(c) *I under τ' is a separable space.*

(d) *I under τ' is a first countable space.*

(e) *I under τ' is not a second countable space.*

Proof. In this proof the notation is that used in defining τ'.

(a) Select a sequence $\{s_i\}$, $i = 1, 2, \ldots$, such that for each i, $s_i \in N$ and $\lim_{i\to\infty} s_i = 0$ under τ_0. It is apparent that such a sequence exists. Since τ' is finer than τ_0, 0 is the only possible limit point of the sequence. However, 0 has a τ'-neighborhood such that it contains no element of N. It follows that 0 is not a limit point of the sequence, and, hence, I under τ' is not countably compact. Applying the well-known theorem that every compact subset of a space is countably compact, it follows immediately that (I, τ') is not a compact space.

(b) Let U be a neighborhood of b_k of the form $[b_k \cup M] \cap (a, b)$ where $0 \leqslant a < b_k < b \leqslant 1$. Assume that there exists a neighborhood of b_k, call it V, such that $\overline{V} \subset U$. It is apparent that V would have a subset, call it W, of the same form as U. Now since all open sets of τ' are unions of finite intersections of subbase elements, there exists a positive integer j such that $(a_j, b_j) \subset W$. Observe that a_j is a limit point of W, hence of V, and $a_j \notin U$, Therefore $\overline{V} \not\subset U$.

(c) Observe that each open set of (I, τ') contains an interval. Therefore, each open set of (I, τ') contains a rational number. Hence, the set of rational numbers contained in I is a countable dense subset of (I, τ'), and I under τ' is a separable space.

(d) Observe from the definition of τ' that if $P \in I$, then one of the following forms a countable base, or local countable base, at p for τ'.

(1) $I \cap (p-1/n,\ p+1/n)$, $n = 1, 2, \ldots$

(2) $(p-1/n, p+1/n) \cap \{M \cup p\}$, $n = 1, 2, \ldots$

(3) $(p-1/n,\ p+1/n) \cap \{N \cup p\}$, $n = 1, 2, \ldots$

It follows that (I, τ') is a first countable space.

(e) Observe that each element $x \in K$ has a neighborhood of the form $U \cap \{M \cup x\}$, U an open set of (I, τ_0). Now if x_0 is a particular element of K, observe that no union of finite intersections of sets which are open in (I, τ'), excluding those of the form $U \cap \{M \cup x_0\}$ where U is an open set of $\{I, \tau_0\}$, is equal to a set of the form $U \cap \{M \cup x_0\}$, where U is an open set of (I, τ_0). Hence, since it is well known that K consists of a non-denumerable number of elements, it follows that (I, τ') is not second countable.

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Deduction-preserving "Recursive Isomorphisms" between theories * **

by

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Introduction. In this paper we concern ourselves with recursive mappings between theories which preserve deducibility, negation and implication. Roughly we show that any two axiomatizable theories containing a small fragment of arithmetic—this can be made precise—are "*isomorphic*" by a primitive recursive function which preserves deducibility, negation and implication (and hence theoremhood, refutability and undecidability). We also show that, between any two effectively inseparable theories formulated in the predicate calculus, there exists a recursive "isomorphism" preserving deducibility, negation and implication. We will see that we cannot replace "recursive" by "primitive recursive" in the last result. As a consequence we obtain a partition of all effectively inseparable theories in standard formalization into $\aleph_0$ equivalence classes. The unique maximal element is the equivalence class of those theories containing the small fragment of arithmetic mentioned above. A more precise summary of our results—which, incidentally answer some questions left open by Pour-El [6]—appears below, following some brief notational remarks.

In our opinion interest in the preservation of sentential connectives—particularly implication—can be justified by the following consideration. The preservation of implication implies the preservation of modus ponens and modus ponens is intimately related to the deductive structure of the theories. (Indeed, it is well known (Quine [5]) that the predicate calculus can formulated so that modus ponens is the sole rule of inference.)

All theories considered in this paper will contain the propositional calculus. For definiteness we assume that implication and negation are the sole primitive propositional connectives: $A \vee B$ is an abbreviation

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for $\neg A \to B$; $A \cdot B$ is an abbreviation for $\neg(\neg A \vee \neg B)$. Furthermore in every section except section 3 the theories discussed will be formulated as applied predicate calculi. All theories will be assumed to be both consistent and axiomatizable.

Notation. Let $\mathcal{T}$ be a theory. Associated with $\mathcal{T}$ is a *recursive* set W, the set of (Gödel numbers of) sentences and two recursively enumerable subsets of W, T the set of theorems and R the set of refutable sentences. We assume that W has an infinite complement $\overline{W}$. If $S \subset W$, then by S^A we mean the subset of S obtained by *omitting* all $\varphi \in S$ such that either $\varphi = \psi_1 \to \psi_2$ or $\varphi = \neg\psi_1$ for some ψ_1 and ψ_2 in W. Thus S^A consists of "generalized atoms" relative to negation and implication. Similarly S^- is the subset of S obtained by omitting all $\varphi \in S$ such that $\varphi = \neg\psi$ for some ψ in W. For example, if

$$S = \{(x)(\psi_1 \to \psi_2)\,,\ (x)\psi_1 \to (x)\psi_2\,,\ (x)\neg\psi_1\,,\ \neg(x)\psi_1\}\,,$$

then

$$S^A = \{(x)(\psi_1 \to \psi_2)\,,\ (x)\neg\psi_1\}$$

and

$$S^- = \{(x)(\psi_1 \to \psi_2)\,,\ (x)\psi_1 \to (x)\psi_2\,,\ (x)\neg\psi_1\}\,.$$

In general we identify a formula with its Gödel number. If a distinction is necessary it will be clear from the context.

Survey of results. The results of A, B, and C below are consequences of some basic lemmas which are too complicated to state here. Results in D are of a different nature: they complement those of C.

A. *Recursive mappings between applied predicate calculi.*

I. *If* $\mathcal{T}_1$ is consistent and $\mathcal{T}_2$ is effectively inseparable, *then* there is a 1-1 recursive function f^* mapping W_1 *into* W_2, $\overline{W}_1$ *onto* $\overline{W}_2$ such that for all B, C, in W_1

a. $f^*(B \to C) = f^*(B) \to f^*(C)$,

b. $f^*(\neg B) = \neg f^*(B)$,

c. $B \vdash_{\mathcal{T}_1} C$ *if and only if* $f^*(B) \vdash_{\mathcal{T}_2} f^*(C)$.

Furthermore, f^* maps W_1^A *into* W_2^A. (Theorem 1.)

From a, b, c we conclude immediately that theorems are mapped into theorems, refutables are mapped into refutables and undecidables are mapped into undecidables.

II. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are effectively inseparable, *then* there is a 1-1 recursive function f^* mapping W_1 *onto* W_2, $\overline{W}_1$ *onto* $\overline{W}_2$ such that for all B, C in W_1

a. $f^*(B \to C) = f^*(B) \to f^*(C)$,

b. $f^*(\neg B) = \neg f^*(B)$,

c. $B \vdash_{\mathcal{T}_1} C$ *if and only if* $f^*(B) \vdash_{\mathcal{T}_2} f^*(C)$.

Furthermore, f^* maps W_1^A onto W_2^A. (Theorem 2.)

B. *Recursive mappings between applied propositional calculi.*

We show by example that II *does not hold for all applied propositional calculi.* Nevertheless given two effectively inseparable (e.i.) theories $\mathcal{T}_1$ and $\mathcal{T}_2$, it is possible to find a 1-1 recursive function g mapping W_1 *onto* W_2 preserving negation, deducibility and which *up to deductive equivalence preserves implication.* More precisely,

III. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two e.i. theories. There is a 1-1 negation-preserving recursive function g mapping W_1 onto W_2 such that for all formulas B_1 and C_1 in $\mathcal{T}_1$.

a. $B_1 \vdash_{\mathcal{T}_1} C_1$ *if and only if* $g(B_1) \vdash_{\mathcal{T}_2} g(C_1)$,

b. $\vdash_{\mathcal{T}_2} g(B_1 \to C_1) \equiv . \ g(B_1) \to g(C_1)$,

and for all formulas B_2 and C_2 in $\mathcal{T}_2$,

$$\vdash_{\mathcal{T}_1} g^{-1}(B_2 \to C_2) \equiv . \ g^{-1}(B_2) \to g^{-1}(C_2)\,.$$

(Theorem 4.)

As a trivial consequence of III theorems are mapped onto theorems, refutables are mapped onto refutables and undecidables are mapped onto undecidables.

In contrast I does hold for propositional calculi provided we omit from the conclusion that f^* maps W_1^A into W_2^A. This is stated as theorem 3.

C. *Primitive recursive mappings between theories.*

For many mathematically interesting formal theories it is possible to strengthen results I and II by showing that f^* can be chosen to be primitive recursive. Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are theories in standard formalization possessing a notation for the natural numbers and a binary predicate $\leqslant$. Suppose further that $\mathcal{T}_i$ contains a subtheory $\mathcal{T}_i'$ such that the following hold

(1) for all $n \vdash_{\mathcal{T}_i'} x \leqslant \bar{n} \vee \bar{n} \leqslant x$,

(2) for all $n \vdash_{\mathcal{T}_i'} x \leqslant \bar{n} \to x = \bar{0} \vee \ldots \vee x = \bar{n}$,

(3) every primitive recursive function of one argument is definable in $\mathcal{T}_i'$.

