On systems of interlocking exact sequences

by

Peter J. Hilton (Zürich)

1. Introduction. Let $\mathcal{C}$ be a pointed category, with point o, and let o^X: $X \to o$, o_X: $o \to X$ be the unique morphisms in $\mathcal{C}$ for each X in $|\mathcal{C}|$. Let $\bar{\iota}$, $\underline{\iota}$ embed $\mathcal{C}$ in $\mathcal{C}^2$ by $\bar{\iota}(X) = o^X$, $\underline{\iota}(X) = o_X$. Let $\mathcal{A}$ be an abelian category and let $T = \{T_n, -\infty < n < \infty\}$ be a graded functor from $\mathcal{C}^2$ to $\mathcal{C}$. Set $\bar{\tau}_n = T_n \bar{\iota}$, $\underline{\tau}_n = T_n \underline{\iota}$, and let $\zeta = \{\zeta_n, -\infty < n < \infty\}$ be a graded natural transformation ζ_n: $\bar{\tau}_n \to \underline{\tau}_{n-1}$. Suppose that for any g: $Y \to Z$ in $\mathcal{C}$ the sequence

$$(1.1) \qquad \ldots \to \tau_n(Y) \xrightarrow{\alpha_n} \tau_n(Z) \xrightarrow{\beta_n} T_n(g) \xrightarrow{\gamma_n} \tau_{n-1}(Y) \xrightarrow{\alpha_{n-1}} \ldots$$

is exact, where

$$\tau = \underline{\tau}\,, \qquad \alpha_n = \tau_n(g)\,, \qquad \beta_n = T_n\binom{\iota Y}{1_Z}, \qquad \gamma_n = \zeta_n(Y) \circ T_n\binom{1_Y}{iZ}.$$

Then the authors prove in [1] that, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, then the sequence

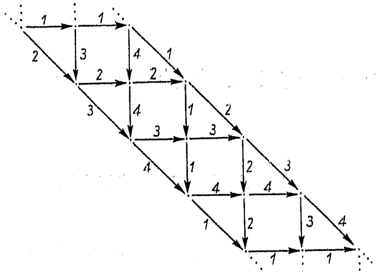
$$(1.2) \qquad \ldots \to T_n(f) \xrightarrow{\varphi_n} T_n(gf) \xrightarrow{\psi_n} T_n(g) \xrightarrow{\omega_n} T_{n-1}(f) \xrightarrow{\varphi_{n-1}} \ldots$$

is exact, where

$$\varphi_n = T_n\binom{1_X}{g}, \qquad \psi_n = T_n\binom{f}{1_Z}, \qquad \omega_n = T_{n-1}\binom{\iota X}{1_Y} \circ \gamma_n\,.$$

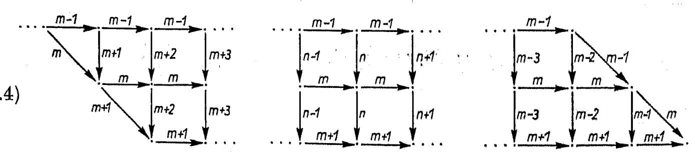
This theorem, generalizing a known result for homology and homotopy groups, was proved by invoking a certain lemma on exact sequences which may be stated as follows:

LEMMA. *Let four (doubly infinite) sequences, labelled* $1, 2, 3, 4$ *be given in the abelian category* $\mathcal{A}$ *which give rise to the commutative diagram* (1.3). *If three of the sequences are exact and if the fourth is differential where it appears horizontally or vertically in* (1.3), *then the fourth sequence is also exact.*

(1.3) 

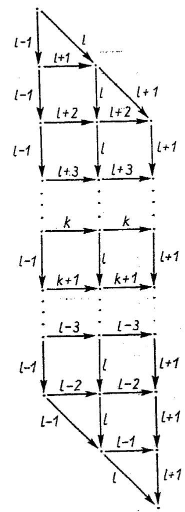
The lemma was proved in [1] by routine diagram-chasing. It was suggested by Mr. B. Wolk that this lemma should admit a generalization to a system of N sequences ($N \geqslant 3$); on the other hand, routine diagram-chasing could scarcely provide a proof in the general case and it is the object of this note to provide a proof of the general result.

