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A connected topology for the unit interval

by

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1. Introduction. This paper presents a solution to a problem proposed by J. Stallings in his paper entitled *Fixed point theorems for connectivity maps* which appeared in Fundamenta Mathematicae 47 (1959), pages 249-263. A knowledge of connected topological spaces and the fundamental theorems pertaining to them is assumed. The following notation and definitions are preliminary to proceeding to the statement of Stallings' problem.

For convenience the closed unit interval $[0, 1]$ will be denoted by I and τ_0 will denote the usual topology on I. Also in notation the term 'interval' shall indicate the usual open interval of (I, τ_0) of the form (a, b) where $a < b$.

1.1. DEFINITION. A family of sets, S, is a *subbase* for a topology τ if and only if each open set of τ is the union of finite intersections of members of S. Such a subbase shall be referred to as a *τ-subbase*.

1.2. DEFINITION. Given a space (X, τ) and an element $x \in X$, then N is a *τ-neighborhood* of x if and only if N is an open set of (X, τ) and $x \in N$.

STALLINGS' PROBLEM. If τ is a topology on $I = [0, 1]$, let τ_L be the topology whose subbase consists of the open sets of τ and of the left-closed intervals $[a, b)$; let τ_R be the topology whose subbase consists of the open sets of τ and of the right-closed intervals $(a, b]$. Suppose that τ is a connected topology for I and that τ is finer than the usual topology for I. Let L and R be subsets of I, $L \cup R = I$, $0 \in L$, $1 \in R$, L open in τ_L and R open in τ_R. Is it necessarily true that $L \cap R \neq \emptyset$?

2. Considering the usual topology on I. One theorem pertaining to the usual topology on I is stated here due to its relationship to Stallings' problem. Its proof, being rather obvious, is omitted.

2.1. THEOREM. *If the τ of Stallings' problem is restricted to τ_0, then Stallings' question has an affirmative answer, that is, $L \cap R \neq \emptyset$.*

3. Characterization of the properties of the required topology. The following four results serve to characterize the properties which must be possessed by a topology on I in order that it satisfy the criteria imposed by Stallings in his problem. Due to the simplicity of their proofs, Theorem 3.1 and Corollary 3.2 are stated without proofs.

3.1. THEOREM. *If $x \in I$ and τ is a connected topology for I finer than τ_0, then for each $\varepsilon > 0$ every τ-neighborhood of x must have elements in $(x-\varepsilon, x)$ for $x \neq 0$ and in $(x, x+\varepsilon)$ for $x \neq 1$.*

3.2. COROLLARY. *If $x \in I$ and τ is a connected topology for I finer than τ_0, then no τ-neighborhood N of x can contain a half closed interval J where the end point J is not 0 or 1 and is a positive distance from $N-J$.*

3.3. DEFINITION. Let S be a topological space. A subset X of S is said to be a *τ-component of S* if and only if it satisfies the following conditions:

(1) X is non-empty.

(2) X is a connected subset of (S, τ).

(3) If Y is any connected subset of (S, τ) satisfying $Y \cap X \neq \emptyset$, then $Y \subset X$.

3.4. DEFINITION. A collection σ of τ-neighborhoods of a point p in a space (S, Δ) is said to form a *local τ-base at p* if and only if, given a Δ-neighborhood U of p in S, there exists a $V \in \sigma$ such that $V \subset U$.

The proof of the following theorem is apparent and hence the theorem is stated without proof.

3.4. THEOREM. *Let τ be a connected topology defined on I which is finer than τ_0 and let x be an element of I. If the τ_0-neighborhoods of x do not form a local τ-base at x, then x must have a τ-neighborhood, call it N, such that $(0, x] \cap C_x = \{x\}$ and/or $[x, 1) \cap C_x = \{x\}$ where C_x is the τ_0-component of N containing x.*

3.6. THEOREM. *Let τ be any connected topology on I which is finer than τ_0, such that for every element x of I, with the exception of a set of elements P, the τ_0-neighborhoods of x form a local τ-base at x. Let L, R, τ_L and τ_R be chosen as in Stallings' problem, and suppose $L \cap R = \emptyset$. Then P contains a non-denumerable number of elements of I.*

Proof. Let U be the τ_0-interior of L. It follows that U is the union of at most $\aleph_0$ disjoint open intervals. Let U' be the set of left end points of the intervals of U. Let U'' be the set of right end points of the intervals of U and $U^* = U' \cup U''$. Let V be the τ_0-interior of R. Define V^*, V' and V'' similarly for V. Observe that $U \subset L$. Assume that $a \in U''$ and $a \in R$. Now $a \in U''$ implies that there exists a $W = (b, a) \in U$ for some $b < a$. Hence due to the connectedness of τ, Theorem 3.1 and the definition of τ_R, every τ_R-neighborhood of a must contain elements of W, hence of U. Which leads to the contradiction that R is not open in τ_R as required in Stallings' problem. Therefore $a \in U''$ implies $a \in L$. Hence

