Toroidal decompositions of E^3 which yield E^3

by

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1. Introduction. The purpose of this paper is to develop a necessary and sufficient condition for a certain type of upper semi-continuous decomposition of E^3 (Euclidean 3-space) to have a decomposition space homeomorphic to E^3.

The type of upper semi-continuous decomposition we wish to consider is the following. For each positive integer i, let T_i be a compact 3-manifold-with-boundary in E^3 such that each component of T_i is a solid polyhedral torus, $T_{i+1} \subset \operatorname{Int} T_i$, and T_{i+1} is inessentially embedded in T_i. Let G be the upper semi-continuous decomposition of E^3 obtained from components of $T_0 = \bigcap_{i=1}^{\infty} T_i$ and points of $E^3 - T_0$. Decompositions of E^3 that can be defined in this manner will be called *toroidal decompositions*, and $\{T_i\}$ will be referred to as the defining sequence of the decomposition. We let E^3/G denote the decomposition space with associated projection map P.

In [5] Bing showed that there is a toroidal decomposition of E^3 such that there are points in the decomposition space without small neighborhoods bounded by 2-spheres; hence the decomposition space of Bing's example is not homeomorphic to E^3. We show that a sufficient condition for a toroidal decomposition of E^3 to yield E^3 is that each point of E^3/G have arbitrarily small neighborhoods bounded by 2-spheres. Decompositions satisfying this sufficient condition will be referred to as locally spherical decompositions. Note that the 2-sphere boundaries in a locally spherical decomposition are not assumed to miss the image of the non-degenerate elements.

It follows from Theorem 1 of [8] that if G is a toroidal decomposition which is locally spherical and if, in addition, these 2-sphere boundaries in E^3/G can be chosen to miss $P(T_0)$, then E^3/G is homeomorphic to E^3. However, the toroidal decomposition obtained in [2] is such that E^3/G is homeomorphic to E^3 but the image of the non-degenerate elements, $P(T_0)$, is a wild Cantor set; then by [4] it follows that at least one point of $P(T_0)$ has no small neighborhood whose boundary is a sphere which misses $P(T_0)$. In Section 6 there are some questions and observations concerning these locally spherical decomposition spaces.

2. Statement of main theorem and outline of proof. We wish to establish the following theorem in this paper.

THEOREM A. *Let G be a toroidal decomposition of E^3. For E^3/G to be homeomorphic to E^3 it is necessary and sufficient that for each point x of E^3/G and each neighborhood U of x there is a neighborhood U' of x such that $U' \subset U$ and $\operatorname{Bd} U'$ is a 2-sphere.*

The necessity of the condition is obvious. By Theorems 3 and 6 of [1] the sufficiency of the condition will follow if we can show that for any $\varepsilon > 0$ and any positive integer n there is a homeomorphism h of E^3 onto E^3 such that (1) if $x \in E^3 - T_n$, then $h(x) = x$ and (2) if $g \in G$, then $\operatorname{diam} h(g) < \varepsilon$. Sections 3, 4, and 5 of this paper will be devoted to showing the existence of h.

3. Preliminaries. We assume then that we have a toroidal decomposition G of E^3 which has a locally spherical decomposition space, that T_n is some member from the defining sequence $\{T_i\}$, and that ε is some positive number. Recall that $T_0 = \bigcap_{i=1}^{\infty} T_i$.

The following lemma shows the existence of a subsequent defining stage T_m, $m > n$, with certain properties which will be important in constructing the homeomorphism h of Section 2.

LEMMA 1. *There is a subsequent defining stage T_m in T_n and a finite collection $U_1, U_2, \ldots, U_a$ of open sets such that*

(1) *each $\operatorname{Cl} U_i \subset \operatorname{Int} T_n$,*

(2) *for each i, each element of G that intersects U_i is contained in U_i,*

(3) *each $\operatorname{Bd} P(U_i)$ is a 2-sphere,*

(4) *each component of T_m is contained in some U_i, and*

(5) *each $P^{-1}(\operatorname{Bd} P(U_i)) - T_0$ is locally polyhedral and in general position relative to $\operatorname{Bd} T_m$.*

Proof. Let x be a point of $P(T_0)$. Then x has an open neighborhood U_x such that $\operatorname{Cl} U_x \subset P(\operatorname{Int} T_n)$ and $\operatorname{Bd} U_x$ is a 2-sphere. Following the procedure of [5], page 439, we may use Theorem 7 of [3] and Theorem VI, 10 of [9] to adjust $P^{-1}(\operatorname{Bd} U_x) - T_0$ slightly so that we may assume it is locally polyhedral. Since $\{U_x \mid x \in P(T_0)\}$ forms an open cover of the compact set $P(T_0)$, it follows that some finite subcollection $U_1', U_2', \ldots, U_a'$ covers $P(T_0)$. Let m, $m > n$, be chosen large enough that the image under P of each component of T_m is contained in some U_i'. Let $U_i = P^{-1}(U_i')$. Since $P^{-1}(\operatorname{Bd} P(U_i)) - T_0$ is locally polyhedral, we may, by a further slight adjustment of $P^{-1}(\operatorname{Bd} P(U_i)) - T_0$, assume it is in general position relative to $\operatorname{Bd} T_m$. Hence $U_1, U_2, \ldots, U_a$ form a finite collection of open sets satisfying the five requirements in Lemma 1.