We first state the result. We have a system of N interlocking sequences labelled $1, 2, \ldots, N$, giving rise to a commutative diagram. The system of interlocking is as in (1.3), except that now each sequence is written as $(N-2)$ horizontal arrows (instead of just two as in (1.3)) followed by a diagonal arrow, followed by $(N-2)$ vertical arrows, followed by a diagonal arrow, whereupon the pattern repeats. Thus a typical horizontal slice from the "leaning ladder" diagram is ([1])

(1.4) 

([1]) We have labelled the arrows simply by the label of the sequence to which the arrow belongs; we will adopt the same convention below in forming compositions of arrows.

and a typical vertical slice from the diagram is

(1.5) 

Of course, the horizontal and vertical slices have the same pattern.

We call the diagram a *leaning ladder diagram of* N *sequences* in $\mathcal{A}$ and prove

THEOREM 1.1. *Let* $(N-1)$ *of the sequences in a leaning ladder diagram of* N *sequences be exact. Then the remaining sequence is exact at every vertex at which it is differential.*

Note that the remaining sequence is certainly differential where it changes direction. Thus, for example, if sequence m is in question then at the vertex on the left of (1.4) where the sequence m becomes horizontal we have

$$m \circ m = m \circ (m+1) \circ (m-1) = (m+2) \circ (m-1) \circ (m-1) = 0 .$$

On the other hand, as we shall show in section 4 by an example, the remaining sequence may fail to be differential at a vertex at which it maintains direction.

Section 2 is devoted to the statement and proof of certain auxiliary results, and section 3 to the proof of Theorem 1.1.

2. Auxiliary results. Lambek [2] has introduced the notion of the *image* and *kernel* of a commutative square. Given a square in the abelian category $\mathcal{A}$,

$$\begin{tikzcd} {} \arrow[r, "\alpha"] \arrow[d, "\beta"'] \arrow[dr, phantom, "S"] & {} \arrow[d, "\varphi"] \\ {} \arrow[r, "\psi"'] & {} \end{tikzcd}$$

we set

$$\operatorname{Im} S = (\operatorname{Im}\varphi \cap \operatorname{Im}\psi)/\operatorname{Im}\varphi\alpha\,, \qquad \operatorname{Ker} S = \ker\varphi\alpha/(\ker\alpha + \ker\beta)\,.$$

Then Lambek proves

Proposition 2.1. *Let the commutative diagram*

$$\begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "S_1"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "S_2"] & {} \arrow[d] \\ {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd}$$

have exact rows. Then $\operatorname{Im} S_1 \cong \operatorname{Ker} S_2$.

Corollary 2.2. *Let the diagram*

$$\begin{tikzcd} & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "S_3"] & {} \arrow[d] \\ {} \arrow[r] \arrow[d] \arrow[dr, phantom, "S_1"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "S_2"] & {} \arrow[d] \\ {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd}$$

have exact rows and columns. Then $\operatorname{Im} S \cong \operatorname{Im} S_3$.

Corollary 2.3. *Let the diagram*

$$\begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "S_4"] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "S_3"] & {} \arrow[d] \\ {} \arrow[r] \arrow[d] \arrow[dr, phantom, "S_1"] & {} \arrow[r] \arrow[d] & {} \\ {} \arrow[r] & {} & \end{tikzcd}$$

have exact rows and columns. Then $\operatorname{Ker} S_1 \cong \operatorname{Ker} S_3$.

Now let

$$\begin{tikzcd} {} \arrow[d] \arrow[dr] & \\ {} \arrow[r] \arrow[u, phantom, "\Delta" near start, shift left=3] & {} \end{tikzcd}$$

Δ be a commutative triangle, then we may fashion out of Δ two commutative squares Δ', Δ'',

$$\begin{tikzcd} {} \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\Delta'"] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Delta''"] & {} \arrow[d, equal] \\ {} \arrow[r] & {} \end{tikzcd}$$

Proposition 2.4. $\operatorname{Im}\Delta' = 0$, $\operatorname{Ker}\Delta'' = 0$.