(3.1) $$U'' \subset L .$$

Similarly,

(3.2) $$V' \subset R .$$

Clearly

(3.3) $$U'' \subset P \quad \text{and} \quad V' \subset P .$$

Choose an element of I, call it y, such that $y \in I-(U \cup V \cup U^* \cup V^*)$. Assume $y \notin P$. Now either $y \in L$ or $y \in R$. It shall be assumed that $y \in L$. The proof is similar if it is assumed that $y \in R$. Therefore either there exists an open interval contained in L containing y or y must be the left end point of an open interval contained in L. However this implies $y \in U$ or $y \in U^*$, a contradiction. Hence,

(3.4) $$y \in I-(U \cup V \cup U^* \cup V^*) \Rightarrow y \in P .$$

It follows from statement (3.4) that if $U = V = \emptyset$, then $P \supset I$ and P is non-denumerable, ending the proof. Hence assume that $U \neq \emptyset$ and observe that the proof follows similarly if it is assumed that $V = \emptyset$.

Define $A = U \cup V$. The number of τ_0-components in A is at most denumerable. If A consists of a finite number of τ_0-components, there exists an interval of U which is closer to 1 than any other interval of U. Let d be the right end point of this maximal open interval of U. From statement (3.1), $d \in L$, and hence $d \neq 1$. Now either there is no maximal open interval of V to the right of d, or there is a first one. If there is a first one, call its left end point e. Then statement (3.2) implies $e \in R$. Therefore either $[d, 1]$ or $[d, e]$, applying statements (3.3) and (3.4), consists only of elements of P since it contains no element of A. Hence either P is non-denumerable and the proof is complete or A must contain a denumerable number of τ_0-components.

If A consists of a denumerable number of τ_0-components, form a set B as follows. Let $B \supset A$. Place $1 \in B$ if 1 is in U^* or V^*. Place 0 in B if 0 is in U^* or V^*. Let $q \in B$, for $q \in I$, if q is the end point of two distinct τ_0-components of A. Notice that B is an open set under τ_0. It follows that $I-B$ is a closed set under τ_0.

Now assume that z is an isolated point of $I-B$ under τ_0. The case where $z \in (0, 1)$ shall be considered since the proof is similar if $z = 0$ or 1. Then there exists an open interval, say (k, h) where $k < z < h$, such that $(k, h)-\{z\} \subset B$. Therefore each element of $(k, h)-\{z\}$ is an element of A or an end point of two distinct τ_0-components of A. Observe that z is not in B, hence z is not in A and is not an end point of two distinct τ_0-components of A. It follows that for some m, where $k < m < z$ or $z < m < h$, that either (m, z) or (z, m) must contain a denumerable number of non-overlapping τ_0-components of A, say K_i, $i = 1, 2, \ldots$, such that K_i is closer to z than K_j if $i > j$, m is an end point of K_1 and such that exactly one point is between K_i and K_{i+1}, $i = 1, 2, \ldots$ Without loss of generality it may be assumed that $K_i \subset (m, z)$, $i = 1, 2, \ldots$ Notice that the τ_0-components of A are elements of U or V and hence each is an open interval. Now for some i suppose that K_i and K_{i+1} are both elements of U. Then K_i can be written in the form (r, s) and K_{i+1} in the form (s, t). Therefore $\{s\} \subset R$ since $\{s\} \subset L$ implies that (r, t) is in the τ_0-interior of L, hence in U, and therefore a τ_0-component of A. Then (r, t) being a τ_0-component of A implies that K_i is not a τ_0-component of A, which is a contradiction. However statement (3.1) says that $\{s\} \subset L$. Therefore K_i and K_{i+1} are not both elements of U for any i. Similarly it can be shown that K_i and K_{i+1} are not both elements of V for any i. Hence there exists a j such that $K_j \in U$ and $K_{j+1} \in V$. Let c be the common end point of K_j and K_{j+1}. Statement (3.1) requires that $\{c\} \subset L$ and statement (3.2) requires that $\{c\} \subset R$. Therefore contradicting the hypothesis that $L \cap R = \emptyset$. It follows that $I - B$ contains no isolated points under τ_0.