The components of T_m are now separated into two disjoint sets as follows.

$A = \{Y \mid Y$ is a component of T_m, and for each i $(1 \leqslant i \leqslant a)$ there is no meridional simple closed curve of $\operatorname{Bd} Y$ in $\operatorname{Bd} Y \cap P^{-1}(\operatorname{Bd} P(U_i))\}$,

$B = \{Y \mid Y$ is a component of T_m and $Y \notin A\}$.

Note that A and B are finite collections of disjoint solid polyhedral tori in E^3 and that the union of these tori is T_m.

In Section 4 we will show that for the positive number ε of Section 2 there is a homeomorphism h_1 of E^3 onto E^3 such that (1) if $x \in E^3 - T_n$, then $h_1(x) = x$ and (2) for each $Y \in A$, $\operatorname{diam} h_1(Y) < \varepsilon$. Then in Section 5 we will show that for any $\varepsilon_1 > 0$ and any element Y of B there is a homeomorphism h_Y of E^3 onto E^3 such that (1) if $x \in E^3 - Y$, then $h_Y(x) = x$ and (2) if $g \in G$ and $g \subset Y$, then $\operatorname{diam} h_Y(g) < \varepsilon_1$. Let $h_2 = \prod_{Y \in B} h_Y$. Since h_1 is uniformly continuous, choose ε_1 so that if M is any closed subset of E^3 with $\operatorname{diam} M < \varepsilon_1$, then $\operatorname{diam} h_1(M) < \varepsilon$. Then $h = h_1 h_2$ will be the homeomorphism promised at the end of Section 2.

4. Construction of the homeomorphism h_1. To show the existence of a homeomorphism h_1 satisfying the requirements stated in Section 3, we need to establish four lemmas.

LEMMA 2. *Let S be a polyhedral 2-sphere and T a solid polyhedral torus such that S is in general position relative to $\operatorname{Bd} T$. Then each simple closed curve in $S \cap \operatorname{Bd} T$ either bounds a disk on $\operatorname{Bd} T$, or circles $\operatorname{Bd} T$ once meridionally and no times longitudinally, or circles $\operatorname{Bd} T$ once longitudinally and no times meridionally.*

Proof. This lemma follows from Theorem 1 of [5].

We continue to use the defining stage T_m, $m > n$, obtained in Section 3. In Lemma 4 we find that for each U_i we can construct a polyhedral 2-sphere S in E^3 such that $P^{-1}(\operatorname{Bd} P(U_i)) \cap \operatorname{Bd} T_m = S \cap \operatorname{Bd} T_m$. It then follows by Lemma 2 that for any component T of T_m, each simple closed curve in $P^{-1}(\operatorname{Bd} P(U_i)) \cap \operatorname{Bd} T$ either bounds a disk on $\operatorname{Bd} T$ or circles $\operatorname{Bd} T$ once meridionally and no times longitudinally, or circles $\operatorname{Bd} T$ once longitudinally and no times meridionally. For the remainder of this paper, a meridional (longitudinal) simple closed curve of the boundary of a solid torus will be one that does not also go around the boundary of the torus longitudinally (meridionally).

LEMMA 3. *In E^3, let C be a continuum, T a solid polyhedral torus, and S a polyhedral 2-sphere such that*

(1) $C \cap T = \emptyset$,

(2) $C \subset \operatorname{Int} S$,

(3) *S is in general position relative to $\operatorname{Bd} T$, and*

(4) *each simple closed curve of* $\operatorname{Bd} T \cap S$ *either bounds a disk on* $\operatorname{Bd} T$ *or is a longitudinal simple closed curve of* $\operatorname{Bd} T$.

Then for each $\delta > 0$ *there is a polyhedral 2-sphere* S' *such that*

(1) $C \subset \operatorname{Int} S'$,

(2) $S' \cap T = \emptyset$, *and*

(3) *every point of* S' *is either a point of* S *or is within a distance* δ *of* $\operatorname{Bd} T$.

Proof. It follows from condition (4) of the hypothesis that there is a longitudinal simple closed curve J of $\operatorname{Bd} T$ such that $J \cap S = \emptyset$. Let U be a δ-neighborhood of T such that $C \subset E^3 - U$, and let E be a tubular neighborhood of J such that $E \subset U$ and $E \cap S = \emptyset$. There is a homeomorphism H of E^3 onto E^3 such that (1) if $x \in E^3 - U$, then $H(x) = x$ and (2) $H(E) = T$. Then $S' = H(S)$ is the required 2-sphere of Lemma 3.

LEMMA 4. *For each* Y, $Y \in A$, *there is a polyhedral 2-sphere* S *such that*

(1) $S \subset \operatorname{Int} T_n$,

(2) $Y \subset \operatorname{Int} S$, *and*

(3) S *does not intersect any element of* A.

