On convex metric spaces IV

by

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§ 1. Introduction. The classical Bolzano-Weierstrass theorem says that every bounded closed subset of the Euclidean space is compact; in the present paper we prove and extend the following generalization:

Every bounded closed subset of a convex complete locally compact metric space is compact.

This form of the generalized Bolzano-Weierstrass theorem has been announced in [3]. We adopt the notation from [1]. Suppose that (X, ϱ) is a complete locally compact metric space. Some purely metric conditions are necessary in order that the Bolzano-Weierstrass theorem hold for (X, ϱ). Indeed, the real line with the metric $\operatorname{Min}\{1, |x-y|\}$ constitutes an example of a bounded complete locally compact metric space which is neither compact nor convex. A theorem below (see § 2) relaxes the condition of convexity, which is replaced by the concept of almost star-like metrization. Our considerations have arisen from a paper by C. Ryll-Nardzewski [4] and are related to some ideas which he has expressed; this is needed for a generalization of certain results of [2] and [4].

As a consequence of the Bolzano-Weierstrass theorem for almost star-like metric spaces we obtain (see § 3) the existence of metric segments joining an arbitrary point with a point in the space. During a seminar discussion Dr. Nitka observed that if a metric space (X, ϱ) satisfies the latter condition, then in order to have the Bolzano-Weierstrass theorem for (X, ϱ) it is enough to assume that (X, ϱ) is complete and peripherally compact, i.e. that there exists an open basis consisting of sets whose boundaries are compact. However, we do not know whether the Bolzano-Weierstrass theorem is true for every almost star-like complete peripherally compact metric space.

At the end of the paper (see § 4) we give a rather paradoxical example of a metric space in connection with the concept of convexity.

§ 2. The generalized Bolzano-Weierstrass theorem. A metric space (X, ϱ) is called *star-like* at a point $o \in X$ if for each $q \in X$, and each θ satisfying $0 \leqslant \theta \leqslant \varrho(o, q)$, there exists $p \in X$ such that

$$\varrho(o, p) = \theta = \varrho(o, q) - \varrho(p, q) .$$

A metric space (X, ϱ) is called *almost star-like* at a point $o \in X$ if for each $q \in X$, and each θ satisfying $0 \leqslant \theta \leqslant \varrho(o, q)$, and each $\varepsilon > 0$, there exists $p \in X$ such that

$$\varrho(o, p) - \varepsilon \leqslant \theta \leqslant \varrho(o, q) - \varrho(p, q) + \varepsilon .$$

We denote by $B(t)$ the ball $\{x:\ \varrho(o, x) \leqslant t\}$, and by t_0 the supremum of real numbers t such that the ball $B(t)$ is totally bounded ($0 \leqslant t_0 \leqslant \infty$).

2.1. *If a metric space (X, ϱ) is almost star-like at $o \in X$, and $t_0 < \infty$, then each infinite set $Y \subset B(t_0 + 1/n)$ contains an infinite subset Z with a diameter $\delta(Z) \leqslant 7/n$.*

Proof. In the case $t_0 = 0$ we put $Z = Y$. Assuming $t_0 > 0$ we choose positive real numbers t', t'' such that

$$t_0 - 1/n < t' < t'' < t_0 .$$

The ball $B(t')$ is totally bounded. If infinitely many points of Y are in $B(t')$, we easily find a suitable set Z because Y must then contain a Cauchy sequence. If that is not so, let us take different points $q_i \in Y \setminus B(t')$ where $i = 1, 2, \ldots$ Since the space is almost star-like at o, we get points $p_i \in X$ such that

$$\varrho(o, p_i) - (t'' - t') \leqslant t' \leqslant \varrho(o, q_i) - \varrho(p_i, q_i) + (t'' - t')$$

for $i = 1, 2, \ldots$ Thus $p_i \in B(t'')$ and

$$\varrho(p_i, q_i) \leqslant \varrho(o, q_i) + t'' - 2t' \leqslant t_0 + 1/n + t'' - 2t' < 3/n$$

for $i = 1, 2, \ldots$ But $B(t'')$ is also totally bounded. Consequently, there exists an infinite set I of positive integers such that $\varrho(p_i, p_j) < 1/n$ for $i, j \in I$. We define $Z = \{q_i:\ i \in I\}$.

