Algebraic independence and measure

by

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0. The main result of this paper (Theorem 1) extends the result of [3] to the case where the supposition "first category" is replaced by "measure 0". This solves some problems mentioned in [3], § 4.2 and in [4]. The first of them is the following question of P. Erdös (see [1], [7]): Does there exist a field of real numbers of the power $2^{\aleph_0}$ which does not contain any Liouville number? It was known (see e.g. [3], § 4.2) that there exists such a field of power $\aleph_1$, and thus using the continuum hypothesis the original problem was solved. Here we will give a proof without this hypothesis using only the fact that the set of Liouville numbers is of measure 0 and we get a field generated by a perfect set (while the argument quoted above did not give Borel generating sets). Our result implies also the existence of a perfect set whose set of distances is disjoint with a given set of measure 0; a slightly stronger theorem was proved recently in [4] and [6] (for related facts see also [3], § 4.5). Another simple corollary of our result seems new: Given a set X of measure 0 on the plane $\mathcal{R}^2$, there exists a perfect set $P \subseteq \mathcal{R}$ such that $P^2 \cap X \subseteq D$, where $D = \{(x, x)\colon x \in \mathcal{R}\}$.

The main results of this paper were announced in [5].

1. We assume that the notions introduced in [3] are familiar to the reader, but this paper will be almost self-contained if we repeat the following definition.

$\mathfrak{R} = \langle A, R_i \rangle_{i \in I}$ being a relational structure, a set $X \subseteq A$ is called *independent in* $\mathfrak{R}$ if

$$(x_1, \ldots, x_{r(i)}) \in R_i \Rightarrow \big(f(x_1), \ldots, f(x_{r(i)})\big) \in R_i$$

for every $i \in I$, $x_1, \ldots, x_{r(i)} \in X$ and every function $f\colon X \to A$, where $r(i)$ denotes the rank of R_i, i.e. $R_i \subseteq A^{r(i)}$.

From now on A denotes some fixed m_0-dimensional Euclidean space $\mathcal{R}^{m_0}$. For any measurable set $X \subseteq A^n$, $|X|_n$ denotes the $m_0 n$-dimensional Lebesgue measure of X. p is called a *metric density point* of X if $p \in A^n$ and

$$\lim_{r \to 0+} |B_p^{(r)} \cap X|_n / |B_p^{(r)}|_n = 1\,,$$

where $B_p^{(r)} = \{q: \|p-q\| \leqslant r\}$. $M_n(X)$ denotes the set of all metric density points of X. M_n will also be applied to sets contained in $m_0 n$-dimensional hyperplanes in A^{n+k}.

Perfect means: non-empty, closed and without isolated points.

In this paper A could be replaced by some more general topological measure-space for which there is a good concept of metric density point but we do not attempt to perform this generalization in detail referring the reader to [8], § 11 and [2], Sec. 61 (5) for the necessary technique.

2. THEOREM 1. *Let* $\mathfrak{R} = \langle A, R_1, R_2, \ldots\rangle$ *be a relational structure whose set of relations is countable and closed under identifications of variables* (i.e. $\overline{\mathfrak{R}} = \mathfrak{R}$ in the notations of [3]) *and such that for every* i *either* $|R_i|_{r(i)} = 0$ *or* $R_i = A^{r(i)}$. *Then there exists a perfect set* $P \subseteq A$ *which is independent in* $\mathfrak{R}$.

Proof. We will need some auxiliary notions and propositions.

A set $X \subseteq A^n$ is called *diagonal-measurable* if for every equivalence relation $E \subseteq \{1, \ldots, n\}^2$ the set

$$X \cap \{(x_1, \ldots, x_n): x_i = x_j \text{ for } (i,j) \in E\}$$

is measurable with respect to the $m_0 d$-dimensional measure, where $m_0 d$ is the dimension of the hyperplane $\{(x_1, \ldots, x_n): x_i = x_j \text{ for } (i,j) \in E\}$, which means that $d = \mathrm{card}(\{1, \ldots, n\}/E)$.

A set $F \subseteq A^n$ is called *fat* if it is diagonal-measurable and for every $(a_1, \ldots, a_n) \in F$ and every equivalence relation $E \subseteq \{(i,j): a_i = a_j\}$ we have

$$(a_1, \ldots, a_n) \in M_d\big(F \cap \{(x_1, \ldots, x_n): x_i = x_j \text{ for } (i,j) \in E\}\big),$$

where $d = \mathrm{card}(\{1, \ldots, n\}/E)$.

This definition clearly implies the following proposition.