Proof. By Lemma 1, Y is contained in some U_i, $1 \leqslant i \leqslant a$, where $\operatorname{Bd} P(U_i)$ is a 2-sphere and $\operatorname{Cl} U_i \subset \operatorname{Int} T_n$. Note that $P(T_0)$ is compact and 0-dimensional. Let $P(T_0) \cap \operatorname{Bd} P(U_i) = \bigcap_{j=1}^{\infty} D_j$ where each D_j is the union of a finite collection of disjoint disks on $\operatorname{Bd} P(U_i)$ and $D_{j+1} \subset \operatorname{Int} D_j$.

There is a positive integer r such that $P^{-1}(D_r) \subset \operatorname{Int} T_m$. Let R be an unbounded polyhedral ray starting at a point of $\operatorname{Bd} Y$, and, except for the initial point of R, R is contained in $E^3 - T_m$. Since $P^{-1}(\operatorname{Bd} P(U_i) - \operatorname{Int} D_r)$ is a disk-with-holes and is the only portion of $\operatorname{Bd} U_i$ that R intersects, we may assume that R intersects $P^{-1}(\operatorname{Bd} P(U_i) - \operatorname{Int} D_r)$ in an odd number of points and pierces it at each of these points.

There is a positive integer s such that $T_s \cap \big(P^{-1}(\operatorname{Bd} P(U_i) - \operatorname{Int} D_r)\big) = \emptyset$ and each component of T_s intersects at most one component of $P^{-1}(D_r)$. There is a positive integer u such that $P^{-1}(D_u) \subset \operatorname{Int} T_{s+1}$. From our definition of a toroidal decomposition it follows that each component of T_{s+1} can be shrunk to a point in the component of T_s containing it. Hence each boundary component of $P^{-1}(\operatorname{Bd} P(U_i) - \operatorname{Int} D_u)$ can be shrunk to a point in the component of T_s containing it.

Then for each component E of D_r there is a singular disk E' in $P^{-1}(E) \cup \operatorname{Int} T_s$ such that E' has no singularities in some neighborhood of its boundary and $\operatorname{Bd} E' = P^{-1}(\operatorname{Bd} E)$. By Dehn's Lemma [11], there is a non-singular polyhedral disk E'' in $P^{-1}(E) \cup \operatorname{Int} T_s$ such that $\operatorname{Bd} E'' = \operatorname{Bd} E'$. By construction, no two of these non-singular polyhedral disks intersect, the boundary of each such disk is also one of the boundary components of $P^{-1}(\operatorname{Bd} P(U_i) - \operatorname{Int} D_r)$, and each boundary component of $P^{-1}(\operatorname{Bd} P(U_i) - \operatorname{Int} D_r)$ is the boundary of exactly one such disk. Hence $P^{-1}(\operatorname{Bd} P(U_i) - \operatorname{Int} D_r)$ together with these polyhedral disks form a polyhedral 2-sphere S'. Also, by construction, S' has the following three properties.

(1) $Y \subset \operatorname{Int} S'$, since R intersects (and pierces) S' in an odd number of points.

(2) $S' \subset \operatorname{Int} T_n$, since $P^{-1}(\operatorname{Bd} P(U_i)) \cup T_s \subset \operatorname{Int} T_n$.

(3) S' is in general position relative to $\operatorname{Bd} T_m$, since there is a neighborhood U' of $\operatorname{Bd} T_m$ such that $P^{-1}(\operatorname{Bd} P(U_i)) \cap U' = S' \cap U'$.

Suppose that S' intersects an element Y' of A. By condition (3) above and the definition of the set A, it follows that each component of $S' \cap \operatorname{Bd} Y'$ is a simple closed curve which either bounds a disk on $\operatorname{Bd} Y'$ or circles $\operatorname{Bd} Y'$ longitudinally. By Lemma 3, we may replace S' by a polyhedral 2-sphere S'' such that (1) $Y \subset \operatorname{Int} S''$, (2) $S'' \cap Y' = \emptyset$, and (3) every point of S'' is either a point of S' or near $\operatorname{Bd} Y'$. (Choose the δ of Lemma 3 so that the δ-neighborhood U of Y' is contained in $\operatorname{Int} T_n$ and disjoint from $T_m - Y'$.) Repetition of the argument of this paragraph a finite number of times yields a polyhedral 2-sphere S satisfying the three requirements in Lemma 4.

Relative to the set A defined in Section 3, we now have the following lemma which establishes the existence of h_1.

LEMMA 5. *There is a homeomorphism* h_1 *of* E^3 *onto itself which is the identity on* $E^3 - T_n$ *and takes each element of* A *into a set of diameter less than* ε.

Proof. By Lemma 4, each element of A is contained in the interior of a polyhedral 2-sphere S such that S is contained in $\operatorname{Int} T_n$ and S does not intersect any element of A. It follows from Theorem 2.3 of [4] that there are mutually exclusive polyhedral 2-spheres $K_1, K_2, \ldots, K_q$ in $\operatorname{Int} T_n$ such that each element of A is contained in $\bigcup_{i=1}^{q} \operatorname{Int} K_i$. It then follows that there is a homeomorphism h_1 of E^3 into E^3 such that (1) if $x \in E^3 - T_n$, then $h_1(x) = x$ and (2) for each i, $\operatorname{diam} h_1(K_i \cup \operatorname{Int} K_i) < \varepsilon$. Then h_1 is the required homeomorphism of Lemma 5.