Then II holds for $\mathcal{T}_1$ and $\mathcal{T}_2$ with a primitive recursive f^*.

(An analogous statement may be made for I, when $\mathcal{T}_2$ contains a subtheory $\mathcal{T}_2'$ satisfying (1), (2) and (3)—see a detailed discussion at the end of section 2.)

Thus for example if $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are any two consistent axiomatizable extensions of the theory* **R** *of Undecidable Theories,* II *holds for a primitive recursive* f^*.

D. *A hierarchy of effectively inseparable theories.*

In contrast to the results of the preceding paragraph it is, in general, not possible to choose the f^* of I or II to be primitive recursive. For we prove

IV. Let $\mathcal{F}$ be a recursively enumerable class of general recursive functions. Then there exists an effectively inseparable theory $\mathcal{T}_1$ in standard formalization such that no recursive function which witnesses the effective inseparability of $\mathcal{T}_1$ is in $\mathcal{F}$. (Theorem 6.)

As an immediate corollary we obtain (where **R** is the theory of *Undecidable Theories*, [10])

V. Given an r.e. class $\mathcal{F}$ of general recursive functions, there exists a theory $\mathcal{T}$ in standard formalization such that no 1-1 recursive function mapping W onto $\mathbf{W}_\mathbf{R}$, T onto $\mathbf{T}_\mathbf{R}$, R onto $R_\mathbf{R}$ preserving deducibility, negation and implication is in $\mathcal{F}$ (theorem 7).

We are thus led to a classification of effectively inseparable theories in standard formalization. These theories are divided into $\aleph_0$ equivalence classes with a unique maximum element. In particular, all the mathematically interesting theories described in the first paragraph of C above belong to this maximum element: between any two of these theories the f^* of II may be chosen to be primitive recursive. For details see the last part of the last section of this paper.

Basic definitions. Let $\omega_0, \omega_1, \ldots, \omega_n, \ldots$ be a standard recursive enemeration of all *recursively enumerable* (r.e.) *sets*. Assume that $\omega_0 = \emptyset$. Let j be the well-known primitive recursive pairing function; let K and L be its inverses.

Definition 1. A pair of disjoint r.e. sets (α, β) is *effectively inseparable* (e.i.) if there exists a recursive function g such that if

$$\omega_{K(i)} \supseteq \alpha\,, \quad \omega_{L(i)} \supseteq \beta\,, \quad \omega_{K(i)} \cap \omega_{L(i)} = \emptyset$$

then

$$g(i) \notin \omega_{K(i)} \cup \omega_{L(i)}\,.$$

Definition 2. Let α, β, γ be recursively enumerable sets. Let $\alpha \subseteq \gamma$, $\beta \subseteq \gamma$ and $\alpha \cap \beta = \emptyset$. Then the pair of sets (α, β) is *effectively inseparable relative to* γ if there exists a recursive function g whose range is included in γ and such that if

$$\gamma \supseteq \omega_{K(i)} \supseteq \alpha\,, \quad \gamma \supseteq \omega_{L(i)} \supseteq \beta\,, \quad \omega_{K(i)} \cap \omega_{L(i)} = \emptyset$$

then

$$g(i) \in \gamma - (\omega_{K(i)} \cup \omega_{L(i)})$$

Definition 3. An applied predicate calculus $\mathcal{T}$ is an *effectively inseparable theory* if (T, R) is an effectively inseparable pair of sets.

If $\mathcal{T}$ is a propositional calculus, then the concept of "sentence" may not have meaning since, for example, $\mathcal{T}$ may not possess variables. In that case we identify "sentence" with "formula". Thus definition 3 applies in that case also. (1)

Definition 4. A class $\mathcal{F}$ of general recursive functions of one argument is a *recursively enumerable* (r.e.) *class* if there exists a general recursive function f of two arguments such that $f_0, f_1, \ldots, f_n, \ldots$ is an enumeration of all and only the members of $\mathcal{F}$.

1. Some basic lemmas. In this section we assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two consistent axiomatizable theories formulated in the applied predicate calculus. (See introduction.)

Notation. (See introduction for notation for S_i^A and W_i^A.)

(i) If S_i^A is a subset of $\widetilde{W}_i^A$, then denote by $\widetilde{S}_i$ the set of all truth functional combinations of S_i^A.

(ii) If f maps S_1^A into S_2^A, then denote by f^* the induced mapping of $\widetilde{S}_1$ into $\widetilde{S}_2$ satisfying

$$f^*(E \to F) = f^*(E) \to f^*(F)\,,$$
$$f^*(\neg E) = \neg f^*(E)$$

for all E and F in $\widetilde{S}_1$.

(iii) $T_i \cap S_i^A$ is abbreviated by T_i^*; $R_i \cap S_i^A$ is abbreviated by R_i^*.

Lemma 1. *Suppose that*

(a) S_1^A *and* S_2^A *are infinite* r.e. *subsets of* W_1^A *and* W_2^A *respectively.*

(b) S_2^A *has the following closure property: If* B *is a truth-functional combination of members of* S_2^A, *and* x *is a variable not occurring in* B, *then* $(x)B \in S_2^A$.

(c) (T_2^*, R_2^*) *is* e.i. *relative to* S_2^A.

Then there is a 1-1 *partial recursive function* f *defined on* S_1^A, *which maps* S_1^A *into* S_2^A *so that the induced mapping* f^* *of* $\widetilde{S}_1$ *into* $\widetilde{S}_2$ *is a* 1-1 *partial recursive function satisfying the following.*

$$E \vdash_{\mathcal{T}_1} F \text{ if and only if } f^*(E) \vdash_{\mathcal{T}_2} f^*(F)$$

for all E *and* F *in* $\widetilde{S}_1$.

Proof. Let $F_1, F_2, \ldots, F_n, \ldots$ be an effective enumeration without repetition of the members of S_1^A. Suppose that at stage n we have a finite 1-1 correspondence between the $F_1, \ldots, F_n$ and $G_{i_1}, \ldots, G_{i_n}$ such that F_k corresponds to G_{i_k}. (The G_i's are members of S_2^A.) We seek to extend

(1) Note that this extension of the definition accords well with the original definition. For when $\mathcal{T}$ is an applied predicate calculus which is e.i. by the definition then the set of *formulas* which are theorems is effectively inseparable from the set of refutable formulas.

this correspondence effectively to a 1-1 correspondence between $F_1, \ldots, F_n, F_{n+1}$ and $G_{i_1}, \ldots, G_{i_n}, G_{i_{n+1}}$. The inductive hypothesis is that any truth functional combination of the $F_1, \ldots, F_n$ of the form $F_1^* \cdot \ldots \cdot F_n^*$ where $F_i^* = F_i$ or $F_i^* = \neg F_i$ is consistent with $\mathfrak{T}_1$ if and only if the corresponding combination of the G_{i_k}'s is consistent with $\mathfrak{T}_2$ (e.g. $F_1 \cdot \neg F_2 \cdot F_3 \cdot \ldots \cdot F_n$ is consistent with $\mathfrak{T}_1$ if and only if $G_{i_1} \cdot \neg G_{i_2} \cdot G_{i_3} \cdot \ldots \cdot G_{i_n}$ is consistent with $\mathfrak{T}_2$). Our aim is to choose a $G_{i_{n+1}}$ so that the same holds for $F_1, \ldots, F_{n+1}$ and $G_{i_1}, \ldots, G_{i_{n+1}}$. The method we use is a generalization of Myhill's [4]—cf. [3a] and [9a].

There are 2^n truth-functional combinations of $F_1, \ldots, F_n$ of the desired form. Denote them by $K_1, \ldots, K_{2^n}$. Similarly denote the corresponding truth functional combinations of $G_{i_1}, \ldots, G_{i_n}$ by $K'_1, \ldots, K'_{2^n}$.

Since (T_2^*, R_2^*) is effectively inseparable relative to S_2^A, there is a recursive function g whose range is included in S_2^A such that if

$$S_2^A \supseteq \omega_{K(i)} \supseteq T_2^*, \quad S_2^A \supseteq \omega_{L(i)} \supseteq R_2^*, \quad \omega_{K(i)} \cap \omega_{L(i)} = \emptyset,$$

then

$$g(i) \in S_2^A - (\omega_{K(i)} \cup \omega_{L(i)}).$$

(Of course, K and L are the well-known inverses of the pairing function j, i.e. $Kj(x, y) = x$ and $Lj(x, y) = y$.)

