, where $W \in \mathfrak{I}(E_1 \cup E_2)$.

Let τ' be the T_1-topology on Y generated by sets of the form:

(i) U, where $U \in \mathfrak{I}(S)$,

(ii) $\{x\} \cup V$, where $x \in E_2$ and $V \in \mathfrak{I}(S \cup E_1)$,

(iii) V, where $V \in \mathfrak{I}(S \cup E_1)$.

It may easily be seen that τ' is a T_1-complement for τ. Now, let $X = E_1 \cup E_2$. Then X is an open, dense subset of Y. However, $\tau|X$ has no T_1-complement (for the proof, see [6]).

THEOREM 6. *Let X be a dense subset of a T_1-space (Y, τ). If $\tau_X = \tau|X$ has a T_1-complement τ'_X which is compact on co-finite subsets of X, then τ has a T_1-complement which is compact on co-finite subsets of Y.*

Proof. Let τ' be the T_1-topology whose base $\mathfrak{B}'$ consists of sets of the form:

(i) $\{y\}$ for $y \in Y - X$,

(ii) $\tilde{V} - F$, where $\tilde{V} = (\bar{V} - X) \cup V$, $V \in \tau'_X$, $\bar{V} = \mathrm{cl}_Y V$ and F is a finite subset of $Y - X$.

If $U \in \tau'$, then

$$U = [\bigcup (\tilde{V}_a - F_a)] \cup [\bigcup \{y_\beta\}] \quad \text{and} \quad U \cap X = \bigcup V_a \in \tau'_X .$$

If $V \in \tau'_X$, then $\tilde{V} \cap X = V$, so $\tau'|X = \tau'_X$. Since τ'_X is a T_1-topology, τ' is also T_1.

If $y \in Y - X$ then $\{y\} \in \tau'$. If $y \in X$, then there are sets $U \in \tau_X$ and $V \in \tau'_X$ such that $\{y\} = U \cap V$. But $U = W \cap X$, where $W \in \tau$. Since τ is T_1, $W - \{y\} \in \tau$ and $Y - (W - \{y\})$ is closed, contains V, and hence contains $\bar{V}$. Therefore $\{y\} = W \cap \tilde{V} \in \tau \vee \tau'$ and $\tau \vee \tau' = 1$.

Let $U \in \tau \wedge \tau'$ and suppose $U \neq \emptyset$. Since X is dense in Y, $U \cap X \neq \emptyset$. Thus $U \cap X \in \tau_X \wedge \tau'_X$ and $U \cap X \in \mathfrak{I}(X)$. Now, $U \in \tau'$ implies that $U = [\bigcup (\tilde{V}_a - F_a)] \cup [\bigcup \{y_\beta\}]$, so $U \cap X = \bigcup V_a$. Since τ'_X is compact on co-finite subsets of X, $U \cap X = \bigcup_{i=1}^{n} V_i$. And since X is dense in Y, $\bigcup_{i=1}^{n} \bar{V}_i = Y - K_1$, where K_1 is a finite subset of X. Now,

$$\begin{aligned}\bigcup_{i=1}^{n} \tilde{V}_i &= \bigcup_{i=1}^{n} [(\bar{V}_i - X) \cup V_i] = \{[\bigcup_{i=1}^{n} \bar{V}_i] - X\} \cup \{\bigcup_{i=1}^{n} V_i\} \\ &= \{\overline{\bigcup_{i=1}^{n} V_i} - X\} \cup \{\bigcup_{i=1}^{n} V_i\} = \{Y - X\} \cup \{U \cap X\} \\ &= Y - K_2, \quad \text{where } K_2 \text{ is a finite subset of } X .\end{aligned}$$

But $\bigcup_{i=1}^{n} (\tilde{V}_i - F_i) \subseteq U$, so $\bigcup_{i=1}^{n} \tilde{V}_i \subseteq U \cup F$, where F is a finite subset of $Y - X$. Therefore $\{Y - X\} \cup \{U \cap X\} = Y - K_2 \subseteq U \cup F$ and thus $Y \subseteq U \cup F \cup K_2$. Hence $U \in \mathfrak{I}(Y)$ and $\tau \wedge \tau' = \mathfrak{C}$.