5. Construction of the homeomorphism h_2. In this section we establish the existence, relative to $\varepsilon_1 > 0$, of the homeomorphism $h_2 = \prod_{Y \in B} h_Y$ mentioned in Section 3. As mentioned at the end of Section 3, the proof of Theorem A will be complete upon showing the existence of each h_Y. In what follows assume that all meridional disks are polyhedral.

Lemmas 6 and 7 give conditions which imply the existence of h_Y.

Lemmas 8 and 9 are then used to establish that Y satisfies the hypothesis of Lemma 7.

LEMMA 6. *If for any positive integer u there are u disjoint meridional disks (polyhedral) $E_1, E_2, \ldots, E_u$ of Y such that no element of G in Y intersects more than one of these disks, then there is a homeomorphism h_Y of E^3 onto E^3 which is the identity on E^3-Y and takes each element of G in Y into a set of diameter less than ε_1.*

Proof. For large enough u, let $E_1', E_2', \ldots, E_u'$ be disjoint meridional disks of Y with the following property. There is a solid torus C contained in Y and concentric to Y such that the closure of each component of $Y-\bigcup_{i=1}^{u} E_i'$ (the closure of each component of $Y-\bigcup_{i=1}^{u} E_i'$ is a cube) intersects C in a cube of diameter less than $\varepsilon_1/2$. Since for each neighborhood U of $\mathrm{Bd}\,Y$ there is a homeomorphism H of Y onto Y such that H is fixed on $Y-U$ and for each i $(1 \leqslant i \leqslant u)$, $H(\mathrm{Bd}\,E_i) = \mathrm{Bd}\,E_i'$ (the disks $E_1, E_2, \ldots, E_u$ may have to be reordered to obtain H), we may assume that $\mathrm{Bd}\,E_i = \mathrm{Bd}\,E_i'$.

There is a homeomorphism H_1 of Y onto Y such that H_1 is fixed on $\mathrm{Bd}\,Y$ and for each i, $H_1(E_i) = E_i'$. Since if $g \in G$ and $g \subset Y$, then g intersects at most one of the disks E_i, it follows that g is contained in the closure of two adjacent components of $Y-\bigcup_{i=1}^{u} E_i$, and hence $H_1(g)$ is contained in the closure of two adjacent components of $Y-\bigcup_{i=1}^{u} E_i'$. There is a homeomorphism H_2 of Y onto Y such that H_2 is fixed on $\mathrm{Bd}\,Y$, $H_2(E_i') = E_i'$, and if $g \in G$ and $g \subset Y$, then $H_2H_1(g)$ is contained in both C and the closure of two adjacent components of $Y-\bigcup_{i=1}^{u} E_i'$. Then for each $g \in G$ and $g \subset Y$, it follows that $\mathrm{diam}\,H_2H_1(g) < \varepsilon_1/2+\varepsilon_1/2 = \varepsilon_1$. Extend H_2H_1 to E^3 by choosing $H_2H_1(x) = x$ for $x \in E^3-Y$. Then $h_Y = H_2H_1$ is the homeomorphism required in the conclusion of Lemma 6.

The following special terminology is needed for Lemma 7. Let $C_{i(0)}$ be a component from a defining stage T_i of G. For each positive integer k, $1 \leqslant k \leqslant r$, let $C_{i(k)} = C_{i(0)} \cap T_{i(k)}$, where $i < i(k) < i(k+1)$. For $s \geqslant 2$, let D be the union of s disjoint meridional disks (polyhedral) $D_1, D_2, \ldots, D_s$ of $C_{i(0)}$. We say that D is a *canonical cutting of $C_{i(0)}$ of size s and depth r* if for each k and each component T of $C_{i(k)}$ it follows that (1) $D \cap T$ is either empty or a finite collection of disjoint meridional disks of T and (2) no component of $C_{i(k)}$ intersects more than one component of $D \cap C_{i(k-1)}$.

Let J be a simple closed curve on the boundary of a solid torus T. Then by a *collar for J on* $\mathrm{Bd}\,T$ we mean an annulus on $\mathrm{Bd}\,T$ such that one of its two boundary components is J.

LEMMA 7. *Let D be a canonical cutting of $C_{i(0)}$ of size s and depth r. Let $\Psi_1, \Psi_2, \ldots, \Psi_s$ be disjoint collars on $\mathrm{Bd}\,C_{i(0)}$ for $\mathrm{Bd}\,D_1, \mathrm{Bd}\,D_2, \ldots, \mathrm{Bd}\,D_s$, respectively. In each Ψ_j, $1 \leqslant j \leqslant s$, let $J_{j1} (= \mathrm{Bd}\,D_j), J_{j2}, \ldots, J_{jr} (= \mathrm{Bd}\,\Psi_j - J_{j1})$ be r disjoint meridional simple closed curves of $\mathrm{Bd}\,C_{i(0)}$. Then there are $s\cdot r$ disjoint meridional disks D_{jk} of $C_{i(0)}$ such that $\mathrm{Bd}\,D_{jk} = J_{jk}$ and no component of $C_{i(r)}$ intersects more than one D_{jk}, where $1 \leqslant j \leqslant s$ and $1 \leqslant k \leqslant r$. (Hence no element of G in $C_{i(0)}$ intersects more than one D_{jk}.)*