Let μ be a primitive recursive function such that

$$\omega_{\mu(p)} = \{p\}.$$

Then consider the following formulas (²)

$$(1) \qquad i \in S_1^A \cap \mathrm{Cl}_1(\omega_{\mu p}) \cdot x = g(y) \,.\vee.\, x \in S_2^A \cap \mathfrak{R}_2(\omega_{\mu q}),$$

$$(2) \qquad i \in S_1^A \cap \mathfrak{R}_1(\omega_{\mu p}) \cdot x = g(y) \,.\vee.\, x \in S_2^A \cap \mathrm{Cl}_2(\omega_{\mu q}).$$

By the recursion theorem, there is a primitive recursive function t such that

$$(3) \qquad x \in \omega_{Lt(i,p,q)} \leftrightarrow: i \in S_1^A \cap \mathrm{Cl}_1(\omega_{\mu p}) \cdot x = gt(i, p, q) \,.\vee.\, x \in S_2^A \cap \mathfrak{R}_2(\omega_{\mu q}),$$

$$(4) \qquad x \in \omega_{Kt(i,p,q)} \leftrightarrow: i \in S_1^A \cap \mathfrak{R}_1(\omega_{\mu p}) \cdot x = gt(i, p, q) \,.\vee.\, x \in S_2^A \cap \mathrm{Cl}_2(\omega_{\mu q}).$$

(²) $\mathrm{Cl}_1(\omega_{\mu p})$ is a notation for the set of theorems of the extension of $\mathfrak{T}_1$ obtained by adding $\omega_{\mu(p)}$ as axioms.

$\mathfrak{R}_1(\omega_{\mu p})$ is a notation for the set of refutables of the extension of $\mathfrak{T}_1$ obtained by adding $\omega_{\mu(p)}$ as axioms.

Similarly for $\mathrm{Cl}_2(\omega_{\mu q})$ and $\mathfrak{R}_2(\omega_{\mu q})$.

Suppose that $\mathfrak{T}_1 + \{K_j\}$ is consistent. Then $\mathfrak{T}_2 + \{K'_j\}$ is consistent by the inductive hypothesis. We show under the assumption that $\mathfrak{T}_1 + \{K_j\}$ is consistent that

$$(5) \qquad F_{n+1} \in S_1^A \cap \mathrm{Cl}_1(\omega_{\mu(K_j)}) \to gt(F_{n+1}, K_j, K'_j) \in S_2^A \cap \mathrm{Cl}_2(\omega_{\mu(K'_j)}),$$

$$(6) \qquad F_{n+1} \in S_1^A \cap \mathfrak{R}_1(\omega_{\mu(K_j)}) \to gt(F_{n+1}, K_j, K'_j) \in S_2^A \cap \mathfrak{R}_2(\omega_{\mu(K'_j)}),$$

$$(7) \qquad F_{n+1} \in S_1^A - [\mathrm{Cl}_1(\omega_{\mu(K_j)}) \cup \mathfrak{R}_1(\omega_{\mu(K_j)})] \to gt(F_{n+1}, K_j, K'_j) \in S_2^A - [\mathrm{Cl}_2(\omega_{\mu(K'_j)}) \cup \mathfrak{R}_2(\omega_{\mu(K'_j)})].$$

Indeed, suppose that (5) does not hold; then

$$(8) \qquad F_{n+1} \in S_1^A \cap \mathrm{Cl}_1(\omega_{\mu(K_j)}) \quad \text{but} \quad gt(F_{n+1}, K_j, K'_j) \notin S_2^A \cap \mathrm{Cl}_2(\omega_{\mu(K'_j)}).$$

Thus from (3) and (4), we obtain

$$\omega_{Lt(F_{n+1}, K_j, K'_j)} = \{gt(F_{n+1}, K_j, K'_j)\} \cup \big(\mathfrak{R}_2(\omega_{\mu(K'_j)}) \cap S_2^A\big),$$

$$\omega_{Kt(F_{n+1}, K_j, K'_j)} = S_2^A \cap \mathrm{Cl}_2\, \omega_{\mu(K'_j)}.$$

Now, since $\mathfrak{T}_2 + \{K'_j\}$ is consistent,

$$\big(S_2^A \cap \mathrm{Cl}_2(\omega_{\mu(K'_j)})\big) \cap \big(S_2^A \cap \mathfrak{R}_2(\omega_{\mu(K'_j)})\big) = \emptyset.$$

Thus by (8)

$$\omega_{Kt(F_{n+1}, K_j, K'_j)} \cap \omega_{Lt(F_{n+1}, K_j, K'_j)} = \emptyset.$$

Since (T_2^*, T_2^*), is e.i. with respect to S_2^A,

$$gt(F_{n+1}, K_j, K'_j) \in S_2^A - (\omega_{Kt(F_{n+1}, K_j, K'_j)} \cup \omega_{Lt(F_{n+1}, K_j, K'_j)}),$$

i.e.

$$gt(F_{n+1}, K_j, K'_j) \in S_2^A - [\{gt(F_{n+1}, K_j, K'_j)\} \cup \mathrm{Cl}_2(\omega_{\mu K'_j}) \cup \mathfrak{R}_2(\omega_{\mu K'_j})]$$

which yields a contradiction. Thus (5) is proved. In a similar manner we can prove (6) and (7). From (5), (6), and (7) we obtain

$$(9) \qquad F_{n+1} \in S_1^A \cap \mathrm{Cl}_1(\omega_{\mu(K_j)}) \leftrightarrow gt(F_{n+1}, K_j, K'_j) \in S_2^A \cap \mathrm{Cl}_2(\omega_{\mu(K'_j)}),$$

$$(10) \qquad F_{n+1} \in S_1^A \cap \mathfrak{R}_1(\omega_{\mu(K_j)}) \leftrightarrow gt(F_{n+1}, K_j, K'_j) \in S_2^A \cap \mathfrak{R}_2(\omega_{\mu(K'_j)}),$$

$$(11) \qquad F_{n+1} \in S_1^A - (\mathrm{Cl}_1\, \omega_{\mu(K_j)} \cup \mathfrak{R}_1\, \omega_{\mu(K_j)}) \leftrightarrow gt(F_{n+1}, K_j, K'_j) \in S_2^A - [\mathrm{Cl}_2\, \omega_{\mu(K'_j)} \cap \mathfrak{R}_2\, \omega_{\mu(K'_j)}].$$

Let us abbreviate $gt(F_{n+1}, K_j, K'_j)$ by H_j. Note that (9), (10), and (11) imply

$$K_j \vdash_{\mathfrak{T}_1} F_{n+1} \leftrightarrow K'_j \vdash_{\mathfrak{T}_2} H_j \quad \text{by (9)},$$

$$K_j \vdash_{\mathfrak{T}_1} \neg F_{n+1} \leftrightarrow K'_j \vdash_{\mathfrak{T}_2} \neg H_j.$$

Thus under the assumption that K_j is consistent we have shown that

(12) $K_j \cdot F_{n+1}$ *is consistent with* $\mathcal{T}_1 \leftrightarrow K'_j \cdot H_j$ *is consistent with* $\mathcal{T}_2$.

(13) $K_j \cdot \neg F_{n+1}$ *is consistent with* $\mathcal{T}_1 \leftrightarrow K'_j \cdot \neg H_j$ *is consistent with* $\mathcal{T}_2$.

Note that if K_j is not consistent, then by the inductive assumption K'_j is not consistent. Thus (12) and (13) hold trivially for inconsistent K_j's. Hence (12) and (13) hold for all K_j's.

Now let x be the first variable not occurring in $G_{i_1}, \ldots, G_{i_n}, H_1, \ldots, H_{2^n}$. Let $G_{i_{n+1}}$ be the formula

$$(x)\,[K'_1 \cdot H_1 . \vee . \ldots . \vee . K'_{2^n} \cdot H_{2^n}].$$

Clearly $G_{i_{n+1}} \in S_2^A$ by hypothesis (b). Furthermore it follows from the fact that

$$\vdash_{\mathcal{T}_2} K'_j \supset . [K'_1 \cdot H_1 . \vee . \ldots . \vee . K'_{2^n} \cdot H_{2^n}] \equiv H_j$$

that we have

$$\vdash_{\mathcal{T}_2} K'_j \supset . G_{i_{n+1}} \equiv H_j .$$

Thus we have

$$K_j \cdot F_{n+1} \text{ is consistent with } \mathcal{T}_1 \leftrightarrow K'_j \cdot G_{i_{n+1}} \text{ is consistent with } \mathcal{T}_2 .$$

$$K_j \cdot \neg F_{n+1} \text{ is consistent with } \mathcal{T}_1 \leftrightarrow K'_j \cdot \neg G_{i_{n+1}} \text{ is consistent with } \mathcal{T}_2 .$$

Thus $G_{i_{n+1}}$ has been chosen so that the inductive hypothesis holds between $F_1, \ldots, F_n, F_{n+1}$ and $G_{i_1}, \ldots, G_{i_n}, G_{i_{n+1}}$. As an immediate consequence of this we have

$$F_p \vdash_{\mathcal{T}_1} F_q \quad \text{if and only if} \quad G_{i_p} \vdash_{\mathcal{T}_2} G_{i_q},$$
$$F_p \vdash_{\mathcal{T}_1} \neg F_q \quad \text{if and only if} \quad G_{i_p} \vdash_{\mathcal{T}_2} \neg G_{i_q}.$$
$$\neg F_p \vdash_{\mathcal{T}_1} F_q \quad \text{if and only if} \quad \neg G_{i_p} \vdash_{\mathcal{T}_2} G_{i_q},$$
$$\neg F_p \vdash_{\mathcal{T}_1} \neg F_q \quad \text{if and only if} \quad \neg G_{i_p} \vdash_{\mathcal{T}_2} \neg G_{i_q},$$

Now let B_1 and C_1 be two truth functional combinations of the members of S_1^A. Let B_2 and C_2 be the corresponding truth functional combinations obtained by replacing each F_p by G_{i_p} throughout. It is an easy matter to show that

$$B_1 \vdash_{\mathcal{T}_1} C_1 \quad \text{if and only if} \quad B_2 \vdash_{\mathcal{T}_2} C_2 .$$

Thus the conclusion of this lemma holds.