It now follows that $I - B$ is a perfect set. Hence $I - B$ is either the null set or contains a non-denumerable number of elements. It follows from statements (3.3) and (3.4) that every element of $I - B$, with the exception of at most a denumerable number of elements of $U' \cup V''$, is in P. If $I - B$ is non-null this results in P being non-denumerable.

Suppose that $I - B$ is the null set. Since $U \neq \emptyset$, pick one maximal open interval of U. Call its right end point g. From statement (3.1), $g \neq 1$. g is not in V' for if it were, by statement (3.2), $g \in R$ and, by statement (3.1), $g \in L$. Hence $g \notin B$ which implies $g \in I - B$ and $I - B$ is non-null.

Therefore it follows that P consists of a non-denumerable number of elements and the proof is complete.

4. The tangled topology (τ') for I. The following is a topology for I which shall be referred to as the tangled topology or τ'. It will be shown that the tangled topology satisfies all of the conditions set forth for τ of Stallings' problem and insures that his question has a negative answer.

The tangled topology for I. Remove the middle 1/3 intervals of I as in the formation of the Cantor set and label them as follows:

$$(a_1, b_1) = (1/3, 2/3), \quad (c_1, d_1) = (1/9, 2/9), \quad (c_2, d_2) = (7/9, 8/9),$$
$$(a_2, b_2) = (1/27, 2/27), \quad (a_3, b_3) = (7/27, 8/27),$$
$$(a_4, b_4) = (19/27, 20/27), \quad (a_5, b_5) = (25/27, 26/27).$$

Continue the above process, taking out the middle 1/3 of intervals not yet labeled at the nth stage and calling them (a_i, b_i) if n is odd or (c_i, d_i) if n is even. Observe that no two intervals for even values of n are assigned the same subscript. Similarly, no two intervals for odd values of n are assigned the same subscript. Define:

$$M = \bigcup_{i=1}^{\infty} I_i \quad \text{where} \quad I_i = (a_i, b_i).$$
$$N = \bigcup_{i=1}^{\infty} J_i \quad \text{where} \quad J_i = (c_i, d_i).$$
$$K = I - \bigcup_{i=1}^{\infty} \bar{I}_i - \bigcup_{i=1}^{\infty} \bar{J}_i - \{0\} - \{1\}.$$

For a subbase B for the topology τ' in the following manner; $S \in B$ if and only if one of the following holds:

(i) S is open in the usual topology of I.

(ii) $S = M \cup \{p\}$ where $p \in K \cup \{0\} \cup \bigcup_{i=1}^{\infty} \{b_i\}$.

(iii) $S = N \cup \{p\}$ where $p \in \{1\} \cup \bigcup_{i=1}^{\infty} \{c_i\}$.

4.1. THEOREM. *I is connected under τ'.*

Proof. Assume that τ' does not leave I connected, that is, there exists sets A and B such that $A \cup B = I$, $A \cap B = \emptyset$, A and B are both open and closed proper subsets of I. Since $0 \in A$ or $0 \in B$, without loss of generality, it may be assumed that $0 \in B$. Therefore the set A must have a greatest lower bound, call it a.

Case I. $a \in A$. It is apparent, since $0 \in B$, that $a > 0$. Hence by the definition of τ' every τ'-neighborhood of a must contain points of $[0, a)$ and, hence, points of B. Therefore a is a limit point of B and B is not a closed subset of I, which contradicts the assumption.

Case II. $a \in B$.

(A) If $a \in M \cup N \cup \bigcup_{i=1}^{\infty} \{a_i\} \cup \bigcup_{i=1}^{\infty} \{c_i\} \cup \bigcup_{i=1}^{\infty} \{d_i\}$, then each τ'-neighborhood of a contains an interval of the form $[a, b)$ for some $b > a$. B is an open set of (I, τ'), therefore a would not be, as defined, the greatest lower bound of A.

(B) If $a \in K \cup \{0\} \cup \bigcup_{i=1}^{\infty} \{b_i\}$, then for some $\varepsilon > 0$, $(a, a+\varepsilon) \cap M$ must be in B since $a \in B$, and therefore the end points of M are in B. Also, from the definition of the τ'-neighborhoods of any point $b \in K$, b must be in B if it is in $(a, a+\varepsilon)$. Hence, since a is the greatest lower bound of A, A must contain a sequence of intervals of N approaching a from the right. Choose one of the intervals in this sequence, call it (c, d), such that $(c, d) \subset (a, a+\varepsilon/2)$. Every τ'-neighborhood of d contains an interval of the form (e, f) where $e < d < f$. Hence, from the construction of M and N, the interval (d, f) contains points of B for every f. Therefore d is a limit point of B and, since B is closed, $d \in B$. This results in a contradiction since d is also a limit point of A.