Proof. We prove Lemma 7 by induction on r with arbitrary s, $s \geqslant 2$. We see that the requirements of the conclusion are fulfilled for $r = 1$ and all s by choosing $D_{j1} = D_j$, $1 \leqslant j \leqslant s$. Assume that the theorem is true for some integer $r = t$ and all s and that we have a canonical cutting $D = \bigcup_{j=1}^{s} D_j$ of $C_{i(0)}$ of size s and depth $t+1$.

We first fix our attention on D_1; the considerations for D_2 through D_s will be essentially the same. Since the following construction can be performed independently on each component of $C_{i(1)}$ that intersects D_1, we assume for simplicity that D_1 intersects just one component T of $C_{i(1)}$. It follows from the definition of canonical cutting that $\Delta_1 = D_1 - \mathrm{Int}\,T$ is a disk-with-holes. Let $\mathrm{Bd}\,D_1 = J_{11}, K_{11}, K_{21}, \ldots, K_{u1}$ be the boundary components of Δ_1 ordered so that if we start at a point of K_{11} and go around $\mathrm{Bd}\,T$ longitudinally in some fixed direction Ω we pass from K_{j1} to $K_{j+1,1}$ $(1 \leqslant j \leqslant u-1)$. (See Figure 1.) Slightly to one side of Δ_1 we may, by geometric methods, construct t mutually disjoint copies $\Delta_2, \Delta_3, \ldots, \Delta_{t+1}$ of Δ_1 such that for each k, $1 \leqslant k \leqslant t+1$, (1) $J_{1k} \subset \mathrm{Bd}\,\Delta_k$ and $\mathrm{Int}\,\Delta_k \subset (\mathrm{Int}\,C_{i(0)} - C_{i(1)})$, (2) the boundary components of Δ_k are $J_{1k}, K_{1k}, \ldots, K_{uk}$, and there are disjoint collars $\Gamma_1, \Gamma_2, \ldots, \Gamma_u$ on $\mathrm{Bd}\,T$ of $K_{11}, K_{21}, \ldots, K_{u1}$, respectively, such that for each Δ_k and each Γ_j $(1 \leqslant j \leqslant u)$, $\Gamma_j \cap \Delta_k = K_{jk}$, and (3) there are annuli $\Phi_1, \Phi_2, \ldots, \Phi_u$ on $\mathrm{Bd}\,T$ such that for p even and $2 \leqslant p \leqslant u$ we have $\Phi_p \cap (\bigcup_{j=1}^{u} \Gamma_j) = K_{p1} \cup K_{p+1,1}$ (where $p+1 = 1$ if $p = u$) and for q odd and $1 \leqslant q \leqslant u-1$ we have $\Phi_q \cap (\bigcup_{j=1}^{u} \Gamma_j) = K_{q,t+1} \cup K_{q+1,t+1}$. (See Figure 1 for the case $t+1 = 3$ and $u = 4$.)

Let F_j be the subdisk of D_1 bounded by K_{j1}. Since D_1 is a component of a canonical cutting of $C_{i(0)}$ of size s and depth $t+1$, $\bigcup_{j=1}^{u} F_j$ forms a canonical cutting of T of size u and depth t. For p even we may assume $\mathrm{Bd}\,F_p = K_{p,t+1}$ by adding Γ_p to F_p and pushing $\Gamma_p - K_{p,t+1}$ slightly to $\mathrm{Int}\,T$ (preserving the property that $\bigcup_{j=1}^{u} F_j$ is a canonical cutting of size u and depth t). By the hypothesis of the induction, there are $u\cdot t$ disjoint meridional disks E_{jk} of T, where either j is even and $2 \leqslant k \leqslant t+1$ or j is odd and $1 \leqslant k \leqslant t$, such that (1) $\operatorname{Bd} E_{jk} = K_{jk}$ and (2) no component of $C_{i(t+1)}$ intersects more than one E_{jk}. For p even, $2 \leqslant p \leqslant u$, let E_{p1} be a meridional disk of T obtained by adding Φ_p to $E_{p+1,1}$ (where $p+1 = 1$ if $p = u$) and pushing $\Phi_p - K_{p1}$ slightly to $\operatorname{Int} T$. For q odd, $1 \leqslant q \leqslant u-1$,

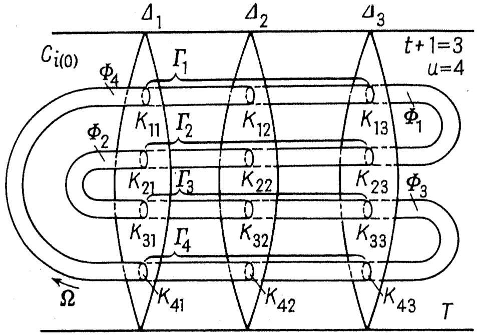

Fig. 1

let $E_{q,t+1}$ be a meridional disk of T obtained by adding Φ_q to $E_{q+1,t+1}$ and pushing $\Phi_q - K_{q,t+1}$ slightly to $\operatorname{Int} T$. We now fill in the holes of Δ_k by setting $D_{1k} = \Delta_k \cup (\bigcup_{j=1}^{u} E_{jk})$.