Corollary 2.5. *Given the commutative diagram*

$$\begin{tikzcd} {} \arrow[r, "\alpha_1"] \arrow[d] \arrow[dr, phantom, "S"] & {} \arrow[d] \arrow[dr, "\alpha_2"] & \\ {} \arrow[r, "\beta_1"'] & {} \arrow[r, "\beta_2"'] & {} \end{tikzcd}$$

in which the α *and* β *sequences are exact, then* $\operatorname{Im} S = 0$.

Corollary 2.6. *Given the commutative diagram*

$$\begin{tikzcd} {} \arrow[d, "\beta_1"'] \arrow[dr, "\alpha_1"] & \\ {} \arrow[r] \arrow[d, "\beta_2"'] \arrow[dr, phantom, "S"] & {} \arrow[d, "\alpha_2"] \\ {} \arrow[r] & {} \end{tikzcd}$$

in which the α *and* β *sequences are exact, then* $\operatorname{Ker} S = 0$.

We now prove

Proposition 2.7. *Given the commutative diagram*

$$\begin{tikzcd} {} \arrow[r, "\theta_1"] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "A"] & {} \arrow[r, "\theta_2"] \arrow[d, "\beta_1"] \arrow[dr, phantom, "B"] & {} \arrow[d, "\gamma_1"] \\ P \arrow[r, "\varphi_1"] \arrow[d, "\alpha_2"'] \arrow[dr, phantom, "C"] & Q \arrow[r, "\varphi_2"] \arrow[d, "\beta_2"] \arrow[dr, phantom, "D"] & R \arrow[d, "\gamma_2"] \\ {} \arrow[r, "\psi_1"'] & {} \arrow[r, "\psi_2"'] & {} \end{tikzcd}$$

where α *is exact at* P, γ *is exact at* R, φ *is exact at* Q, $\operatorname{Im} B = 0$, $\operatorname{Ker} C = 0$ *and* $\beta_2\beta_1 = 0$. *Then* β *is exact at* Q.

Proof. We appeal to the embedding theorem to allow us to refer to elements. Thus let $x \in Q$ with $\beta_2 x = 0$. Then $\gamma_2\varphi_2 x = 0$, so there exists y with $\varphi_2 x = \gamma_1 y$. Since $\operatorname{Im} B = 0$, there exists z with $\varphi_2 x = \varphi_2\beta_1 z$, so that there exists u with

$$x = \beta_1 z + \varphi_1 u\,.$$

Then $0 = \beta_2 x = \beta_2\beta_1 z + \beta_2\varphi_1 u = \beta_2\varphi_1 u$. Since $\operatorname{Ker} C = 0$, $u = u' + u''$ with $\varphi_1 u' = 0$, $\alpha_2 u'' = 0$, so that $u'' = \alpha_1 t$ for some t. Thus

$$x = \beta_1 z + \varphi_1 u = \beta_1 z + \varphi_1 u'' = \beta_1 z + \varphi_1\alpha_1 t = \beta_1 z + \beta_1\theta_1 t = \beta_1(z + \theta_1 t)\,,$$

and the proposition is proved.

Remarks. (i) Lambek introduced the notions of the image and kernel of a commutative square for more general categories than abelian categories. Propositions 2.1, 2.4 and 2.7 remain valid in the more general context (for example, in the category of groups).

(ii) It is a curious feature of Proposition 2.7 that, in order to prove

$$\ker\beta_2 \subseteq \operatorname{im}\beta_1\,,$$

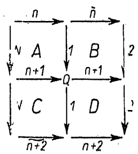
one must assume

$$\ker\beta_2 \supseteq \operatorname{im}\beta_1\,.$$

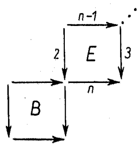
We will construct in section 4 an example in which the remaining hypotheses of Proposition 2.7 are satisfied but neither inclusion holds.

