line $M \in \boldsymbol{M}$ which passes through p' and intersects the interior of R'. We find an interior point p of R' such that $p \in M \cap X$. It is not difficult to check that $\varrho(o, p) = \theta$ and $\varrho(o, q) = \theta + \varrho(p, q)$.

Remark. In view of 3.1, a metric space satisfying 4.3 cannot be complete. However, we do not know whether there exists a non-degenerate zero-dimensional separable metric space which is star-like at each point, and which is topologically complete, i.e. homeomorphic with a complete metric space. Such a space would exist if one could construct a G_δ on the plane such that all conditions from 4.1 are fulfilled.

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Connectivity retracts of finitely coherent Peano continua

by

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The principle result of this paper is

THEOREM 1. *Every connectivity retract of a k-coherent Peano continuum is an m-coherent Peano continuum, where $m < k$.*

An auxiliary result essential to the proof of Theorem 1 is

LEMMA 1. *If X is a k-coherent Peano continuum and $H \subset X$ is totally disconnected, then each quasicomponent of $X - H$ is connected.*

In [3], it is shown that every connectivity retract of a continuum is a continuum, and there is described a Peano continuum (locally connected, metric) which has a connectivity retract that is not a Peano continuum. From Theorem 1, such a continuum must have infinite coherence. In [1], the special case of Theorem 1 for unicoherent continua ($k = 0$) was established. Lemma 1 is the key to the generalization of that argument and should be useful in other results on connectivity functions. One may readily construct examples which show that neither the condition that X be locally connected nor the condition that X be finitely coherent nor the condition that H be totally disconnected may be omitted from the hypothesis of Lemma 1.

In view of Theorem 1 and the fact that for finite polyhedra, the fixed point property is preserved by connectivity retraction ([3], Th. 3.13), we raise the

QUESTION. *Is there a k-coherent Peano continuum that has a connectivity retract that is not a continuous retract?*

1. Preliminaries. Let X and Y denote topological spaces and $f\colon X \to Y$ a transformation. Then f is a *connectivity function* if for each connected $C \subset X$, $\{(x, f(x))\colon x \in C\}$ is connected in the product space $X \times Y$. If $Y \subset X$ and f is a connectivity function and for each $x \in Y$, $f(x) = x$, then Y is a *connectivity retract* of X.

STALLINGS' LEMMA. *Suppose that X is a compact metric semi-locally-connected space and $f\colon X \to Y$ is a connectivity function where Y is a T_1 space. Then if C is a closed subset of Y and G' denotes the collection of components of $f^{-1}(C)$, the set G consisting of G' together with all of the degenerate subsets of $X - f^{-1}(C)$ is a monotone uppersemicontinuous decomposition of X and G' as a subset of the decomposition space is totally disconnected.*

Discussion of connectivity functions and Stallings' Lemma may be found in [1], [2], [3], [5] and [6]. If M is a subset of a topological space, $\mathrm{bd}(M)$ denotes the boundary of M, $\bar{M}$ denotes the closure of M, and the quasicomponent of M containing $P \in M$ is the set of all x in M such that M is not the sum of two mutually separated sets, one containing x and the other containing P.

Suppose that X is a continuum. If $C \subset X$, let $b(C)$ be the number that is one less than the number of components in C (possibly infinite, in which case $b(C) = \infty$). The *coherence* of X is the supremum of $\{b(U \cap V)\colon U$ and V are continua and $U \cup V$ is $X\}$. Also, X is k-coherent if X has *finite* coherence k.

LEMMA A. *If a subset H of a k-coherent Peano continuum X separates points a and b of X in X, there is a closed subset C of H with not more than k components which separates a from b in X. In particular, if H is totally disconnected, C is finite.*

LEMMA B. *Suppose that X is a Peano continuum. For each non-negative integer n, let S_n denote the statement "there is a subset F of X with $n+1$ points and there are disjoint closed subsets A and B of X such that neither A nor B separates two points of F in X, but $A \cup B$ separates each two points of F in X." Then if γ is a non-negative integer, X has coherence γ if and only if S_γ is true and $S_{\gamma+1}$ is false.*

LEMMA C. *If E is a subcontinuum of a k-coherent Peano continuum X and $\{C_\lambda\}$ are the components of $X - E$, then $\Sigma_\lambda b\big(\mathrm{bd}(C_\lambda)\big) \leqslant k$.*

It will be observed that Lemmas A and B are analogs of certain "Phragmèn-Brouwer" properties of unicoherent continua. Lemma A is contained in [7], p. 390, Th. 1. It seems that Lemma B has not appeared in this form. However, each implication either has a rather standard argument or follows, with suitable interpretation, from [7], p. 402, Lemma 7.4. Lemma C is implied by [7], p. 396, Th. 5.