Then D_{11} and $D_{1,t+1}$ are singular disks and $D_{12}, D_{13}, \ldots, D_{1t}$ are non-singular polyhedral disks. By our construction no component of $C_{i(t+1)}$ intersects more than one D_{1k}. By Dehn's Lemma [11] or by direct geometric methods since the singularities are of an elementary nature, we may assume that D_{11} and $D_{1,t+1}$ are non-singular polyhedral disks. (See Figure 2.) Thus $D_{11}, D_{12}, \ldots, D_{1,t+1}$ are disjoint meridional disks of $C_{i(0)}$ and no component of $C_{i(t+1)}$ intersects two of them. Repeating a similar procedure on each of the disks $D_2, D_3, \ldots, D_s$ yields the required $s(t+1)$ disks, and the proof of Lemma 7 is complete.

In the following two lemmas we complete the proof of Theorem A by showing that for each Y, $Y \in B$, and each positive integer r there is a canonical cutting of Y of size 2 and depth r. Recall from Section 3 that Y is a solid polyhedral torus contained in T_m and there is an open set U_i such that (1) $\operatorname{Bd} P(U_i)$ is a 2-sphere, (2) $\operatorname{Bd} Y$ and $P^{-1}(\operatorname{Bd} P(U_i)) - T_0$ are in general position, and (3) $\operatorname{Bd} Y \cap [P^{-1}(\operatorname{Bd} P(U_i))]$ contains a meridional simple closed curve of $\operatorname{Bd} Y$.

LEMMA 8. *Let r be a positive integer. Then there are two disjoint meridional disks E_1 and E_2 of Y and r defining stages $C_{m(1)}, C_{m(2)}, \ldots, C_{m(r)}$ such that for each k, where $1 \leqslant k \leqslant r$ and $m < m(k) < m(k+1)$, we have*

(1) *$C_{m(k)} = Y \cap T_{m(k)}$,*

(2) *$E_1 \cup E_2$ is in general position relative to $\operatorname{Bd} C_{m(k)}$,*

(3) *no component of $C_{m(1)}$ intersects both E_1 and E_2, and*

(4) *no component of $C_{m(k+1)}$ intersects more than one component of $E_1 \cup E_2 - \operatorname{Bd} C_{m(k)}$.*

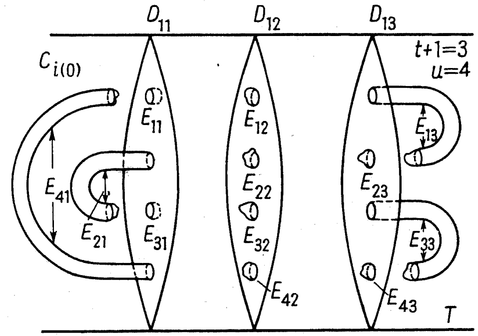

Fig. 2

Proof. There is a positive integer $m(1)$, $m(1) > m$, such that no component of $C_{m(1)}$, where $C_{m(1)} = Y \cap T_{m(1)}$, intersects more than one component of $P^{-1}(\operatorname{Bd} P(U_i)) - \operatorname{Bd} Y$. We may adjust $P^{-1}(\operatorname{Bd} P(U_i)) - T_0$ slightly so as to assume $P^{-1}(\operatorname{Bd} P(U_i)) - T_0$ is in general position relative to $\operatorname{Bd} C_{m(1)}$. Repetition of this argument yields integers $m(2), m(3), \ldots, m(r)$ such that (1) $m(1) < m(2) < \ldots < m(r)$, (2) for each k, $1 \leqslant k \leqslant r$, $P^{-1}(\operatorname{Bd} P(U_i))$ is in general position relative to $\operatorname{Bd} C_{m(k)}$, and (3) for each k, $1 \leqslant k \leqslant r-1$, no component of $C_{m(k+1)}$ intersects more than one component of $P^{-1}(\operatorname{Bd} P(U_i)) - \operatorname{Bd} C_{m(k)}$.

As in Lemma 4, let $P(T_0) \cap \operatorname{Bd} P(U_i) = \bigcap_{j=1}^{\infty} D_j$, where each D_j is the union of a finite number of disjoint disks on $\operatorname{Bd} P(U_i)$ and $D_{j+1} \subset \operatorname{Int} D_j$. There is a positive integer u such that $P^{-1}(D_u) \subset \operatorname{Int} T_{m(r)}$. As in the proof of Lemma 4, we may fill in the holes of $P^{-1}(\operatorname{Bd} P(U_i) - \operatorname{Int} D_u)$ with polyhedral disks contained in $\operatorname{Int} T_{m(r)}$ to obtain a polyhedral sphere S such that S is in general position relative to each $\operatorname{Bd} C_{m(k)}$ and no component of $C_{m(k+1)}$ intersects more than one component of $S - \operatorname{Bd} C^{m(k)}$.