LEMMA 2. *Suppose that:*

(a) S_1^A *and* S_2^A *are infinite* r.e. *subsets of* W_1^A *and* W_2^A *respectively.*

(b) *Both* S_1^A *and* S_2^A *have the following closure property: If* B_i *is a truth functional combination of members of* S_i^A *and* x *is a variable not occurring in* B_i, *then* $(x)B_i \in S_i^A$.

(c) (T_i^*, R_i^*) *is* e.i. *relative to* S_i^A.

Then there is a 1-1 *partial recursive function* f *defined on* S_1^A *mapping* S_1^A **onto** S_2^A *such that the induced mapping* f^* *of* $\tilde{S}_1$ *onto* $\tilde{S}_2$ *is a* 1-1 *partial recursive function satisfying the following*

$$E \vdash_{\mathcal{T}_1} F \quad \text{if and only if} \quad f^*(E) \vdash_{\mathcal{T}_2} f^*(F)$$

for all E *and* F *in* $\tilde{S}_1$.

Proof. The proof is an obvious modification of the proof of lemma 1. Let $F_1, F_2, \ldots, F_n, \ldots$ be a recursive enumeration without repetition of the members of S_1^A; let $G_1, \ldots, G_n, \ldots$ be a recursive enumeration without repetition of the members of S_2^A. Suppose that after stage k we have a 1-1 correspondence between $F_{i_1}, \ldots, F_{i_k}$ and $G_{i_1}, \ldots, G_{i_k}$. At stage $k+1$, we seek to extend this correspondence effectively to a 1-1 correspondence between $F_{i_1}, \ldots, F_{i_k}, F_{i_{k+1}}$ and $G_{i_1}, \ldots, G_{i_k}, G_{i_{k+1}}$ so that any truth functional combination of $F_{i_1}, \ldots, F_{i_{k+1}}$ of the form described in lemma 1 is consistent with $\mathcal{T}_1$ if and only if the corresponding truth functional combination of $G_{i_1}, \ldots, G_{i_{k+1}}$ is consistent with $\mathcal{T}_2$. The inductive hypothesis is, of course, that any truth functional combination of $F_{i_1}, \ldots, F_{i_k}$ of the kind described in lemma 1 is consistent with $\mathcal{T}_1$ if and only if the corresponding truth functional combination of $G_{i_1}, \ldots, G_{i_k}$ is consistent with $\mathcal{T}_2$.

Case 1. $k+1$ *is even.* Generate $F_1, F_2, \ldots, F_n, \ldots$ until the first F is found such that $F \neq F_{i_j}$ for $j = 1, \ldots, k$. Suppose that it is F_s. Let $F_{i_{k+1}} = F_s$. Let g_2 be a function which witnesses the effective inseparability of (T_2^*, R_2^*) with respect to S_2^A. Obtain t_2 by the recursion theorem as in equations (3) and (4) of the proof of lemma 1. Let $K_1, \ldots, K_{2^k}$ be all possible truth functional combinations of the F_{i_j} (for $j = 1, \ldots k$) of the desired form (see lemma 1); let $K'_1, \ldots, K'_{2^k}$ be the corresponding truth functional combinations of G_{i_j}'s. Let x be the alphabetically first variable not occurring in any of $g_2 t_2(F_{i_{k+1}}, K_j, K'_j)$ for $j = 1, \ldots, 2^k$, or in any of G_{i_j} for $j = 1, \ldots, k$. Let

$$G_{i_{k+1}} = (x)\,[K'_1 \cdot g_2 t_2(F_{i_{k+1}}, K_1, K'_1) . \vee . \ldots . \vee . K'_{2^k} \cdot g_2 t_2(F_{i_{k+1}}, K_{2^k}, K'_{2^k})] .$$

Note that $G_{i_{k+1}} \neq G_{i_j}$ for $j = 1, \ldots, k$. Furthermore we have, of course, that any truth functional combination of $F_{i_1}, \ldots, F_{i_{k+1}}$ is consistent with $\mathcal{T}_1$ if and only if the corresponding truth functional combination of $G_{i_1}, \ldots, G_{i_{k+1}}$ is consistent with $\mathcal{T}_2$.

Case 2. $k+1$ *is odd.* Generate $G_1, G_2, \ldots, G_n, \ldots$ until the first G is found, such that $G \neq G_{i_j}$ for $j = 1, \ldots, k$. Suppose that it is G_s. Then let $G_{i_{k+1}} = G_s$. Our aim is to find an $F_{i_{k+1}}$ so that any truth functional combination of the F_{i_j} (of the desired form) for $j = 1, \ldots, k+1$ is consistent with $\mathcal{T}_1$ if and only if the corresponding truth functional combination of the G_{ij}'s for $j = 1, \ldots, k+1$ is consistent with $\mathfrak{C}_2$. This can be accomplished by reversing the roles of the F's and the G's in aase 1 and using the effective inseparability of (T_1^*, R_1^*) with respect to S_1^A in place of the effective inseparability of (T_2^*, R_2^*) with respect to S_2^A. The proof is entirely analogous to the proof of case 1. Thus we obtain a 1-1 partial recursive function f defined on S_1^A and mapping S_1^A onto S_2^A. Furthermore the induced mapping f^* is a 1-1 partial recursive function defined on $\widetilde{S}_1$ and mapping $\widetilde{S}_1$ onto $\widetilde{S}_2$. As in the proof of lemma 1, it can be shown that

$$E \vdash_{\mathfrak{C}_1} F \quad \text{if and only if} \quad f^*(E) \vdash_{\mathfrak{C}_2} f^*(F)$$

for all E, F in $\widetilde{S}_1$.

2. Recursive and primitive recursive mappings between applied predicate calculi. This section is devoted to applications of lemmas 1 and 2. In order to obtain these applications it is necessary to state two simple facts concerning effective inseparability. The first one, which we state in lemma 3 is well known. The second, which we give as lemma 4 does not appear to have been explicitly stated in the literature.

LEMMA 3. *Let (α, β) be an effectively inseparable pair of sets; let (γ, δ) be a disjoint pair of* r.e. *sets. Suppose that f is a recursive function such that*

$$x \in \alpha \leftrightarrow f(x) \in \gamma,$$
$$x \in \beta \leftrightarrow f(x) \in \delta;$$

then (γ, δ) is an effectively inseparable pair of sets.

LEMMA 4. *Let (α, β) be an effectively inseparable pair of sets. Let δ be a recursive set such that $\delta \supseteq \alpha$, $\delta \supseteq \beta$. Then (α, β) is effectively inseparable relative to δ.*

Proof. Obvious.

On the basis of lemmas 3 and 4 we see that if $\mathfrak{C}$ is an applied predicate calculus which is effectively inseparable then

(14) $\qquad (T^A, R^A)$ is e.i. relative to W^A.

For let C be any formula in W. Let x be alphabetically the first variable not occurring in C. Let C' be $(x)C$. Define f by

$$f(D) = \begin{cases} D' & \text{if} \quad D \in W, \\ 0 & \text{otherwise}. \end{cases}$$

Since (T, R) is e.i., we see by lemma 3 (using the f above) that (T^A, R^A) is e.i. Now applying lemma 4 we see that (14) holds.

We are now ready to state two of our main results.

THEOREM 1. *If $\mathfrak{C}_1$ is consistent and $\mathfrak{C}_2$ is effectively inseparable then there is a* 1-1 *recursive function f^* mapping W_1 into W_2, $\overline{W}_1$ onto $\overline{W}_2$ such that for all B, C in W_1*

a. $f^*(B \to C) = f^*(B) \to f^*(C)$,

b. $f^*(\neg B) = \neg f^*(B)$,

c. $B \vdash_{\mathfrak{C}_1} C$ *if and only if* $f^*(B) \vdash_{\mathfrak{C}_2} f^*(C)$

Furthermore f^* maps W_1^A into W_2^A.

Proof. We employ lemma 1. Let the S_1^A and S_2^A of lemma 1 be W_1^A and W_2^A. Then the $\widetilde{S}_1$ and $\widetilde{S}_2$ of lemma 1 become W_1 and W_2. By (14), hypothesis c of lemma 1 holds. Since a and b are automatically satisfied, we obtain from the conclusion of lemma 1, a 1-1 partial recursive function mapping W_1 into W_2 and satisfying a, b, and c of the conclusion of this theorem. Since W_1 is recursive with infinite recursive complement, this function may be extended to a recursive function satisfying the conclusion of this theorem.