2. Proof of Lemma 1. Suppose that X is a k-coherent Peano continuum, H is a totally disconnected subset of X, and C is a quasicomponent of $X - H$ which is the sum of two mutually separated non-empty sets A and B. It will be shown that $X - H$ is the sum of two mutually separated sets $\alpha \supset A$ and $\beta \supset B$ and thus violate the hypothesis that C is a quasicomponent of $X - H$.

For any quasicomponent Q of a subset T of a compact metric space S, there is a continuum Q' such that $Q \subset Q' \subset Q \cup (S - T)$. Hence, there is a continuum M such that $C \subset M \subset C \cup H$. In this instance, since H is totally disconnected, C is dense in M and M is simply $\bar{C}$. Let D denote the set of all components of $X - M$. From Lemma C, all but k of the members of D have connected boundaries and the boundary of no member of D has more than k components. Using the fact that H is totally disconnected, it will be shown that the components of the boundaries of the members of D are degenerate.

Suppose this is not so and Δ is a member of D whose boundary has a nondegenerate component. Then, by considering $n = (k+1)(k+2)$ arcs from a single point z of Δ to distinct accessible boundary points of Δ, one can obtain a continuum $Z \subset \Delta$ containing z, and n disjoint arcs $\{(x_i, y_i)\}_{i=1}^{n}$ such that for $i = 1, \ldots, n$, y_i is in Z but not in H, x_i is in $\mathrm{bd}(\Delta)$ and $(x_i, y_i) - \{x_i, y_i\} \subset \Delta - Z$. From Lemma A, for $i = 1, \ldots, n$, there is a finite $F_i \subset H$ such that F_i separates y_i from a point of C in X and must then separate y_i from all of C in X. Let $F = \bigcup_{i=1}^{n} F_i$ and $Y = \{y_1, \ldots, y_n\}$. Then F separates Y from C in X and there is $G \subset F$ such that G separates Y from C in X and no proper subset of G separates Y from C in X. Observe that for $i = 1, \ldots, n$, G must intersect the arc (x_i, y_i) and the component of $X - G$ containing y_i must have at least two boundary points, one in (x_i, y_i) and another in $Z \cup (x_j, y_j)$ for some $j \neq i$. Let E denote the component of $X - G$ that contains C. Then from Lemma C and because $G = \mathrm{bd}(E)$ is finite, no more than k components of $X - \bar{E}$ have nondegenerate boundaries. Therefore, some component of $X - \bar{E}$ must contain at least $k+2$ points of Y, but such a component would have at least $k+2$ boundary points which would contradict Lemma C.

It follows now that every member of D has a finite boundary and not more than k members of D have a nondegenerate boundary. Let $D' = \{\Delta_1, \ldots, \Delta_m\}$ denote those elements of D whose boundaries intersect both A and B. It will next be shown that for each $\Delta_i \in D'$, $\bar{\Delta}_i - H$ is the sum of two mutually separated set $U_i \supset A \cap \bar{\Delta}_i$ and $V_i \supset B \cap \bar{\Delta}_i$.

Suppose $\Delta_i \in D'$ and $x \in \mathrm{bd}(\Delta_i) \cap (A \cup B)$ and Q is the quasicomponent of $\bar{\Delta}_i - H$ containing x. Then just as $M = \bar{C}$ is connected, $\bar{Q}$ is connected, and if Q is nondegenerate, Q contains a point y of $\Delta_i - H$. But x and y do not belong to the same quasicomponent of $X - H$ and hence do not belong to the same quasicomponent of $\bar{\Delta}_i - H$, which involves a contradiction. We conclude that Q is $\{x\}$. Therefore, if $a \in A \cap \bar{\Delta}_i$ and $b \in B \cap \bar{\Delta}_i$, $\bar{\Delta}_i - H$ is the sum of two mutually separated sets, one containing a, the other containing b. Since there is only a finite collection of such pairs $\{a, b\}$, by a rather standard process one can obtain two mutually separated sets $U_i \supset A \cap \bar{\Delta}_i$ and $V_i \supset B \cap \bar{\Delta}_i$ such that $U_i \cup V_i = \bar{\Delta}_i - H$.