Since $\operatorname{Bd} Y \cap S$ contains a meridional simple closed curve of $\operatorname{Bd} Y$, there are two disjoint disks E_1 and E_2 on S such that (1) $\operatorname{Bd} E_1$ and $\operatorname{Bd} E_2$ are meridional simple closed curves on $\operatorname{Bd} Y$ and (2) each of $\operatorname{Int} E_1 \cap \operatorname{Bd} Y$ and $\operatorname{Int} E_2 \cap \operatorname{Bd} Y$ is the sum of a finite collection of simple closed curves which bound disks on $\mathrm{Bd}\,Y$. By cutting E_1 and E_2 off on $\mathrm{Bd}\,Y$ (this is a geometric procedure which was used in Theorem 1 of [5]), we may assume that $\mathrm{Int}\,E_1$ and $\mathrm{Int}\,E_2$ are contained in $\mathrm{Int}\,Y$. The resulting disks E_1 and E_2 are the disks required in the conclusion of Lemma 8.

For Lemma 9 we continue to use E_1, E_2, and $C_{m(k)}$ $(1 \leqslant k \leqslant r)$ of Lemma 8.

LEMMA 9. *There are two disjoint meridional disks D_1 and D_2 of Y such that $D_1 \cup D_2$ forms a canonical cutting of Y of size 2 and depth r (relative to the defining stages $C_{m(k)}$ of Lemma 8).*

Proof. By the geometric procedure used in the proof of [5], Therem 1, we may eliminate the simple closed curves in $(E_1 \cup E_2) \cap (\bigcup_{k=1}^{r} \mathrm{Bd}\,C_{m(k)})$ which bound disks on $\bigcup_{k=1}^{r} \mathrm{Bd}\,C_{m(k)}$.

Suppose that there is a component T of some $C_{m(k)}$ such that either $E_1 \cap \mathrm{Bd}\,T$ or $E_2 \cap \mathrm{Bd}\,T$, say $E_1 \cap \mathrm{Bd}\,T$, contains a longitudinal simple closed curve of $\mathrm{Bd}\,T$. Using the technique in the proof of Lemma 3, we may push E_1 off T (choose the δ of Lemma 3 so that the δ-neighborhood U of T is contained in $\mathrm{Int}\,C_{m(k-1)}$ and disjoint from $C_{m(k)}-T$). Hence, for each component T of each $C_{m(k)}$, we may assume each component of $(E_1 \cup E_2) \cap \mathrm{Bd}\,T$ is a meridional simple closed curve of $\mathrm{Bd}\,T$.

Assume $T \cap E_1 \neq \emptyset$, where T is a component of $C_{m(1)}$. Let U be a neighborhood of T such that $U \subset \mathrm{Int}\,Y$ and $U \cap (C_{m(1)}-T) = \emptyset$. Let θ be the component of $E_1 - \mathrm{Int}\,T$ which contains $\mathrm{Bd}\,E_1$, and let $J_1, J_2, \ldots, J_s$ be the boundary components of θ other than $\mathrm{Bd}\,E_1$. For each i, let K_i be the subdisk of E_1 bounded by J_i. Since each J_i is a meridional simple closed curve of $\mathrm{Bd}\,T$, in each K_i there is a meridional disk Δ_i' of T. For each i, let N_i be an annulus on $\mathrm{Bd}\,T$ with boundary components $\mathrm{Bd}\,K_i$ and $\mathrm{Bd}\,\Delta_i'$. (If $K_i = \Delta_i'$, then let $N_i = \mathrm{Bd}\,K_i$.) Let $\theta' = \theta \cup \left(\bigcup_{i=1}^{s} (N_i \cup \Delta_i')\right)$. Then θ' is a singular disk, and if we push each $N_i - \mathrm{Bd}\,\Delta_i'$ into $U-T$, we may assume that the singularities of θ' are contained in $U-T$ and that $\theta' \cap T = \bigcup_{i=1}^{s} \Delta_i'$.

In U, let U' be a neighborhood of the singularities of θ'. By Dehn's Lemma [11] or by direct geometric methods since the singularities can be made to be of an elementary nature, we may replace θ' by a non-singular polyhedral disk E_1' such that $E_1' \subset \theta' \cup U'$ and $\mathrm{Bd}\,E_1' = \mathrm{Bd}\,\theta' = \mathrm{Bd}\,E_1$. Then $E_1' \cap T$ is the union of a finite number of disjoint meridional disks of T. Applying the above argument to each component of $C_{m(1)}$, then to each component of $C_{m(2)}$, and so on up to $C_{m(r)}$, we obtain disjoint meridional disks D_1 and D_2 of Y such that (1) for each component T of each $C_{m(k)}$, $(D_1 \cup D_2) \cap T$ is either empty or the sum of a finite number of disjoint meridional disks of T and (2) no component of $C_{m(k+1)}$ intersects more than one component of $(D_1 \cup D_2) \cap C_{m(k)}$. Then $D_1 \cup D_2$ forms a canonical cutting of T of size 2 and depth r, and the proof of Lemma 9 is complete.