3. Proof of Theorem 1.1. We may assume without loss of generality that the sequences $2, 3, \ldots, N$ are exact and we study the exactness of sequence 1. We first consider a vertex Q at which the sequence does not change direction [2]; we may assume the sequence is vertical at this vertex so we have the diagram, for some n, $2 \leqslant n \leqslant N-2$,

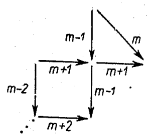
(3.1)



where $\tilde{n}$ belongs to sequence n if $n > 2$, and $\tilde{n}$ is the identity if $n = 2$; and where $\widetilde{n+2}$ belongs to sequence $(n+2)$ if $N-n > 2$, and $\widetilde{n+2}$ is the identity if $N-n = 2$. Thus if $n = 2$, $\operatorname{Im}B = 0$. Now assume $n > 2$; we may then proceed northeast from square B, obtaining a sequence of squares



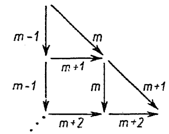
terminating in



if $\quad n = 2m-2\,,$

(2) This implies $N \geqslant 4$.

and in



if $\quad n = 2m-1\,.$

Thus by applying Corollary 2.6 and Proposition 2.1, or Corollary 2.5, and, successively, Corollary 2.2, we infer that $\operatorname{Im}B = 0$. (Notice that, until we reach square E we may infer that the image *and kernel* of each square are zero; but to proceed from E to B we must go south and then west since we do not know that sequence 1 is exact; that is, we must invoke Corollary 2.2 rather than Corollary 2.3.) Thus we infer in any case that, in (3.1),

(3.2) $$\operatorname{Im}B = 0\,.$$

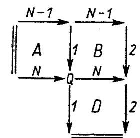
An entirely analogous argument, proceeding southwest from C (if $N-n > 2$), enables us to infer

(3.3) $$\operatorname{Ker}C = 0\,.$$

We now apply Proposition 2.7 to diagram (3.1) to infer that sequence 1 is exact at Q if it is differential at Q.

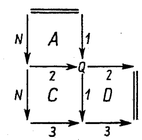
It remains to consider a turning point of sequence 1. Observe that, as we have already remarked, sequence 1 is certainly differential at such a vertex Q. Let us suppose that sequence 1 turns from the vertical to the diagonal at Q. Then we have the diagram

(3.4)



As before we infer that $\operatorname{Im}B = 0$. We now observe that we may complete (3.4) by assigning the zero object to the bottom left hand corner. This certainly gives a commutative square C since $1 \circ N = 2 \circ N \circ N = 0$. Moreover, plainly, $\operatorname{Ker}C = 0$. Thus we may again apply Proposition 2.7 to infer that sequence 1 is exact at Q. An entirely analogous argument, based on the diagram

(3.5)



establishes that sequence 1 is also exact where it turns from the diagonal to the vertical; and the other two cases of turning points of sequence 1 each yield diagrams isomorphic to one of (3.4), (3,5). This completes the proof of the theorem.

COROLLARY (of proof) 3.1. *In a leaning ladder diagram of N exact sequences, all the squares have zero kernel and zero image.*

Remarks. (i) Plainly the theorem and the corollary remain true in the context of the more general categories considered by Lambek (for example, the category of groups).