(C) If $a = 1$, $A = \emptyset$. However, A was restricted to be non-null. A contradiction.

Therefore it follows that τ' leaves I connected.

It is apparent that the connected subsets of I under τ_0 are the intervals contained in I. It is also apparent that if τ is a finer connected topology for I than τ_0, then any connected subset of I under τ is also a connected subset of I under τ_0. The following theorem shows that the reverse is also true.

4.2. THEOREM. *If τ is a finer connected topology for I than τ_0, then any connected subset of I under τ_0 will be a connected subset of I under τ.*

Proof. Observe that from the statement of the theorem it follows that I is connected under τ, hence it is necessary only to consider proper subsets of I. Let J be a proper subset of I which is connected under τ_0, that is, J is an interval contained in I.

Case I. J is a closed interval. Let $J = [a, b]$ where $0 \leqslant a < b \leqslant 1$. Assume that J is not a connected subset of I under τ. Therefore there exists A and B such that $A \cup B = J$, $A \neq \emptyset$, $B \neq \emptyset$, $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$. Now either $a \in A$ or $a \in B$ and either $b \in A$ or $b \in B$. Assume $a \in A$ and $b \in B$. The proof is similar if any of the other three possible combinations are considered.

Now define $[0, a) \cup A = C$ and $(b, 1] \cup B = D$. Observe that $C \cup D = I$, $C \neq \emptyset$, $D \neq \emptyset$, $\bar{C} \cap D = \emptyset$ and $C \cap \bar{D} = \emptyset$, which contradicts the fact that I is connected under τ. Hence, if J is a closed interval of I, J is connected under τ.

Case II. J is a half closed interval. Assume that J does not contain its right end point. The proof is similar if it is assumed that J does not contain its left end point. Let $J = [a, b)$ where $0 \leqslant a < b \leqslant 1$. Observe that J can be written as $\bigcup_{i=1}^{\infty} [a, b - 1/n]$. By Case I of this theorem, for each n, $[a, b - 1/n]$ is connected under τ. Since a is in each of the intervals, it follows that $J = \bigcup_{i=1}^{\infty} [a, b - 1/n]$ is a connected subset of I under τ.

Case III. J is an open interval, that is, J contains neither of its end points. Let $J = (a, b)$ where $0 \leqslant a < b \leqslant 1$. Then $J = (a, c] \cup [c, b)$ where $a < c < b$. It follows from case II of this theorem that both $(a, c]$ and $[c, b)$ are connected subsets of I under τ. Hence, since they have the point c in common, this union is a connected subset of I under τ.

4.3. COROLLARY. *Any connected subset of I under τ_0 is a connected subset of I under τ'.*

Proof. Theorem 4.1 establishes that τ' is a connected topology for I. Observe that in the definition of τ' all open sets of I under τ_0 are also open sets of I under τ'; hence τ' is a finer topology for I than τ_0. Therefore it follows from Theorem 4.2 that any connected subset of I under τ_0 is a connected subset of I under τ'.

With the information contained in the preceding theorems it is now possible to show that the question asked by Stallings has a negative answer. Observe that Theorem 4.1 insures that τ' is a connected topology for I. From the definition of τ' it is apparent that τ' is finer than τ_0, hence τ' satisfies the conditions of the τ of Stallings' problem. Now define the following sets where the symbols $M, N, K, b_i, c_i, a_i, d_i$ have the same meaning as in the definition of τ' and where τ'_L and τ'_R have the same meaning as in the statement of Stallings' problem.

$$P = M \cup \{0\} \cup K \cup \bigcup_{i=1}^{\infty} \{b_i\},$$

$$Q = N \cup \{1\} \cup \bigcup_{i=1}^{\infty} \{c_i\},$$

$$L' = P \cup \bigcup_{i=1}^{\infty} \{a_i\},$$

$$R' = Q \cup \bigcup_{i=1}^{\infty} \{d_i\}.$$

It is apparent that P is an open set in I under τ' and, hence, in I under τ'_L and that Q is an open set in I under τ' and, hence, in I under τ'_R. It follows immediately that L' is open in τ'_L and that R' is open in τ'_R. Observe that $L' \cup R' = I$, $0 \in L'$ and $1 \in R'$. Hence L' and R' satisfy, respectively, the requirements of the L and R of Stallings' problem. It is quite apparent that $L' \cap R' = \emptyset$ which results in the negative answer to Stallings' problem.

It is of interest to observe that τ' is an uncomplicated a topology as any which answers Stallings' problem. Theorems 3.1 and 3.5, along with Corollary 3.2, tend to describe the additional open sets which were added to τ_0 to form τ' while Theorem 3.6 assures that is necessary for I under a topology satisfying Stallings' conditions to have a non-denumerable number of points at which the τ_0-neighborhood system does not form a local base.