THEOREM 2. *Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two effectively inseparable theories. Then there is a* 1-1 *recursive function f^* mapping W_1* **onto** *W_2, $\overline{W}_1$ onto $\overline{W}_2$ such that for all B, C in W_1*

a. $f^*(B \to C) = f^*(B) \to f^*(C)$,

b. $f^*(\neg B) = \neg f^*(B)$,

c. $B \vdash_{\mathfrak{C}_1} C$ *if and only if* $f^*(B) \vdash_{\mathfrak{C}_2} f^*(C)$.

Furthermore f^* maps W_1^A onto W_2^A.

Proof. Similar to theorem 1 using lemma 2 in place of lemma 1.

It is easy to see that the function of relative effective inseparability for (14) may be chosen to be primitive recursive both in W and in the function of effective inseparability of (T, R). This note leads to the following generalizations of theorems 1 and 2.

Remark 1. (*Primitive recursive mappings between theories.*) A slight modification of the proof of lemma 1 shows that the f^* of theorem 1 can be chosen to be primitive recursive in W_1, W_2, and in the function g of effective inseparability of (T_2, R_2). Similarly the function f^* of theorem 2 can be chosen to be primitive recursive in W_1, W_2 and in both functions of effective inseparability—the function for (T_1, R_1) and the function for (T_2, R_2). Thus in theorem 1, if W_1, W_2 and the function of effective inseparability for $\mathfrak{C}_2$ are primitive recursive, then f^* is primitive recursive. Similarly in theorem 2, if W_1, W_2 and both functions of effective inseparability are primitive recursive, the f^* is primitive recursive.

We now apply these observations to some mathematically interesting formal theories. Let (α, β) be Kleene's e.i. pair of sets [3]. Then for a theory $\mathfrak{T}$ in standard formalization possessing a notation for the natural numbers, the following holds (cf. [6], [7]).

Suppose that $\mathfrak{T}$ possesses a binary predicate $\leqslant$ satisfying the following

(1) for all n $\vdash_{\mathfrak{T}} x \leqslant \bar{n} \vee \bar{n} \leqslant x$,

(2) for all n $\vdash_{\mathfrak{T}} x \leqslant \bar{n} \rightarrow x = \bar{0} \vee \ldots \vee x = \bar{n}$.

Suppose further, that every primitive recursive function of one argument is definable in $\mathfrak{T}$.

Then there is a formula Φ with one free variable such that

$$n \in \alpha \rightarrow \vdash_{\mathfrak{T}} \Phi(\bar{n}), \quad n \in \beta \rightarrow \vdash_{\mathfrak{T}} \neg\Phi(\bar{n}).$$

The above result shows that under the hypothesis of the result a disjoint pair (α^*, β^*) of r.e. supersets of Kleene's e.i. pair (α, β) is reducible to (T, R) by a primitive recursive function. Since Kleene's pair is effectively inseparable by a primitive recursive function, so is (α^*, β^*). It follows (by the proof of lemma 3) that (T, R) is effectively inseparable by a primitive recursive function. Thus in theorem 1 if $\mathfrak{T}_2$ satisfies the hypothesis of the above result—or merely contains a subtheory which satisfies this hypothesis—and if W_1 and W_2 are primitive recursive, then f^* may be chosen to be primitive recursive. Similarly, in theorem 2, if $\mathfrak{T}_1$ and $\mathfrak{T}_2$ both contain subtheories which satisfy the hypothesis of the above result and if W_1 and W_2 are primitive recursive, then the f^* of theorem 2 may be chosen to be primitive recursive. Now many of the interesting formal theories in standard formalization which satisfy the above result have W as a primitive recursive set—for example, the theories **R**, **Q**, and **P** of *Undecidable Theories* [10], *Z-F set theory*, *Gödel-Bernays set theory*, etc. *Between any two of these theories the mapping f^* of theorem 2 may be chosen to be primitive recursive.*

Remark 2. It may be supposed that theorem 2 can be extended so that if $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are two effectively inseparable theories each possessing a common effectively inseparable subtheory $\mathfrak{T}$ (which is a proper subtheory of both) then the f^* of theorem 2 (conclusion 1) can be chosen to map T onto T and R onto R. That this is not necessarily the case may be seen by the following example.

Let $\mathfrak{T}$ be Peano Arithmetic; let $\mathfrak{T}_1$ be a proper finite extension of $\mathfrak{T}$; let $\mathfrak{T}_2$ be an extension of $\mathfrak{T}$ which is not a finite extension. Of course, for $\mathfrak{T}_1$ we have the existence of a fixed A_1 such that for all C_1

$$\vdash_{\mathfrak{T}_1} C_1 \quad \text{if and only if} \quad \vdash_{\mathfrak{T}} A_1 \supset C_1 .$$

Since $\mathfrak{T}_2$ does not have this property the result follows.

It ought to be remarked that one can replace vacuous quantification by non-vacous quantification in the above proofs of lemmas 1 and 2.

3. Propositional calculi. It is natural to inquire whether theorems 1 and 2 hold for theories formulated in the propositional calculus (with modus ponens as the sole rule of inference). That theorem 2 does not hold follows from an example given by Pour-El [6]. We restate this example here for the sake of completeness.

EXAMPLE. There exist two e.i. theories $\mathfrak{T}_1$ and $\mathfrak{T}_2$ formulated as applied propositional calculi such that *no* 1-1 recursive function f mapping W_1 onto W_2, T_1 onto T_2 satisfies the following

$$f(E \rightarrow F) = f(E) \rightarrow f(F), \quad f(\neg E) = \neg f(E).$$

Proof. $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are assumed to be applied propositional calculi based on negation and implication with modus ponens as the sole rule of inference. Suppose further that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ possess a unary predicate B, the successor function S and $\mathbf{0}$. Let (α, β) be an effectively inseparable pair of sets. Suppose that $0 \notin \alpha \cup \beta$ and $1 \notin \alpha \cup \beta$.

AXIOMS FOR $\mathfrak{T}_2$.

1. *Axiom schemata for the propositional calculus*;
2. $B(\mathbf{n})$ *for* $n \in \alpha$;
3. $\neg B(\mathbf{n})$ *for* $n \in \beta$.

AXIOMS FOR $\mathfrak{T}_1$.

1. *Axiom schemata for the propositional calculus*;
2. $B(\mathbf{n})$ *for* $n \in \alpha$;
3. $\neg B(\mathbf{n})$ *for* $n \in \beta$;
4. $B(\mathbf{0}) \rightarrow B(\mathbf{1})$.

It is easy to see that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are e.i. theories. Note that any 1-1 recursive function mapping W_1 onto W_2, preserving negation and implication must map atomic sentences onto atomic sentences. Thus $B(\mathbf{0})$ is mapped onto $B(\mathbf{n}_0)$ and $B(\mathbf{1})$ is mapped onto $B(\mathbf{n}_1)$ where $n_0 \neq n_1$. Hence the theorem $B(\mathbf{0}) \rightarrow B(\mathbf{1})$ must be mapped onto the non-theorem $B(\mathbf{n}_0) \rightarrow \rightarrow B(\mathbf{n}_1)$.

Note that we have proved something stronger. There is no 1-1 function mapping W_1 onto W_2, T_1 onto T_2 such that both implication and negation are preserved.

In contrast most of theorem 1 does hold for propositional calculi. We state this result as theorem 3. Note that the g of theorem 3 does not map W_1^A into W_2^A.

THEOREM 3. *Let $\mathfrak{T}_1$ be consistent and $\mathfrak{T}_2$ be effectively inseparable. Then there is a 1-1 recursive function g mapping W_1 into W_2 such that for all B and C in W_1*

a. $g(B\to C) = g(B)\to g(C)$,

b. $g(\neg B) = \neg g(B)$,

c. $B \vdash_{\mathcal{T}_1} C$ *if and only if* $g(B) \vdash_{\mathcal{T}_2} g(C)$.

The proof is via a slight modification of lemma 1:

Suppose that W_1^A and W_2^A are infinite recursive sets. Suppose further that $\mathcal{T}_2$ is an e.i. theory. Then there exists a 1-1 partial recursive function f mapping W_1^A into W_2 such that the induced mapping f^* (which preserves implication and negation) is a 1-1 partial recursive function preserving deducibility. The proof of this lemma is analogous to the proof of lemma 1. In particular the inductive assumption is similar. Assume we have a correspondence between $F_1, \dots, F_n$ and $G_{i_1}, \dots, G_{i_n}$—the F's are members of W_1^A, the G's are members of W_2. We seek to extend this correspondence to a correspondence between $F_1, \dots, F_n, F_{n+1}$ and $G_{i_1}, \dots, G_{i_n}, G_{i_{n+1}}$. Define K_j, K_j', H_j in a manner analogous to lemma 1. (In particular $H_j = g_2 t_2(F_{n+1}, K_j, K_j')$ where g_2 is the function of effective inseparability of (T_2, R_2) relative to W_2.) Let G_n^+ be the first member of W_2^A which does not occur in $G_{i_1}, \dots, G_{i_n}$. Define $G_{i_{n+1}}$ by

$$K_1'\cdot H_1\cdot(G_n^+ \vee \neg G_n^+)\ .\vee. \dots .\vee.\ K_{2^n}'\cdot H_{2^n}(G_n^+ \vee \neg G_n^+)\,.$$

This gives a 1-1 partial recursive function f mapping W_1^A into W_2 which preserves deducibility. The induced mapping f^* also preserves deducibility. (The inclusion of $(G_n^+ \vee \neg G_n^+)$ in $G_{i_{n+1}}$ assures that the induced mapping is 1-1!) Thus theorem 3 holds. (³)

It follows immediately that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are both effectively inseparable there are two 1-1 recursive functions g_1 and g_2 such that for all B_1 and C_1 in W_1 and for all B_2 and C_2 in W_2

(a) $g_1(B_1\to C_1) = g_1(B_1)\to g_1(C_1)$,
$g_1(\neg B_1) = \neg g_1(B_1)$,
$B_1 \vdash_{\mathcal{T}_1} C_1$ if and only if $g_1(B_1) \vdash_{\mathcal{T}_2} g_1(C_1)$;

(b) $g_2(B_2\to C_2) = g_2(B_2)\to g_2(C_2)$,
$g_2(\neg B_2) = \neg g_2(B_2)$,
$B_2 \vdash_{\mathcal{T}_2} C_2$ if and only if $g_2(B_2) \vdash_{\mathcal{T}_1} g_2(C_2)$.