Let $\mathfrak{G}$ be an open covering of E in τ', where $E \in \mathfrak{I}(Y)$. Each $G \in \mathfrak{G}$ is the union of base elements of τ' so $\mathfrak{G}' = \{B_a\colon G = \bigcup B_a, B_a \in \mathfrak{B}', G \in \mathfrak{G}\}$ also covers E and thus covers $E \cap X$. The family $\mathfrak{G}'' = \{B_a \cap X\colon B_a \in \mathfrak{G}'\}$ covers $E \cap X$ and is a subcollection of τ'_X so a finite number of $B_a \cap X$ cover $E \cap X$; that is,

$$E \cap X = \bigcup_{i=1}^{n} B_i \cap X = \bigcup_{i=1}^{n} V_i .$$

Since $\tilde{V}_i - F_i \subseteq B_i$, for $i = 1, 2, \ldots, n$,

$$\begin{aligned}\bigcup_{i=1}^{n} B_i \supseteq \bigcup_{i=1}^{n} (\tilde{V}_i - F_i) &= \{\overline{\bigcup_{i=1}^{n} V_i} - X\} \cup \{E \cap X\} - F' \\ &= \{Y - X\} \cup \{E \cap X\} - F' \in \mathfrak{I}(Y) ,\end{aligned}$$

where F' is a finite subset of $Y - X$. Each $B_i \subseteq G_i \in \mathfrak{G}$ so $\bigcup_{i=1}^{n} G_i$ covers all but a finite number of elements of E. Therefore a finite subcollection of $\mathfrak{G}$ covers E and τ' is compact on co-finite subsets of Y.

The following example shows that the converse of Theorem 6 is not true:

Let $Y = E_1 \cup E_2 \cup E_3 \cup E_4$, where E_1, E_2, E_3, and E_4 are mutually disjoint infinite sets. Let τ be a T_1-topology on Y generated by sets of the form:

(i) $\{x\}$ for $x \in E_1$,

(ii) $\{x\} \cup U$, where $x \in E_2$ and $U \in \mathfrak{I}(E_1)$,

(iii) $\{x\} \cup V$, where $x \in E_3$ and $V \in \mathfrak{I}(E_4)$,

(iv) V, where $V \in \mathfrak{I}(E_4)$.

Let τ' be the T_1-topology on Y generated by sets of the form:

(i) U, where $U \in \mathfrak{I}(Y)$,

(ii) V, where $V \in \mathfrak{I}(E_2)$,

(iii) W, where $W \in \mathfrak{I}(E_3)$,

(iv) $\{x\} \cup V \cup W$, where $x \in E_4$, $V \in \mathfrak{I}(E_2)$, and $W \in \mathfrak{I}(E_3)$.

It can easily be seen that $\tau \vee \tau' = 1$ and $\tau \wedge \tau' = \mathfrak{C}$. Moreover, since any co-finite subset of Y must contain points of E_1 and since any open set in τ' containing points of E_1 must itself be co-finite, τ' is compact on co-finite subsets of Y. However, if $X = E_1 \cup E_4$, then X is a dense subset of Y and $\tau_X = \tau|X$ has the following properties: $E_1, E_4 \in \tau_X$, $\tau_X|E_1 = 1$, and $\tau_X|E_4 = \mathfrak{C}$. Therefore τ_X has no T_1-complement (for proof, see [6]).

Since the real numbers with the usual topology $\mathfrak{R}$ satisfy the hypotheses of Theorems 2 and 4, if $\mathfrak{R}$ has a T_1-complement $\mathfrak{R}'$, then $\mathfrak{R}'$ cannot have a countable base and must be countably compact on all co-finite sets of real numbers.

Similarly, if $\mathfrak{R}_Q$ denotes the usual topology on the rational numbers, $\mathfrak{R}'_Q$ is a T_1-complement for $\mathfrak{R}_Q$ only if $\mathfrak{R}'_Q$ is compact on all co-finite subsets of rational numbers and $\mathfrak{R}'_Q$ is not Hausdorff.