(i) *If* $F \subseteq A^m$ *is fat,* $(i_1, \ldots, i_m) \in \{1, \ldots, n\}^m$ *and* $(a_1, \ldots, a_n) \in A^n$ *are such that*

$$(a_{i_1}, \ldots, a_{i_m}) \in F,$$

then

$$(a_1, \ldots, a_n) \in M_n\big(\{(x_1, \ldots x_n): (x_{i_1}, \ldots, x_{i_m}) \in F\}\big).$$

Let us put $D_i = \big\{(x_1, \ldots, x_{r(i)}): \big(f(x_1), \ldots, f_{r(i)}(x)\big) \in R_i$ for every function $f: \{x_1, \ldots, x_{r(i)}\} \to A\big\}$ and

$$S_i = (A^{r(i)} \setminus R_i) \cup D_i.$$

(ii) S_i *are fat sets and, for every set* $X \subseteq A$, X *is independent in* $\mathfrak{R}$ *if and only if* $X^{r(i)} \subseteq S_i$ *for every* i.

This follows from the properties of the sequence $R_1, R_2, \ldots$ which are stipulated in Theorem 1.

(iii) *If* $Q \subseteq U \subseteq A^n$, *where* Q *is closed and* U *is fat, then there exists a fat set* F *such that* $Q \subseteq F$ *and* $\overline{F} \subseteq U$. (1)

Notice first that

(*) If $X, Y \subseteq A^k$ are fat sets then $X \cap Y$ is fat.

Now we prove (iii) by induction on n. For $n = 1$ the argument would be an obvious simplification of the second step of our inductive proof which is the following. Suppose that $n > 1$ and the result holds for $n-1$. Then in each hyperplane $H_{ij} = \{(x_1, \ldots, x_n): x_i = x_j\}$ $(i<j)$ we find a set F_{ij} fat in H_{ij} and such that $Q \cap H_{ij} \subseteq F_{ij}$ and $\overline{F}_{ij} \subseteq U \cap H_{ij}$. Let be

$$F_0 = \bigcup_{a_0} F_{a_0} \setminus \bigcup_{a_0 \neq a_1} (H_{a_0} \cap H_{a_1}) \cup \bigcup_{a_0 \neq a_1} (F_{a_0} \cap F_{a_1})$$
$$\setminus \bigcup_{a_0 \neq a_1 \neq a_2 \neq a_0} (H_{a_0} \cap H_{a_1} \cap H_{a_2}) \cup \bigcup_{a_0 \neq a_1 \neq a_2 \neq a_0} (F_{a_0} \cap F_{a_1} \cap F_{a_2})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\setminus \bigcap_{1 \leqslant i < j \leqslant n} H_{ij} \cup \bigcap_{1 \leqslant i < j \leqslant n} F_{ij},$$

where all a_k run over the set of pairs (i,j) with $1 \leqslant i < j \leqslant n$. It follows from (*) that each set $F_0 \cap H_{ij}$ is fat in H_{ij} and of course $Q \cap H_{ij} \subseteq F_0 \cap H_{ij}$ and $\overline{F}_0 \subseteq U$. For every natural number m let V_m be a neighbourhood of radius $1/m$ around $\overline{F}_0 \cup Q$. We choose closed sets $C_m \subseteq U \cap V_m$ such that $\overline{F}_0 \cup Q \subseteq C_m$, and for every ball $B \subseteq V_m$ of radius $1/m$ we have

$$|B \cap C_m|_n / |B \cap U|_n \geqslant 1 - 1/m$$

(the existence of such C_m is visible). Now we put $C = \bigcup_{m=1}^{\infty} C_m$ and notice that C is closed, $C \subseteq U$ and $Q \subseteq M_n(C)$. Hence, by the construction of C, the set $F = M_n(C) \setminus \bigcup_{1 \leqslant i < j \leqslant n} H_{ij} \cup F_0$ is fat and satisfies (iii).

(iv) $|\{(x_1, \ldots, x_n): \{x_1, \ldots, x_n\}$ is not independent in $\mathfrak{R}\}|_n = 0$.

This follows from the suppositions of Theorem 1 (it helps to apply the statement (i) $\Leftrightarrow$ (iv) of [3], § 2, (1)).

(v) *If* $F_k \subseteq A^{r(k)}$ $(k = 1, \ldots, n)$ *is a system of fat sets and* $Q = \{q_1, \ldots, q_s\} \subseteq A$ *is a finite set such that* $Q^{r(k)} \subseteq F_k$ *for* $k = 1, \ldots, n$ *then, for every* $r > 0$,

$$|B^{(r)}_{(q_1, q_1, q_2, \ldots, q_s)} \cap \{(x_0, x_1, \ldots, x_s): x_0 \neq x_1 \text{ and } \{x_0, x_1, \ldots, x_s\}^{r(k)} \subseteq F_k \text{ for } k = 1, \ldots, n\}|_{s+1} > 0.$$

To prove this we put $q_0 = q_1$ and then, given a sequence $i_1, \ldots, i_{r(k)}$ $(k \leqslant n,\ 0 \leqslant i_k \leqslant s)$, we have by supposition $(q_{i_1}, \ldots, q_{i_{r(k)}}) \in F_k$. Hence since F_k is fat and by (i) the set

$$B^{(t)}_{(q_0, \ldots, q_s)} \cap \{(x_0, x_1, \ldots, x_s): (x_{i_1}, \ldots, x_{i_{r(k)}}) \notin F_k\}$$

(1) $\overline{F}$ denotes the topological closure of F.

is of relatively small measure in $B^{(t)}_{(q_0,\ldots,q_s)}$ when $t\downarrow 0$. This clearly implies (v), since the set of sequences $i_1,\ldots,i_{r(k)}$ that we have to consider is finite.