2.2. THEOREM. *If a complete locally compact metric space (X, ϱ) is almost star-like, then each bounded subset in (X, ϱ) is totally bounded.*

Proof. We have to prove that $t_0 = \infty$. Suppose on the contrary that $t_0 < \infty$. There exist numbers $\varepsilon_n > 0$ and infinite sets $Y_n \subset B(t_0 + 1/n)$ such that $\varrho(p, q) > \varepsilon_n$ for $p, q \in Y_n$ and $p \neq q$ $(n = 1, 2, \ldots)$. By 2.1, there exist infinite sets $Z_n \subset Y_n$ such that $\delta(Z_n) \leqslant 7/n$. Let $z_n \in Z_n$. It follows from 2.1 that the sequence $z_1, z_2, \ldots$ must contain a Cauchy sequence. The latter converges to a point $z \in X$. Each neighbourhood of z contains one of the sets Z_n, which is impossible because X is locally compact.

The following examples refute some modifications of 2.2 which could be conjectured.

2.3. *There exist bounded convex metric spaces (X', ϱ') and (X'', ϱ'') such that* (i) *(X', ϱ') is locally compact but neither complete nor totally bounded, and* (ii) *(X'', ϱ'') is complete but neither locally compact nor totally bounded.*

Proof. Take plane sets

$$X' = \{(x, 0):\ 0 < x \leqslant 1\} \cup \bigcup_{i=1}^{\infty} \{(1/i, y):\ 0 \leqslant y \leqslant 1\} ,$$

$$X'' = \bigcup_{i=1}^{\infty} \{(x, x/i):\ 0 \leqslant x \leqslant 1\}$$

with distances $\varrho'(p, q)$, $\varrho''(p, q)$ defined as the lengths of the arcs which join p and q in X', X'', respectively. Then conditions (i) and (ii) are satisfied.

§ 3. Connectedness of almost star-like spaces. The space of rational numbers with the ordinary metric is almost star-like at each point. Thus non-degenerate non-complete almost star-like spaces may even be zero-dimensional. Moreover, it will be shown in the next paragraph that there exists a non-degenerate zero-dimensional separable metric space which is star-like at each point.

3.1. *If a complete metric space (X, ϱ) is almost star-like at $o \in A \subset X$, then for each $q \in X \setminus A$, and each $\eta > 0$, there exists $p \in \operatorname{Fr} A$ such that*

$$\varrho(o, p) + \varrho(p, q) \leqslant \varrho(o, q) + \eta .$$

Proof. In the case $\varrho(q, A) = 0$ we put $p = q$. Assuming $\varrho(q, A) > 0$ we inductively define points $p_0, p_1, \ldots$ such that $\varrho(p_n, A) > 0$ for $n = 0, 1, \ldots$ Let $p_0 = q$. If $n > 0$ and p_{n-1} is defined, then the positive numbers

$$\theta = \varrho(o, p_{n-1}) - \tfrac{1}{2}\varrho(p_{n-1}, A) , \qquad \varepsilon = \tfrac{1}{5}\operatorname{Min}\{\theta, \eta/2^n, \varrho(p_{n-1}, A)\}$$

satisfy the inequality $0 \leqslant \theta - \varepsilon \leqslant \varrho(o, p_{n-1})$, and the space (X, ϱ) being almost star-like at o, we get a point p_n such that

$$\varrho(o, p_n) - \varepsilon \leqslant \theta - \varepsilon \leqslant \varrho(o, p_{n-1}) - \varrho(p_n, p_{n-1}) + \varepsilon .$$

This yields

$$\varrho(o, p_n) \leqslant \varrho(o, p_{n-1}) - \tfrac{1}{2}\varrho(p_{n-1}, A) \tag{1}$$

and

$$\varrho(p_n, p_{n-1}) \leqslant \varrho(o, p_{n-1}) - \theta + 2\varepsilon < \varrho(p_{n-1}, A) ,$$

whence $\varrho(p_n, A) > 0$ and

$$\varrho(o, p_n) + \varrho(p_n, p_{n-1}) \leqslant \varrho(o, p_{n-1}) + \eta/2^n . \tag{2}$$

It follows from (1) that

$$\varrho(o, p_n) \leqslant \varrho(o, q) - \tfrac{1}{2}\sum_{i=0}^{n-1} \varrho(p_i, A)$$

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and from (2) that

$$\varrho(o, p_n)+\varrho(p_n, q) \leqslant \varrho(o, q)+\sum_{i=1}^{n} \eta/2^i < \varrho(o, q)+\eta .$$

Consequently, we obtain

$$\sum_{i=1}^{\infty} \varrho(p_i, p_{i-1}) \leqslant \sum_{i=0}^{\infty} \varrho(p_i, A) \leqslant 2\varrho(o, q) < \infty ,$$

which means that $p_0, p_1, \ldots$ is a Cauchy sequence; let p be its limit. Since $p_n \in X\setminus A$ and $\varrho(p_i, A)$ converges to zero as i tends to infinity, the point p satisfies the conclusion of 3.1.