An interesting contrast to the example given at the beginning of this section is provided by theorem 4 below. For, given two e.i. theories $\mathcal{T}_1$ and $\mathcal{T}_2$ it is possible to find a 1-1 recursive function g mapping W_1 *onto* W_2 *preserving negation, deducibility* (and hence theoremhood and refutability) and which *up to deductive equivalence preserves implication.*

(³) Recall that $A\vee B$ is an abbreviation for $\neg A\to B$; $A\cdot B$ is an abbreviation for $\neg(\neg A\vee\neg B)$. Theorem 3 would hold with any of the usual choices of connectives as primitives.

THEOREM 4. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two* e.i. *theories. There is a* 1-1 *negation-preserving recursive function g mapping W_1 onto W_2 such that for all formulas B_1 and C_1 in $\mathcal{T}_1$*

(1) $B_1 \vdash_{\mathcal{T}_1} C_1$ *if and only if* $g(B_1) \vdash_{\mathcal{T}_2} g(C_1)$,

(2) $\vdash_{\mathcal{T}_2} g(B_1\to C_1) \equiv .\ g(B_1)\to g(C_1)$

and for all formulas B_2 and C_2 in $\mathcal{T}_2$,

(3) $\vdash_{\mathcal{T}_1} g^{-1}(B_2\to C_2) \equiv .\ g^{-1}(B_2)\to g^{-1}(C_2)$.

Proof. The proof uses an obvious modification of lemma 2. Replace S_1^A by W_1^-; S_2^A by W_2^- (see end of introduction for notation). Use the fact that (T_1^-, R_1^-) and (T_2^-, R_2^-) are effectively inseparable relative to W_1^- and W_2^-. The formula corresponding to $F_{i_{k+1}}$ is $(K_1'\cdot H_1\cdot\ .\vee. \dots .\vee.\ \cdot K_{2^k}'\cdot\cdot H_{2^k})$ in case 1 (when n is even). An analogous change is made in case 2 (when n is odd). We thus obtain a 1-1 recursive function h mapping W_1^- onto W_2^- such that (1) of theorem 4 holds. Extend this to a 1-1 recursive function g mapping W_1 onto W_2 preserving negation so that (1) theorem 4 holds. It is an immediate consequence of the construction that (2) and (3) hold.

4. A classification of effectively inseparable theories.

In section 2 we showed that, for many pairs of mathematically interesting formal theories $\mathcal{T}_1$ and $\mathcal{T}_2$, theorem 2 holds with a primitive recursive f^*. It is natural to ask whether this strengthening can be obtained for all e.i. pairs of theories. We will show that the answer is negative, even if we confine ourselves to theories in standard formalization. Our discussion will give rise to a hierarchy of effectively inseparable theories in standard formalization (see theorem 7 below).

The main result of this section, theorem 6, is based on theorem 5, concerning effectively inseparable sets, which may be of some independent interest. (⁴)

THEOREM 5. *Given an* r.e. *class $\mathcal{F}$ of general recursive functions of a single argument we can find an effectively inseparable pair of sets (γ, δ) such that no recursive function which witnesses the effective inseparability of (γ, δ) is in $\mathcal{F}$.* (⁴)

(⁴) A special case of this result obtained by letting $\mathcal{F}$ be the set of all primitive recursive functions was obtained by McLaughin. This result also generalizes results of Rogers [8] and Fischer (*Theory of Provable Recursive Functions*, MIT Doctoral Dissertation 1962). Rogers showed that if $\mathcal{S}$ is a consistent extension of elementary number theory then there is a creative set which possesses no productive function which is a p-function of $\mathcal{S}$. Letting $\mathcal{F}$ be the r.e. class of all p-functions of $\mathcal{S}$, we can, by a slight modification of theorem 5, obtain Rogers' result. Indeed, it seems that many of the results in Fischer's thesis can be generalized by replacing the class of p-functions by an arbitrary r.e. class of general recursive functions. For, as in theorem 5, the recursive enumerability of $\mathcal{F}$ together with a suitable diagonalization argument combine to give the result in more general form.

Proof. We assume that $\mathcal{F}$ has the property that every recursive function of one argument which is primitive recursive in $\mathcal{F}$ is a member of $\mathcal{F}$. (If $\mathcal{F}$ does not have this property then we enlarge $\mathcal{F}$ to obtain an r.e. class $\mathcal{F}^*$ which is closed with respect to this property. The proof then proceeds with $\mathcal{F}^*$ in place of $\mathcal{F}$.) Since $\mathcal{F}$ is an r.e. class of recursive functions of one argument, there is a recursive function f of two arguments such that $f_0, f_1, \ldots, f_n, \ldots$ is an enumeration of the members of $\mathcal{F}$. Define the function g by

$$g(y) = \sum_{\substack{n \leqslant y \\ x \leqslant y}} \big(f(n, x)+1\big)\ ;$$

g is a strictly increasing recursive function. Hence ϱg—the range of g—is recursive.

Let (α, β) be Kleene's effectively inseparable pair of sets. Let $\gamma = g(\alpha)$; let $\delta = g(\beta) \cup \overline{\varrho g}$. Trivially, γ and δ are r.e. sets. Furthermore, (γ, δ) is an e.i. pair of sets. For let t be a primitive recursive function such that

$$\omega_{t(i)} = \{x |\ g(x) \in \omega_i\}\ .$$

Let h be a primitive recursive function witnessing the effective inseparability of (α, β). Then the function h^* defined by

$$h^*(i) = ghj\Big(\big(tK(i)\big), t\big(L(i)\big)\Big)$$

is a function of effective inseparability for (γ, δ). Note that the function h^* is primitive recursive in the universal function f of $\mathcal{F}$.

It is now easy to see that there does not exist a function $s \in \mathcal{F}$ which witnesses the effective inseparability of (γ, δ). For suppose the contrary. We will show that this assumption leads to the existence of a strictly increasing function $w \in \mathcal{F}$ such that $\varrho w \subseteq \overline{\gamma \cup \delta} \subseteq \varrho g$. We will thus obtain a contradiction.

Let e_0 and e_1 be indices for γ and δ respectively. Let v be a primitive recursive function such that

$$\omega_{v(i)} = \omega_i \cup \{sj(e_0, i)\}\ .$$

Now the following sequence of elements of $\overline{\gamma \cup \delta}$

$$sj(e_0, e_1), sj\big(e_0, v(e_1)\big), \ldots, sj\big(e_0, v^{n-1}(e_1)\big), \ldots$$

contains no repetitions. This is an immediate consequence of the fact that for each $n > 0$

$$\omega_{e_0} \cap \omega_{v^n(e_1)} = \emptyset\ ,$$

$$\omega_{v^n(e_1)} = \omega_{e_1} \cup \{sj(e_0, e_1)\} \cup \ldots \cup \Big\{sj\big(e_0, v^{n-1}(e_1)\big)\Big\}\ .$$

(These last two equalities are proved by a straightforward induction on n. Of course, v^n is defined by: $v^0(p) = p$, $v^{n+1}(p) = v\big(v^n(p)\big)$.)

Now define w by

$$w(0) = sj(e_0, e_1)$$

$$w(n+1) = sj\,[e_0, v^{\mu k(k \leqslant w(n)+2\cdot sj(e_0, v^k(e_1)) > w(n))}(e_1)]$$

Clearly w is a strictly increasing function such that $\varrho w \subseteq \overline{\gamma \cup \delta} \subseteq \varrho g$. Furthermore w is primitive recursive in s, and hence $w \in \mathcal{F}$. But this is a contradiction, since—by construction of g—if ϱg is enumerated without repetition, in order of magnitude, then this enumeration grows faster than ϱf for any $f \in \mathcal{F}$. Thus, since $\varrho w \subseteq \varrho g$, the same is true for ϱw.