(ii) The case $N = 3$ admits the following interpretation. Let $\varphi = (\varphi_n)$, $\psi = (\psi_n)$ be two exact sequences and let $\alpha = (\alpha_n)$ be a morphism from φ to ψ, thus

(3.6)
$$\begin{tikzcd} \cdots \arrow[r] \arrow[d, "\alpha_{n-2}"] & \cdot \arrow[r, "\varphi_{n-1}"] \arrow[d, "\alpha_{n-1}"] & \cdot \arrow[r, "\varphi_n"] \arrow[d, "\alpha_n"] & \cdot \arrow[r, "\varphi_{n+1}"] \arrow[d, "\alpha_{n+1}"] & \cdots \\ \cdots \arrow[r, "\psi_{n-1}"'] & \cdot \arrow[r, "\psi_n"'] & \cdot \arrow[r, "\psi_{n+1}"'] & \cdot \arrow[r] & \cdots \end{tikzcd}$$

Suppose that α_n is an isomorphism if n is even. Then the sequence

(3.7)
$$\cdots \xrightarrow{\alpha_{2n-1}} \cdot \xrightarrow{\beta_{2n}} \cdot \xrightarrow{\alpha_{2n+1}} \cdot \xrightarrow{\beta_{2n+2}} \cdots$$

is exact, where

$$\beta_{2n} = \varphi_{2n+1} \alpha_{2n}^{-1} \psi_{2n} .$$

4. An example. Plainly for $N \neq 3$ every sequence turns at each vertex so no supplementary condition that the sequence in question be differential is required to prove that it is exact. But for $N = 4$ one sees by the following example of (1.3) that the supplementary condition is necessary. Let $\mathcal{A}$ be the category of abelian groups and consider the leaning ladder of four sequences

(4.1)
$$\begin{tikzcd} & 0 \arrow[d] \arrow[dr] & & & \\ 0 \arrow[r] \arrow[dr] & 0 \arrow[r] \arrow[d] & A \arrow[d] \arrow[dr] & & \\ & B \arrow[r] \arrow[dr] & C \arrow[r] \arrow[d] & A \arrow[d] \arrow[dr] & \\ & & B \arrow[r] \arrow[dr] & 0 \arrow[r] \arrow[d] & 0 \\ & & & 0 & \end{tikzcd}$$

All the remaining vertices of the diagram are occupied by zero groups: $A = Z_2 = (a)$; $B = Z_2 = (b)$; $C = Z_2 \oplus Z_2 = (c_1, c_2)$; sequence 2 contains the identity on A; sequence 4 contains the identity on B; sequence 3 contains

$$0 \to B \xrightarrow{\gamma} C \xrightarrow{\delta} A \to 0$$

where $\gamma(b) = c_1$, $\delta(c_1) = 0$, $\delta(c_2) = a$; and sequence 1 contains

$$0 \to A \xrightarrow{\alpha} C \xrightarrow{\beta} B \to 0$$

where $\alpha(a) = c_2$, $\beta(c_1) = b$, $\beta(c_2) = b$.

Then sequences 2, 3, 4 are exact, and all commutativity relations hold so that we have, in fact, a leaning ladder diagram. On the other hand sequence 1 fails to be differential; indeed, neither of the inclusions

$$\operatorname{Im} \alpha \subseteq \operatorname{Ker} \beta ,$$
$$\operatorname{Im} \alpha \supseteq \operatorname{Ker} \beta$$

holds. It is clear that we may adapt this example to provide a proof of the necessity of the differentiality condition at any vertex of a leaning ladder diagram of N sequences at which the sequence in question maintains direction.

Added in proof: C. T. C. Wall (*On the exactness of interlocking sequences*, L'Enseignement Mahtématique 12 (1966), pp. 95-100) also proves the Lemma proved in [1] and quoted in the Introduction, and provides an example, different from ours, to show the necessity of the differentiality condition. Wall's paper does not consider the generalization to more than 4 sequences which is the topic of this note. Wall attributes the 4-sequence diagram to Kervaire, presumably in connection with the study of homotopy spheres.

References

[1] B. Eckmann and P. J. Hilton, *Composition functors and spectral sequences*, Comm. Math. Helv. 41 (1966-67), pp. 187-221.

[2] J. Lambek, *Goursat's Theorem and homological algebra*, Can. Math. Bull. 7 (1964), pp. 597-608.

E. T. H., ZÜRICH, SCHWEIZ
and CORNELL UNIVERSITY, ITHACA, N. Y.

Reçu par la Rédaction le 7. 1. 1967