Since X is metric, there exist disjoint open sets α' and β' in X such that $A \subset \alpha'$ and $B \subset \beta'$. Then let

$$D_1 = \{\Delta \in D - D'\colon \mathrm{bd}(\Delta) \text{ intersects } M \cap \alpha' \text{ but not } B\},$$
$$D_2 = (D - D') - D_1,$$
$$\alpha = A \cup U_1 \cup \ldots \cup U_m \cup (\bigcup D_1 - H),$$
$$\beta = B \cup V_1 \cup \ldots \cup V_m \cup (\bigcup D_2 - H).$$

It is immediate that $\alpha \cup \beta$ is $X - H$ and that α and β are disjoint and contain A and B, respectively. It will be shown that α does not contain a limit point of β.

Suppose that a point P of α is a limit point of β. If P is in a member Δ' of D_1, Δ' is an open set in A containing P but no point of β, which is a contradiction. If for some $i = 1, \ldots, m$, P is in $U_i - A$, Δ_i is an open set in X containing P and no point of $\beta - V_i$. It would follow that P would be a limit point of V_i which would contradict the fact that U_i and V_i are mutually separated. Now suppose $P \in A$. There is a connected open set O in X such that $P \in O \subset \alpha'$, O does not intersect $V_i \cup \ldots \cup V_m$, and O does not intersect any of the finite number of elements of D that have non-degenerate boundaries that intersect B but not A. Then O contains a point R of β and R must belong to a member Δ'' of D_2. There is an arc T from R to P lying in O and the first point of that arc that belongs to M is a boundary point of Δ'' and belong to $M \cap \alpha'$. Since Δ'' belongs to D_2, Δ'' does not belong to D_1 or to D' and consequently must have a boundary point in B and no boundary point in A, and thus the last condition on O is violated. This exhausts the possibilities and we have that α contains no limit point of β. The proof that β contains no limit point of α is similar. Finally, α and β form the separation of $X - H$ mentioned in the first paragraph of this argument and the proof is complete.

3. Proof of Theorem 1. Suppose that X is a k-coherent Peano continuum, Y is a subspace of X and $f\colon X \to Y$ is a connectivity function such that for each x in Y, $f(x) = x$. Then Y is a continuum ([3], Th. 3.5). It will be shown first that Y is locally connected and then that Y has coherence $m \leqslant k$.

Y is locally connected. Suppose that P is a point of Y at which Y is not locally connected. Then there are open sets R and D containing P such that $\bar{D}$ is a subset of R and there is a sequence $M_1, M_2, \ldots$ of components of $Y \cap \bar{D}$ which converges to a non-degenerate continuum M containing P but no point of $\bigcup_{i=1}^{\infty} M_n$ and such that no component of $Y \cap \bar{R}$ intersects M and one of $M_1, M_2, \ldots$ ([4], p. 90, Th. 11). Let G' denote the collection of all components of $f^{-1}(Y - R)$ and let G denote G' together with the degenerate subsets of $f^{-1}(R \cap Y)$. From Stallings' Lemma, G is

a monotone uppersemicontinuous decomposition of X and G' is totally disconnected in G. Since the elements of G are connected, the associated map $T\colon X \to G$ is monotone and ([8], p. 153, Th. 8.6) G has finite coherence not greater than k. Since $M_1, M_2, \ldots$ converges to M in X, $T(M_1), T(M_2), \ldots$ converges to $T(M)$ in G. Also, since each of $M, M_1, M_2, \ldots$ is a subset of $Y \cap R$ and f is the identity on Y, no one of $M, M_1, M_2, \ldots$ intersects $f^{-1}(Y - R)$. Consequently, $T(M)$ is non-degenerate and each of $T(M)$, $T(M_1), T(M_2), \ldots$ is a subset of $G - G'$.

Let Q be the quasicomponent of $G - G'$ that contains $T(M)$. There exist $k+2$ disjoint connected open subsets $U_1, \ldots, U_{k+2}$ of G that intersect $T(M)$ and there is an integer i such that $T(M_i)$ intersects each of $U_1, \ldots, U_{k+2}$. If G' separates a point x of $T(M)$ from a point y of $T(M_i)$ in G, from Lemma A, there would be a set $F \subset G'$ with not more than $k+1$ points that separates x from y in G. But since $T(M)$ and $T(M_i)$ do not intersect G', F would have to intersect each of $U_1, \ldots, U_{k+2}$ which is impossible. It follows then that Q must contain $T(M_i)$.

From Lemma 1, Q must be connected. Since T is monotone, $T^{-1}(Q)$ is connected: therefore $f(T^{-1}(Q))$ is connected. Since $Q \subset G - G'$, $f(T^{-1}(Q)) \subset Y \cap R$, and since $T(M) \cup T(M_i) \subset Q$ and f is the identity on Y, $M \cup M_i \subset f(T^{-1}(Q))$ and the component of $Y \cap R$ containing M must intersect M_i. This contradicts an original stipulation on M and $M_1, M_2, \ldots$, and it follows that Y is locally connected.