3.2. *Almost star-like complete metric spaces are connected.*

The following theorem gives more information than 3.2 for the class of locally compact spaces.

3.3. *If a complete locally compact metric space (X, ϱ) is almost star-like at $o \in X$, then each point $q \in X\setminus\{o\}$ can be joined with o by a metric segment $\overline{oq}$.*

Proof. Let P_n be the finite set constructed as follows ($n = 1, 2, \ldots$). Put $p_n^0 = q$ and take a point

$$p_n^m \in \operatorname{Fr} B[(n-m)\varrho(o, q)/n]$$

such that

$$\varrho(o, p_n^m)+\varrho(p_n^m, p_n^{m-1}) \leqslant \varrho(o, p_n^{m-1})+1/n^2$$

according to 3.1 ($m = 1, \ldots, n$). Let $P_n = \{p_n^0, \ldots, p_n^n\}$.

Clearly all sets P_n are contained in the ball $B[\varrho(o, q)+1]$, which is compact by 2.2. Thus the sequence $P_1, P_2, \ldots$ has a convergent subsequence. Without loss of generality we can assume that the sequence $P_1, P_2, \ldots$ itself converges, and write

$$P = \operatorname*{Lim}_{n\to\infty} P_n .$$

It turns out that P is a segment $\overline{oq}$. Indeed, let us first notice that if p, p' are points of P_n, then

$$|\varrho(o, p)-\varrho(o, p')| \leqslant \varrho(p, p') \leqslant |\varrho(o, p)-\varrho(o, p')|+1/n$$

($n = 1, 2, \ldots$). Hence P meets the boundary of each ball $B(t)$, where $0 \leqslant t \leqslant \varrho(o, q)$, at exactly one point, i.e. we have $\operatorname{Fr} B(t) \cap P = \{p_t\}$. Furthermore, if $0 \leqslant t < t' \leqslant \varrho(o, q)$ and $\varepsilon > 0$, there exist an integer $n > 5/\varepsilon$ and points $p, p' \in P_n$ such that $\varrho(p, p_t) < \varepsilon/5$ and $\varrho(p', p_{t'}) < \varepsilon/5$. This gives the inequalities

$$\varrho(p, p')-2\varepsilon/5 < \varrho(p_t, p_{t'}) < \varrho(p, p')+2\varepsilon/5 ,$$

$$|t-t'|-2\varepsilon/5 < |\varrho(o, p)-\varrho(o, p')| < |t-t'|+2\varepsilon/5 ,$$

because $|\varrho(o, p_t)-\varrho(o, p_{t'})| = |t-t'|$. Consequently, we obtain

$$|t-t'|-\varepsilon < \varrho(p_t, p_{t'}) < |t-t'|+\varepsilon ,$$

which implies $\varrho(p_t, p_{t'}) = |t-t'|$. The set P is thus isometric with a segment of the real line, and so $P = \overline{oq}$.

§ 4. An example of metric space. We start with a construction of a set on the plane E^2. We denote by $\boldsymbol{L}$ the collection of all straight lines $\{(x, y)\colon x = a\}$ and $\{(x, y)\colon y = a\}$, and by $\boldsymbol{M}$ the collection of all straight lines $\{(x, y)\colon y = x+a\}$ and $\{(x, y)\colon y = -x+a\}$, where a is an arbitrary real number.

4.1. *There exists a set $X \subset E^2$ such that $L \cap X$ contains one point or is empty, and $M \cap X$ is dense in M, for every $L \in \boldsymbol{L}$ and $M \in \boldsymbol{M}$.*

Proof. Denote by γ the minimum ordinal of cardinality continuum. Let $\{J_\alpha\colon \alpha < \gamma\}$ be the collection of all intervals of lines from $\boldsymbol{M}$. We define points $p_\alpha \in J_\alpha$ inductively. Let p_1 be a point of J_1, and suppose that points p_α are defined for $\alpha < \beta$ where $\beta < \gamma$. The set Q of points at which the lines $L \in \boldsymbol{L}$ passing through p_α ($\alpha < \beta$) intersect J_β is of cardinality less than the continuum. Thus there exists a point $p_\beta \in J_\beta\setminus Q$. We take $X = \{p_\alpha\colon \alpha < \gamma\}$.