Theorem 6 below is an analogue of theorem 5 for theories $\mathcal{T}$ in standard formalization. Note that the proof is not as straightforward as may be expected. One reason is that $W-(T \cup R)$ has quite different properties with respect to recursiveness from the $\overline{\gamma \cup \delta}$ of the preceding theorem. More specifically, $W-(T \cup R)$ contains a subset which can be generated in order of magnitude by a primitive recursive function. (For let $\varphi \in W-(T \cup R)$. Then $\varphi, \varphi\cdot\varphi, \varphi\cdot\varphi\cdot\varphi, \ldots$ is such a sequence.) In contrast, the proof of theorem 5 hinges on the fact that if a subset of $\overline{\gamma \cup \delta}$ is enumerated in order of magnitude, it cannot be so enumerated by a member of $\mathcal{F}$. The analogy which we use is that $W-(T \cup R)$ does not contain an infinite r.e. sequence of sentences $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$ such that

(1) φ_{i+1} is undecidable in $\mathcal{T} \cup \{\varphi_0, \ldots, \varphi_i\}$ for each i;

(2) the function g defined by $g(n) = \varphi_n$ is in $\mathcal{F}$.

Even with this modification there are complications because $W-(T \cup R)$ contains not only sentences of the forms Φ_n and $\neg\Phi_n$ but also sentences *not* of these forms—see next paragraph for notation. A brief outline of the proof appears below.

Let $\mathcal{T}^*$ be the decidable theory of one equivalence relation S. Let Φ_n be a statement asserting that there is an equivalence class of S consisting of n members. Let Ψ_n be a statement asserting that there is at most one equivalence class having n members. With this notation we can state and prove theorem 6.

THEOREM 6. *Let $\mathcal{F}$ be an* r.e. *class of general recursive functions of one argument. Then there exists an effectively inseparable theory $\mathcal{T}$ in standard formalization such that no recursive function which witnesses the effective inseparability of $\mathcal{T}$ is in $\mathcal{F}$.*

Outline of the proof. We assume that a function of e.i. of $\mathcal{T}$ constructed below is in $\mathcal{F}$ First we obtain an infinite r.e. sequence $X_0, \ldots, X_n, \ldots$ of undecidable sentences such that for each i, X_{i+1} is undecidable in $\mathcal{T} \cup \{X_0, \ldots, X_i\}$. This sequence is obtained primitive recursively from $\mathcal{F}$. From this sequence we obtain in two stages another r.e.

sequence $X_0'', \ldots, X_n'', \ldots$ having the property that each X_i'' contains a Φ_{p_i} which is undecidable in $\mathcal{T}$ where p_i is an increasing (not necessarily recursive) function of i. The sequence $X_0'', \ldots, X_n'', \ldots$ is obtained primitive recursively from $\mathcal{F}$. Thus

$$p_0, p_1, \ldots$$

is a strictly increasing sequence of elements of $\overline{\gamma \cup \delta}$ bounded by a member of $\mathcal{F}$, contradicting the construction of $\gamma \cup \delta$.

Proof of Theorem 6. As in the proof of theorem 5 we can assume without loss of generality that every recursive function of one argument, primitive recursive in $\mathcal{F}$ is a member of $\mathcal{F}$. (If $\mathcal{F}$ does not satisfy these conditions enlarge $\mathcal{F}$ to obtain an r.e. class $\mathcal{F}'$ such that $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{F}'$ does satisfy these conditions. Work with $\mathcal{F}'$ instead of $\mathcal{F}$.) We will also assume that in the standard enumeration $\omega_0, \omega_1, \omega_2, \ldots, \omega_n, \ldots$ of r.e. sets we have $\omega_0 = \emptyset$.

Let (γ, δ) be the effectively inseparable pair of sets obtained from $\mathcal{F}$ by theorem 5. We may assume without loss of generality that $0 \notin \gamma \cup \delta$. For the theory $\mathcal{T}$ we take the extension of the decidable theory of one equivalence relation obtained by adding the following non-logical axioms

Ψ_n for all n,

Φ_n for $n \in \gamma$,

$\neg \Phi_n$ for $n \in \delta$.

Note that every sentences X of $\mathcal{T}$ is equivalent to a truth function of a finite number of the Φ_n's and this truth function can be calculated from X. (5)

Clearly $\mathcal{T}$ is effectively inseparable. Hence there is a recursive function g^+ such that if

$$W \supseteq \omega_{K(i)} \supseteq T, \quad W \supseteq \omega_{L(i)} \supseteq R, \quad \omega_{K(i)} \cap \omega_{L(i)} = \emptyset$$

then

$$g^+(i) \in W - (\omega_{K(i)} \cup \omega_{L(i)}) .$$

We show that $g^+ \notin \mathcal{F}$. Assume $g^+ \in \mathcal{F}$. Let a be a Gödel number of T; let b be a Gödel number of R. Let $c = j(a, b)$. Define g^* by

$$g^*(0) = g^+(c), \quad g^*(n+1) = g^+(n+1) .$$

Note that since $\mathcal{F}$ is closed under primitive recursive operations, $g^* \in \mathcal{F}$. We will show that under that assumption that $g^* \in \mathcal{F}$, we obtain a contradiction. Thus $g^+ \notin \mathcal{F}$. From this, the conclusion of the theorem follows easily.

(5) See Shoenfield [9] and Janiczak [2] for the decision procedure.

The function g^* has the property that if

$$W \supseteq \omega_{K(i)} \supseteq T, \quad W \supseteq \omega_{L(i)} \supseteq R, \quad \omega_{K(i)} \cap \omega_{L(j)} = \emptyset ,$$

then

$$g^*(i) \in W - (\omega_{K(i)} \cup \omega_{L(i)}) .$$

Define primitive recursive functions t, h_1, h_2 and h by

(0) $\omega_{t(i)} = \{\neg\varphi \mid \varphi \in \omega_i \cap W\}$;

(1) $\omega_{h_1(i)} =$ set of theorems of $\mathcal{T}$ obtained by adding $\omega_{K(i)} \cup \{g^*(i)\}$ as axioms;

(2) $\omega_{h_2(i)} =$ set of refutable sentences of $\mathcal{T}$ obtained by adding $\omega_{tL(i)} \cup \{g^*(i)\}$ as axioms;

(3) $h(i) = j\big(h_1(i), h_2(i)\big)$.

Note that

(4) $\omega_{h_1 h^n(i)} =$ set of theorems obtained by adding $\omega_{Kh^n(i)} \cup \{g^* h^n(i)\}$ as axioms.

(5) $\omega_{h_2 h^n(i)} =$ set of refutable sentences obtained by adding $\omega_{tLh^n(i)} \cup \{g^* h^n(i)\}$ as axioms.

We see that

(6) $Kh^{n+1}(i) = h_1 h^n(i)$ for all n;

(7) $Lh^{n+1}(i) = h_2 h^n(i)$ for all n,

i.e.

$$tLh^{n+1}(i) = th_2 h^n(i)$$

for

(8) $h\big(h^n(i)\big) = j\big(h_1 h^n(i), h_2 h^n(i)\big)$ by (3).

Thus

(9) $\omega_{h_1 h^n(i)} =$ set of theorems obtained by adding $\{g^*(i)\} \cup, \ldots, \cup \{g^* h^n(i)\} \cup \omega_{K(i)}$ as axioms.

(10) $\omega_{h_2 h^n(i}$ = set of refutable sentences obtained by adding $\{g^*(i)\} \cup \ldots \cup \{g^* h^n(i)\} \cup \omega_{tL(i)}$ as axioms.

(9) and (10) are obtained by a simple induction on n. If $n = 0$, the result holds trivially. Assuming that (9) holds for p, we show that (9) holds for $p+1$. For

$\omega_{h_1 h^{p+1}} =$ set of theorems obtained by adding $\omega_{\overline{Kh}^{p+1}(i)} \cup \{g^* h^{p+1}(i)\}$ as axioms.

$\omega_{h_1 h^{p+1}(i)} =$ set of theorems obtained by adding $\omega_{h_1 h^p(i)} \cup \{g^* h^{p+1}(i)\}$ as axioms.

$\omega_{h_1 h^{p+1}(i)} =$ set of theorems obtained by adding $\omega_{K(i)} \cup \{g^*(i)\} \cup \ldots \cup \{g^* h^p(i)\} \cup \{g^* h^{p+1}(i)\}$ as axioms.

Analogously for (10).

Now 0 is a Gödel number of the empty set. Since $K(0) = L(0) = 0$, $\omega_{K(0)} = \omega_{L(0)} = \emptyset$. Hence $\omega_{tL(0)} = \emptyset$. Thus (9) and (10) become

$\omega_{h_1 h^n(0)} =$ set of theorems obtained by adding $\{g^*(0)\} \cup \ldots \cup \{g^* h^n(0)\}$ as axioms.

$\omega_{h_2 h^n(0)} =$ set of refutable sentences obtained by adding $\{g^*(0)\} \cup \cup \ldots \cup \{g^* h^n(0)\}$ as axioms.