Y has coherence $\leqslant k$. Suppose that Y has coherence greater than k. Then from Lemma B, there is a set F of $k+2$ distinct points of Y and there are disjoint closed subsets A and B of Y such that neither A nor B separates two points of F in Y, but $A \cup B$ separates each two points of F in Y. Let G' denote the collection of components of $f^{-1}(A \cup B)$ and let G be G' together with all of the degenerate subsets of $X - f^{-1}(A \cup B)$. As in the previous part, G is a monotone uppersemicontinuous decomposition of X, G' is totally disconnected, and the coherence of the decomposition space G is not greater than k. Also, $T\colon X \to G$ will denote the decomposition map. Since f preserves connected sets and A and B are mutually separated, each element of G' is a component of $f^{-1}(A)$ or of $f^{-1}(B)$, so that $T(f^{-1}(A))$ and $T(f^{-1}(B))$ are disjoint.

Since A [respectively B] does not separate two points of F in Y, there is a connected subset α [β] of $Y - A$ [$Y - B$] that contains F. Since f is the identity on Y, α [β] does not intersect $f^{-1}(A)$ [$f^{-1}(B)$] and $T(\alpha)$ [$T(\beta)$] is a connected subset of G that contains $T(F)$ and does not intersect $T(f^{-1}(A))$ [$T(f^{-1}(B))$]. Consequently, neither $T(f^{-1}(A))$ nor $T(f^{-1}(B))$ separates two points of $T(F)$ in G.

Suppose now that $T(f^{-1}(A)) \cup T(f^{-1}(B))$, which is G', separates each two points of $T(F)$ in G. From Lemma B, and because F is finite, there is a finite subset H of G' which separates each two points of $T(F)$ in G.

Then if A' is $H \cap T(f^{-1}(A))$ and B' is $H \cap T(f^{-1}(B))$, A' and B' are disjoint closed subsets of G, neither A' nor B' separates two points of $T(F)$ (which has $k+2$ points) in G, and $A' \cup B'$ separates each two points of $T(F)$ in G. This contradicts Lemma B, since the coherence of G is $\leqslant k$. Consequently, there are two points of $T(F)$ that are not separated in G by G' and therefore belong to the same quasicomponent Q of $G-G'$.

From Lemma 1, Q is connected. Since T is monotone, $T^{-1}(Q)$ is connected. Then $f(T^{-1}(Q))$ is a connected subset of $Y-(A \cup B)$ that contains two points of F and this involves a contradiction. It follows that the coherence of Y is less than or equal to k.

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On a method of construction of abstract algebras

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1. In this note we consider abstract algebras with finitary operations without nullary fundamental operations [1] and of a fixed type. First we recall the definition of a direct system of algebras (see [3], chapter 3):

(i) *I is a given poset (partially ordered set) whose ordering relation is denoted by $\leqslant$.*

(ii) *For each $i \in I$ an algebra $\mathfrak{A}_i = \langle A_i\colon \langle F_t^{(i)}\rangle_{t\in T}\rangle$ is given, all algebras $\mathfrak{A}_i$ being of the same type.*

(iii) *For each pair i, j of elements of I with $i \leqslant j$ a homomorphism $\varphi_{ij}\colon \mathfrak{A}_i \to \mathfrak{A}_j$ is given. The resulting set of homomorphisms must satisfy the following conditions:*

(a) *$i \leqslant j \leqslant k$ implies $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, and*

(b) *φ_{ii} is the identity map for $i \in I$.*

The system $\langle I, \langle \mathfrak{A}_i\rangle_{i\in I}, \langle \varphi_{ij}\rangle_{i\leqslant j;\, i,j\in I}\rangle$ is called a *direct system* of algebras.

We shall consider only direct systems $\mathcal{A}$ with the l.u.b.-property, i.e. systems which satisfy additionally the condition:

(iv) *The ordering relation of I induces a partial order with the least upper bound property* [2].

For every such direct system $\mathcal{A}$ we define an algebra $\mathfrak{A} = S(\mathcal{A})$ which we shall call the *sum* of the direct system $\mathcal{A}$.

We may clearly assume that the carriers of the algebras $\mathfrak{A}_i$ are mutually disjoint, as otherwise we could obtain this by taking isomorphic copies.

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[1] This is not a serious restriction. In fact, if a fundamental operation F_t is nullary, then one can replace it by a unary operation $F_t'(x) = F_t$, without essential changes in the algebraic structure of the algebra in question.

[2] We recall that an ordered set has the *least upper bound property* if every two of its elements have a least common upper bound.