(iv) and (v) yield easily the following proposition.

(vi) *Under the suppositions of* (v) *for every* $r>0$ *there exists a sequence* $q_1', q_1''\in B^{(r)}_{q_1},\ldots,q_s',q_s''\in B^{(r)}_{q_s}$ *such that* $q_i'\neq q_i''$ *for* $i=1,\ldots,s$ *and*

$$\{q_1', q_1'', \ldots, q_s', q_s''\}^{r(k)}\subseteq F_k \quad \textit{for} \quad k=1,\ldots,n$$

and the set $\{q_1', q_1'', \ldots, q_s', q_s''\}$ *is independent in* $\mathfrak{R}$.

Now applying (i), ..., (vi) we will construct by induction a system of points $p_{b_1,\ldots,b_n}\in A$ $(b_k=0,1,\ n=1,2,\ldots)$ and a sequence of sets $F_1, F_2, \ldots$ such that

(1) $\overline{F}_i\subseteq S_i$ and F_i are fat sets;

(2) $p_{b_1,\ldots,b_i,0}\neq p_{b_1,\ldots,b_i,1}$ and

$$\|p_{b_1,\ldots,b_i}-p_{b_1,\ldots,b_i,b_{i+1}}\|\leqslant \|p_{b_1,\ldots,b_i}-p_{b_1',\ldots,b_i'}\|/3\cdot 2^i\,,$$

for $(b_1,\ldots,b_i)\neq(b_1',\ldots,b_i')$;

(3) $P^{r(k)}_{i+1}\subseteq F_k$ for $k=1,\ldots,i+1$, where $P_i=\{p_{b_1,\ldots,b_j}\colon (b_1,\ldots,b_j)\in\{0,1\}^j\}$.

Let $p_0, p_1\in A$ be any two points such that $\{p_0,p_1\}$ is independent in $\mathfrak{R}$ (the existence of such points follows from (iv)), and let F_1 be a set satisfying (1) (its existence follows from (ii) and (iii)). Suppose that $P_1,\ldots,P_i$ and $F_1,\ldots,F_i$ are already constructed and satisfy (1), (2) and (3). By (vi) we get P_{i+1} which is independent in $\mathfrak{R}$ and such that (2) and (3) for $k=1,\ldots,i$ hold. Finally the existence of F_{i+1} satisfying (1) and (3) follows from the independence of P_{i+1}, (ii) and (iii) with $Q=P^{r(i+1)}_{i+1}$ and $U=S_{i+1}$. This concludes our inductive definition of the sequences satisfying (1), (2) and (3).

Now we finish the proof of Theorem 1. We put $P=\lim\limits_{i\to\infty}P_i$. By (2) P is perfect. By (3), $P^{r(i)}\subseteq\overline{F}_i$ for every i. Hence, by (1) and (ii), P is independent in $\mathfrak{R}$. Q.E.D.

3. COROLLARY. *If X is a set of measure 0 of irrational real numbers then there exists a perfect set P such that the field generated by P is disjoint with X.*

This follows from Theorem 1 by the same argument which is given in [3], § 4.2.

Let us prove still a consequence of Theorem 1 of [3] closely related to the above result.

THEOREM 2. *Every perfect set P of real numbers contains a perfect subset which is algebraically independent.*

Proof. Let us show two auxiliary propositions.

(i) *If $w(x_1,\ldots,x_n)$ is a polynomial in n variables and real coefficients which vanishes on a set of the form $D_1\times\ldots\times D_n$, where each D_i has a limit point, then w is the constant* 0.

The proof follows by an easy induction on n which uses only analyticity of w.

(ii) *If P is a perfect set of real numbers and w is a non zero polynomial of n variables then the set*

$$R_w=P^n\cap\{(x_1,\ldots,x_n)\colon w(x_1,\ldots,x_n)=0\}$$

is nowhere dense in P^n.

This follows clearly from (i).

Now Theorem 2 follows from (ii) and Theorem 1 of [3] applied to the relational structure $\langle P, R_w\rangle_{w\in W}$, where W is the set of polynomials with integral coefficients in the variables $x_1, x_2, \ldots$

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