An induction on n shows that, for all n,

$$\omega_{h_1 h^n(0)} \cap \omega_{h_2 h^n(0)} = \emptyset\,.$$

Thus

$$g^* h^{n+1}(0) = g^* j\big(h_1 h^n(0),\, h_2 h^n(0)\big) \in W - (\omega_{h_1 h^n(0)} \cup \omega_{h_2 h^n(0)})\,.$$

Hence for all n, $g^* h^{n+1}(0)$ is undecidable in the extension of $\mathcal{T}$ obtained by adding $\{g^*(0)\} \cup \ldots \cup \{g^* h^n(0)\}$ as axioms. By applying the decision procedure to $g^* h^n(0)$, we obtain a sentence X_n equivalent in $\mathcal{T}$ to the sentence with Gödel number $g^* h^n(0)$ such that X_n is a truth-functional combination of Φ_k's. Note that the sequence $X_0, X_1, \ldots, X_n, \ldots$ is obtained primitive recursively from g^* [6]. We obtain a new r.e. sequence $X'_0, X'_1 \ldots$ such that each X'_n contains a Φ_p not in any X'_k for $k < n$. Furthermore this Φ_p is undecidable in $\mathcal{T}$. Let $X'_0 = X_0$. Assume that $X'_0, \ldots, X'_n$ have been defined. We show how to define X'_{n+1}. Suppose that

$$X'_n = X_p \cdot X_{p+1} \cdot \ldots \cdot X_{p+k} \quad \text{for some } p \text{ and } k\,.$$

Let g be the number of distinct Φ_s's appearing in any of the X'_i for $i = 0, \ldots, n$.

Let

$$X'_{n+1} = X_{p+k+1} \cdot \ldots \cdot X_{p+k+h+2} \quad \text{where} \quad h = \sum_{k=1}^{g} \binom{g}{k} 2^{2^k}\,.$$

(The justification of this appears in the next paragraph.)

Note that if X' is a Boolean combination of $\Phi_{i_1}, \ldots, \Phi_{i_k}$, there are 2^{2^k} possible truth assignments to $\Phi_{i_1}, \ldots, \Phi_{i_k}$ and X'. Thus there are at most $h = \sum_{k=1}^{g} \binom{g}{k} 2^{2^k}$ distinct truth assignmets to any X' containing only Φ_p's appearing in $X'_0, \ldots, X'_n$. Since for all s, $X_{p+k+s+1}$ is undecidable in $\mathcal{T} \cup \cup \{X'_0, \ldots, X'_n, X_{p+k+1}, \ldots, X_{p+k+s}\}$, one of $X_{p+k+1}, \ldots, X_{p+k+h+1}$ must contain a Φ_p not in $X'_0, \ldots, X'_n$ which is undecidable in $\mathcal{T}$. Thus X'_{n+1} contains a Φ_p not in $X'_0, \ldots, X'_n$ which is undecidable in $\mathcal{T}$.

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(6) If the decision procedure were not primitive recursive, we would add the recursive function d which carries out the decision procedure to $\mathcal{F}$ and close under primitive recursion and composition to obtain $\mathcal{F}^*$. We would then work with $\mathcal{F}^*$ instead for $\mathcal{F}$.

We now construct an r.e. sequence $X''_0, X''_1, X''_2, \ldots$ such that each X''_n contains a Φ_{p_n} satisfying the following:

Φ_{p_n} is undecidable in $\mathcal{T}$, Φ_{p_n} does not occur in X''_k for $k < n$ and p_n is a strictly increasing function of n—which is not necessarily recursive.

Let $X''_0 = X'_0$. Assume we have generated $X''_0, \ldots, X''_n$. We show how to generate X''_{n+1}. Suppose that

$$X''_n = X'_p \cdot \ldots \cdot X'_{p+k} \quad \text{for some } p \text{ and } k\,.$$

Define X''_{n+1} by

$$X''_{n+1} = X'_{p+k+1} \cdot \ldots \cdot X'_{p+\sum_{i=0}^{n} \ulcorner X''_i \urcorner + k+2}\,,$$

where $\ulcorner X''_i \urcorner$ is the Gödel number of X''_i. Since each X'_{p+k+i} contains a Φ_{s_i}, undecidable in $\mathcal{T}$ which does not occur in X'_{p+k+j} for $j < i$, there must be a $\Phi_{p_{n+1}}$ occurring in X''_{n+1} which satisfies the required condition.

We see by construction that the sequence $X''_0, X''_1, \ldots$ is obtained primitive recursively from g^*. Now let $\Phi_{p_0}, \Phi_{p_1}, \ldots$ be a sequence (not necessarily recursively enumerable) such that

$$p_i > p_j \quad \text{if} \quad i > j\,,$$

$$\Phi_{p_n} \text{ occurs in } X''_n \text{ and is undecidable in } \mathcal{T}\,.$$

Thus, by construction, $p_0, p_1, \ldots, p_n, \ldots$ is an increasing sequence of elements of $\overline{\gamma \cup \delta}$ such that $p_n \leqslant \ulcorner X''_n \urcorner$. But this is impossible since if any subset of $\overline{\gamma \cup \delta}$ is enumerated in order of magnitude (even non-effectively), this enumeration grows faster than the range of any member of $\mathcal{F}$. Thus $g^* \notin \mathcal{F}$ and the theorem is proved.

Remark 3. A function of effective inseparability for (T, R) may be chosen to be primitive recursive in the universal function f of $\mathcal{F}$ (see theorem 5 for notation).

Let $\mathcal{T}$ be Peano Arithmetic. As an immediate corollary to theorem 6 we obtain

THEOREM 7. *Given an* r.e. *class $\mathcal{F}$ of general recursive functions, there exists a theory $\mathcal{T}$ in standard formalization such that no* 1-1 *recursive function u mapping W onto $W_{\mathcal{T}}$, T onto $T_{\mathcal{T}}$, R onto $R_{\mathcal{T}}$ is in $\mathcal{F}$.*

Proof. As in the proof of theorem 6, we can assume without loss of generality that any function of one argument which is primitive recursive in $\mathcal{F}$ is a member of $\mathcal{F}$. Use the theory $\mathcal{T}$ of theorem 6. Note that if u were in $\mathcal{F}$, then $\mathcal{T}$ would be effectively inseparable by a function primitive recursive in $\mathcal{F}$ and hence in $\mathcal{F}$ (see the proof of theorem 6).

Theorem 7 gives rise to a classification of e.i. theories in standard formalization. As a basis for this classification we give the following reducibility definition for theories. We assume that $\mathcal{F}$ contains all primitive recursive functions and is closed under composition and primitive recursion.

DEFINITION 5. $\mathcal{C}_1$ is *$\mathcal{F}$-reducible to* $\mathcal{C}_2$ ($\mathcal{C}_1 \leqslant_{\mathcal{F}} \mathcal{C}_2$) if there is a 1-1 recursive function $f \in \mathcal{F}$ mapping W_1 into W_2, T_1 into T_2, R_1 into R_2 preserving deducibility, negation and implication.

The reducibility relation of definition 5 gives rise in a natural manner to an equivalence relation: $\mathcal{C}_1 \equiv_{\mathcal{F}} \mathcal{C}_2$ if and only if $\mathcal{C}_1 \leqslant_{\mathcal{F}} \mathcal{C}_2$ and $\mathcal{C}_2 \leqslant_{\mathcal{F}} \mathcal{C}_1$.

Now let $\mathcal{F}$ be the set of primitive recursive functons $\mathcal{F}_p$. We discuss briefly the equivalence classes of effectively inseparable theories under $\equiv_{\mathcal{F}_p}$. First recall that, by remark 1, section 2 for any two theories $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfying the hypothesis of the theorem stated in this remark, the f^* of theorem 2 may be chosen to be primitive recursive. Thus, for example, all consistent extensions of the theory **R** of *Undecidable Theories* [10] belong to the same equivalence class modulo $\equiv_{\mathcal{F}_p}$. Furthermore from lemma 1 we know: if $\mathcal{C}_1$ is an e.i. theory possessing a primitive recursive witness to its effective inseparability (e.g. any extension of **R**) and if $\mathcal{C}_2$ is any other e.i. theory then $\mathcal{C}_2 \leqslant_{\mathcal{F}_p} \mathcal{C}_1$. Now let $\mathcal{C}_1'$ be the theory obtained by applying theorem 6 to $\mathcal{F}_p$. Then $[\mathcal{C}_1'] \lneqq_{\mathcal{F}_p} [\mathcal{C}_1]$. Let g be a function witnessing the effective inseparability of $\mathcal{C}_1'$. Let $\mathcal{F}_p'$ be the r.e. class of functions of one argument obtained from $\mathcal{F}_p \cup \{g\}$ by finite application of composition and primitive recursion. Apply theorem 6 to $\mathcal{F}_p'$ to obtain $\mathcal{C}_1''$. Now $[\mathcal{C}_1''] \lneqq_{\mathcal{F}_p} [\mathcal{C}_1]$ and $[\mathcal{C}_1''] \neq_{\mathcal{F}_p} [\mathcal{C}_1']$. Continuing in this way we obtain $\aleph_0$ distinct equivalence classes of effectively inseparable theories in standard formalization with $[\mathcal{C}_1]$ a maximum element. The detailed structure of the algebra of these equivalence classes remains to be investigated. (7)

Added in proof, Sept. 1967: These results have appeared in summary form in Bull. Amer. Math. Soc. 73 (1967), pp. 145-148.

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