4.2. *If a set $X \subset E^2$ is dense in E^2, and $L \cap X$ contains one point or is empty for every $L \in \boldsymbol{L}$, then X is zero-dimensional.*

Proof. Given an arbitrary point $(x_0, y_0) \in X$ and $\varepsilon > 0$, the open set $\{(x, y)\colon x_0 < x < x_0+\varepsilon, |y-y_0| > \varepsilon\}$ contains a point (a, b) of X. Then the segment $\{(x, y)\colon x = a, |y-y_0| \leqslant \varepsilon\}$ is disjoint with X. It follows that (x_0, y_0) has arbitrarily small rectangular neighbourhoods whose boundaries lie outside X.

4.3. *There exists an uncountable zero-dimensional separable metric space which is star-like at each point.*

Proof. Let X be the set from 4.1 with the metric defined by the formula

$$\varrho[(x_1, y_1), (x_2, y_2)] = |x_1-x_2|+|y_1-y_2| ,$$

which does not change the natural topology. Thus (X, ϱ) is a zero-dimensional separable metric space, according to 4.2. Suppose that o, q are two points of X. Since no line $L \in \boldsymbol{L}$ passing through o contains q, we have a rectangle R whose sides are segments of $L \in \boldsymbol{L}$, and whose opposite vertices are o and q. Let S be the union of two sides of R, joining o and q. Observe that $\varrho(o, q)$ is the length of S. Now, if $0 < \theta < \varrho(o, q)$, let us consider a point $p' \in S$ such that the part of S from o to p' has the length θ. The arc S is divided by p' into two arcs, and at least one of them is a straight line segment $S' \subset S$ with the end point p'. Let $R' \subset R$ be a rectangle such that S' is a side of R'. Since p' is a vertex of R', there exists exactly one

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line $M \in \boldsymbol{M}$ which passes through p' and intersects the interior of R'. We find an interior point p of R' such that $p \in M \cap X$. It is not difficult to check that $\varrho(o, p) = \theta$ and $\varrho(o, q) = \theta + \varrho(p, q)$.

Remark. In view of 3.1, a metric space satisfying 4.3 cannot be complete. However, we do not know whether there exists a non-degenerate zero-dimensional separable metric space which is star-like at each point, and which is topologically complete, i.e. homeomorphic with a complete metric space. Such a space would exist if one could construct a G_δ on the plane such that all conditions from 4.1 are fulfilled.

References

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Connectivity retracts of finitely coherent Peano continua

by

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The principle result of this paper is

THEOREM 1. *Every connectivity retract of a k-coherent Peano continuum is an m-coherent Peano continuum, where $m < k$.*

An auxiliary result essential to the proof of Theorem 1 is

LEMMA 1. *If X is a k-coherent Peano continuum and $H \subset X$ is totally disconnected, then each quasicomponent of $X - H$ is connected.*

In [3], it is shown that every connectivity retract of a continuum is a continuum, and there is described a Peano continuum (locally connected, metric) which has a connectivity retract that is not a Peano continuum. From Theorem 1, such a continuum must have infinite coherence. In [1], the special case of Theorem 1 for unicoherent continua ($k = 0$) was established. Lemma 1 is the key to the generalization of that argument and should be useful in other results on connectivity functions. One may readily construct examples which show that neither the condition that X be locally connected nor the condition that X be finitely coherent nor the condition that H be totally disconnected may be omitted from the hypothesis of Lemma 1.

In view of Theorem 1 and the fact that for finite polyhedra, the fixed point property is preserved by connectivity retraction ([3], Th. 3.13), we raise the

QUESTION. *Is there a k-coherent Peano continuum that has a connectivity retract that is not a continuous retract?*

1. Preliminaries. Let X and Y denote topological spaces and $f\colon X \to Y$ a transformation. Then f is a *connectivity function* if for each connected $C \subset X$, $\{(x, f(x)) \colon x \in C\}$ is connected in the product space $X \times Y$. If $Y \subset X$ and f is a connectivity function and for each $x \in Y$, $f(x) = x$, then Y is a *connectivity retract* of X.