5. Additional properties of the tangled topology (τ').

Since τ' is finer than τ_0, it is apparent that I under τ' forms a Hausdorff space. The following theorem gives additional information about τ'.

5.1. THEOREM. *I under τ' has the following properties.*

(a) *I under τ' is not countably compact, hence, not a compact space.*

(b) *I under τ' is not regular.*

(c) *I under τ′ is a separable space.*

(d) *I under τ′ is a first countable space.*

(e) *I under τ′ is not a second countable space.*

Proof. In this proof the notation is that used in defining τ'.

(a) Select a sequence $\{s_i\}$, $i = 1, 2, \ldots$, such that for each i, $s_i \in N$ and $\lim_{i\to\infty} s_i = 0$ under τ_0. It is apparent that such a sequence exists. Since τ' is finer than τ_0, 0 is the only possible limit point of the sequence. However, 0 has a τ'-neighborhood such that it contains no element of N. It follows that 0 is not a limit point of the sequence, and, hence, I under τ' is not countably compact. Applying the well-known theorem that every compact subset of a space is countably compact, it follows immediately that (I, τ') is not a compact space.

(b) Let U be a neighborhood of b_k of the form $[b_k \cup M] \cap (a, b)$ where $0 \leqslant a < b_k < b \leqslant 1$. Assume that there exists a neighborhood of b_k, call it V, such that $\overline{V} \subset U$. It is apparent that V would have a subset, call it W, of the same form as U. Now since all open sets of τ' are unions of finite intersections of subbase elements, there exists a positive integer j such that $(a_j, b_j) \subset W$. Observe that a_j is a limit point of W, hence of V, and $a_j \notin U$, Therefore $\overline{V} \not\subset U$.

(c) Observe that each open set of (I, τ') contains an interval. Therefore, each open set of (I, τ') contains a rational number. Hence, the set of rational numbers contained in I is a countable dense subset of (I, τ'), and I under τ' is a separable space.

(d) Observe from the definition of τ' that if $P \in I$, then one of the following forms a countable base, or local countable base, at p for τ'.

(1) $I \cap (p-1/n,\ p+1/n)$, $n = 1, 2, \ldots$

(2) $(p-1/n, p+1/n) \cap \{M \cup p\}$, $n = 1, 2, \ldots$

(3) $(p-1/n,\ p+1/n) \cap \{N \cup p\}$, $n = 1, 2, \ldots$

It follows that (I, τ') is a first countable space.

(e) Observe that each element $x \in K$ has a neighborhood of the form $U \cap \{M \cup x\}$, U an open set of (I, τ_0). Now if x_0 is a particular element of K, observe that no union of finite intersections of sets which are open in (I, τ'), excluding those of the form $U \cap \{M \cup x_0\}$ where U is an open set of $\{I, \tau_0\}$, is equal to a set of the form $U \cap \{M \cup x_0\}$, where U is an open set of (I, τ_0). Hence, since it is well known that K consists of a non-denumerable number of elements, it follows that (I, τ') is not second countable.

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Deduction-preserving "Recursive Isomorphisms" between theories * **

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Introduction. In this paper we concern ourselves with recursive mappings between theories which preserve deducibility, negation and implication. Roughly we show that any two axiomatizable theories containing a small fragment of arithmetic—this can be made precise—are "*isomorphic*" by a primitive recursive function which preserves deducibility, negation and implication (and hence theoremhood, refutability and undecidability). We also show that, between any two effectively inseparable theories formulated in the predicate calculus, there exists a recursive "isomorphism" preserving deducibility, negation and implication. We will see that we cannot replace "recursive" by "primitive recursive" in the last result. As a consequence we obtain a partition of all effectively inseparable theories in standard formalization into $\aleph_0$ equivalence classes. The unique maximal element is the equivalence class of those theories containing the small fragment of arithmetic mentioned above. A more precise summary of our results—which, incidentally answer some questions left open by Pour-El [6]—appears below, following some brief notational remarks.

In our opinion interest in the preservation of sentential connectives—particularly implication—can be justified by the following consideration. The preservation of implication implies the preservation of modus ponens and modus ponens is intimately related to the deductive structure of the theories. (Indeed, it is well known (Quine [5]) that the predicate calculus can formulated so that modus ponens is the sole rule of inference.)

All theories considered in this paper will contain the propositional calculus. For definiteness we assume that implication and negation are the sole primitive propositional connectives: $A \vee B$ is an abbreviation

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