Therefore, by Theorem 6, if $\mathfrak{R}_Q$ has a T_1-complement then $\mathfrak{R}$ has a T_1-complement which is compact on co-finite subsets of real numbers, and which is not Hausdorff. However, if $\mathfrak{R}_Q$ has no T_1-complement, no conclusion can be drawn about the existence of a T_1-complement for $\mathfrak{R}$.

References

[1] R. W. Bagley, *On the characterization of the lattice of topologies*, London Math. Soc. Journal 29-30 (1955), pp. 247-249.

[2] M. P. Berri, *The complement of a topology for some topological groups*, Fund. Math. 58 (1966), pp. 159-162.

[3] H. Gaifman, *The lattice of all topologies on a denumerable set* (abstract), Amer. Math. Soc. Notices 8 (1961), pp. 356.

[4] A. K. Steiner, *The Topological Complementation Problem*, Bull. Amer. Math. Soc. 72 (1966), pp. 125-127.

[5] — *The Lattice of Topologies: Structure and Complementation*, Trans. Amer. Math. Soc. 122 (1966), pp. 379-398.

[6] — *Complementation in the Lattice of T_1-Topologies*, Proc. Amer. Math. Soc. 17 (1966), pp. 884-885.

Reçu par la Rédaction 8. 2. 1966

An extension of a theorem of Gaifman-Hales-Solovay

by

Saul A. Kripke (Omaha, Nebraska)

Solovay [1] has found a remarkably simple proof of a theorem of Gaifman [2] and Hales [3]: *There are countably generated complete Boolean algebras of arbitrarily high cardinality*. Here we show that Solovay's methods can be extended to prove the stronger theorem: *Every Boolean algebra can be completely embedded in a countably generated complete Boolean algebra.* (A *complete embedding* of a Boolean algebra B into a complete Boolean algebra B' is a monomorphism of B into B' preserving all suprema that happen to exist in B. If B is itself a complete Boolean algebra, this means that *all* suprema are preserved (1).)

Like Solovay's [1], the present work was suggested by Cohen's notion of *forcing* [5]. We will follow Solovay, however, in making the present proof independent of any knowledge of Cohen's work, leaving the connection with Cohen to be divined by the *cognoscenti* (2).

Let S be any discrete topological space, and let S^ω be the product space of S taken countably many times as a factor. An element of S^ω can be represented as a function f from the positive integers into S. If we are given a finite sequence $\sigma = \langle s_1, \ldots, s_n\rangle$ of elements of S, the set of all functions $f \in S^\omega$ with $f(i) = s_i$ $(i = 1, \ldots, n)$ is an open set of the product space S^ω; call it $\mathfrak{O}(\sigma)$. The sets of form $\mathfrak{O}(\sigma)$ form a basis for the product topology of S^ω. We explicitly include the empty finite sequence; the corresponding basic open set is the entire space S^ω.

Solovay [1] has shown that the regular open algebra of S^ω, where S is any discrete space, is a countably generated complete Boolean algebra.

LEMMA 1. *Let S be a discrete space, and let B' be the regular open algebra of S^ω. For each element $s \in S$ and positive integer n, let $\varrho(n, s)$ be the*

(1) We use the notations of Halmos [4] for finite and infinite Boolean operations, except that the complement of b is denoted by $-b$. For all unexplained terminology, the reader should consult [1], [2], [3], or [4].

In the terminology of [4], p. 34, Ex. 6, Theorem 1 of the present paper would read: *Every Boolean algebra is isomorphic to a regular subalgebra of a countably generated complete Boolean algebra.* (So in particular, every complete Boolean algebra is isomorphic to a complete subalgebra of a countably generated complete Boolean algebra).

(2) The connection between Boolean algebras and forcing will be developed in a forthcoming paper by Scott and Solovay. I have not yet seen this paper, but I have benefited from hearing both authors expound some of their ideas.