It follows from Lemmas 7 and 9 that for $\varepsilon_1 > 0$ there is a homeomorphism h_Y of E^3 onto E^3 such that (1) if $x \in E^3 - Y$, then $h_Y(x) = x$ and (2) if $g \in G$ and $g \subset Y$, then $\mathrm{diam}\,h_Y(g) < \varepsilon_1$. As mentioned at the beginning of Section 5, the existence of h_Y completes the proof of Theorem A.

6. Questions and examples. In the definition of a toroidal decomposition we required that T_{i+1} be inessentially embedded in $\mathrm{Int}\,T_i$. Can Theorem A be proved without this restriction? Note that, even with the restriction that T_{i+1} be inessentially embedded in $\mathrm{Int}\,T_i$, the elements of a toroidal decomposition may not be point-like. On the other hand, without this restriction some of the elements of G may have non-trivial Čech cohomology, and hence, if this happens, E^3/G is not homeomorphic to E^3 (see Theorem 3, Corollary 2 of [10]). It is interesting to note that there is an example [7] of a continuum C in E^3 with trivial Čech cohomology, but all of its embeddings in E^3 fail to be point-like.

The following example, motivated by Bing's example ([6], p. 7), shows that there is a locally spherical decomposition of E^3 such that $E^3/G \neq E^3$.

EXAMPLE. Let A be the Cantor set obtained by removing middle thirds of the interval $[0, 1]$.

For each $a \in A$ there is a figure eight (or circle) F_a in E^3 obtained by adding the circle in the xy-plane with center at the origin and radius a to the circle in the xz-plane with center $(-1, 0, 0)$ and radius $1-a$. We obtain a decomposition G of E^3 by using the F_a's as the non-degenerate elements. Note that the non-degenerate elements are the intersection of a sequence $\{M_i\}$ of 3-manifolds-with-boundary such that for each i, each component of M_i is a cube with two handles and $M_{i+1} \subset \mathrm{Int}\,M_i$. Again, by Theorem 3, Corollary 2 of [10], E^3/G is not E^3, but E^3/G has the property that for each $x \in E^3/G$ and each neighborhood U of x there is a neighborhood U' of x contained in U whose boundary is a 2-sphere. To obtain U' for a given neighborhood U of the point $P(F_a)$ in E^3/G, we select a tubular neighborhood M of F_a such that $P(M) \subset U$ and $\mathrm{Bd}\,M$ contains exactly two figure eights F_a and F_β (or exactly one figure eight F_a if $a = 0$ or $a = 1$) so that $P(\mathrm{Bd}\,M)$ is a 2-sphere. Then $P(M)$ is the required neighborhood U'.

QUESTION. *Let G be a point-like upper semi-continuous decomposition of E^3 such that the closure of the image of the non-degenerate elements is compact and 0-dimensional. Is E^3/G homeomorphic to E^3 if each point of E^3/G has arbitrarily small neighborhoods bounded by 2-spheres?*

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A connected topology for the unit interval

by

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1. Introduction. This paper presents a solution to a problem proposed by J. Stallings in his paper entitled *Fixed point theorems for connectivity maps* which appeared in Fundamenta Mathematicae 47 (1959), pages 249-263. A knowledge of connected topological spaces and the fundamental theorems pertaining to them is assumed. The following notation and definitions are preliminary to proceeding to the statement of Stallings' problem.

For convenience the closed unit interval $[0,1]$ will be denoted by I and τ_0 will denote the usual topology on I. Also in notation the term 'interval' shall indicate the usual open interval of (I,τ_0) of the form (a,b) where $a<b$.

1.1. DEFINITION. A family of sets, S, is a *subbase* for a topology τ if and only if each open set of τ is the union of finite intersections of members of S. Such a subbase shall be referred to as a *τ-subbase*.

1.2. DEFINITION. Given a space (X,τ) and an element $x\in X$, then N is a *τ-neighborhood* of x if and only if N is an open set of (X,τ) and $x\in N$.

STALLINGS' PROBLEM. If τ is a topology on $I=[0,1]$, let τ_L be the topology whose subbase consists of the open sets of τ and of the left-closed intervals $[a,b)$; let τ_R be the topology whose subbase consists of the open sets of τ and of the right-closed intervals $(a,b]$. Suppose that τ is a connected topology for I and that τ is finer than the usual topology for I. Let L and R be subsets of I, $L\cup R=I$, $0\in L$, $1\in R$, L open in τ_L and R open in τ_R. Is it necessarily true that $L\cap R\neq\emptyset$?

2. Considering the usual topology on I. One theorem pertaining to the usual topology on I is stated here due to its relationship to Stallings' problem. Its proof, being rather obvious, is omitted.

2.1. THEOREM. *If the τ of Stallings' problem is restricted to τ_0, then Stallings' question has an affirmative answer, that is, $L\cap R\neq\emptyset$.*

3. Characterization of the properties of the required topology. The following four results serve to characterize the properties which must be possessed by a topology on I in order that it satisfy the