On the homotopy type of compact polyhedra

by

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Dedicated to Professor Karol Borsuk

Freyd [1] has recently given an example to show that we may not cancel summands in the stable homotopy category. More precisely he constructed an example of two (based) spaces X and Y such that

(1) $$X \vee S^9 \sim Y \vee S^9, \quad X \nsim Y,$$

where $\sim$ is the stable homotopy relation. This example was recalled in response to a question of Borsuk who asked about methods of constructing compact polyhedra with isomorphic homology and homotopy groups but different homotopy type. In this paper we generalize Freyd's example (1) and prove by entirely elementary argument that it has the properties attributed to it by Freyd. We then show that the example provides a ready answer to Borsuk's question. In order to demonstrate this we prove the following theorem.

THEOREM 1. *Let X, Y, A, B be compact connected polyhedra furnished with base points, such that*

$$X \vee A \simeq Y \vee B \quad \textit{and} \quad A \simeq B,$$

where $\simeq$ is the based homotopy relation. Then

$$\pi_i(\Sigma X) \cong \pi_i(\Sigma Y), \quad \textit{all } i,$$

where Σ denotes suspension.

This theorem may be regarded as a heuristic dual of the theorem that, for compact connected polyhedra, if $X \times A \simeq Y \times B$ and $A \simeq B$, then the homology groups of X and Y are isomorphic; this follows immediately from the Künneth formula. Our proof uses the so-called Hilton–Milnor formula [2] somewhat in the rôle of the Künneth formula. The Hilton–Milnor formula only enables us to discuss suspension spaces.

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(1) This generalization has been observed by Freyd himself. Another construction is to take the Stiefel manifold $V_{2n,2}$ $(n \neq 1, 2, 4)$ and compare it with $S^{2n-2} \times S^{2n-1}$ (Theorem 1.12 of [3]).

Thus we do not know whether the hypotheses of Theorem 1 enable us to deduce that the homotopy groups of X and Y are isomorphic even when the spaces concerned are 1-connected.

Now let $\alpha \in \pi_{m-1}(S^n)$ be a homotopy element in the stable range (i.e., $m \leqslant 2n-1$) of finite order $k \neq 1, 2, 3, 4, 6$; note that this implies $n \leqslant m-4$. We may then choose an integer $l > 0$ such that

$$(2) \qquad (k, l) = 1\,,$$

$$(3) \qquad l \neq \pm 1 \pmod{k}\,.$$

Set $\beta = l\alpha$, $X = S^n \cup_\alpha e^m$, $Y = S^n \cup_\beta e^m$. We prove

THEOREM 2. *$X \vee S^m \simeq Y \vee S^m$, $X \not\simeq Y$.*

Proof. The elements α and β are each multiples of the other, by (2). Thus the image of β in $\pi_{m-1}(X)$ is zero and the image of α in $\pi_{m-1}(Y)$ is zero, whence

$$S^n \cup_\alpha e^m \cup_\beta e^m \simeq (S^n \cup_\alpha e^m) \vee S^m\,,$$
$$S^n \cup_\beta e^m \cup_\alpha e^m \simeq (S^n \cup_\beta e^m) \vee S^m\,,$$

so that

$$X \vee S^m \simeq Y \vee S^m\,.$$

Now let $f\colon X \simeq Y$. Since $n \leqslant m-2$, we may assume that $fS^n \subset S^n$ and then $f|S^n$ is a map of degree ± 1 onto S^n. Then f maps the Puppe sequence of α to the Puppe sequence of β and we have in particular the commutative diagram

$$(4) \qquad \begin{tikzcd} S^m \arrow[r, "\Sigma\alpha"] \arrow[d, "\bar{f}"] & S^{n+1} \arrow[d, "\Sigma(f|S^n)"] \\ S^m \arrow[r, "\Sigma\beta"] & S^{n+1} \end{tikzcd}$$

where $\bar{f}$ is induced by f and has degree ± 1. Then (4) implies that $\Sigma\beta = \pm\Sigma\alpha$; but we are in the stable range so $\beta = \pm\alpha$, contradicting (3). Thus $X \not\simeq Y$; but an entirely analogous argument shows that $\Sigma^r X \not\simeq \Sigma^r Y$, for all $r \geqslant 0$, so that $X \not\simeq Y$.

THEOREM 3. *Let h be any (generalized) homology or cohomology theory whose coefficients are of finite type. Then under the hypotheses of Theorem 1*

$$h(X) \cong h(Y)\,.$$

Proof. Since $X \vee A \simeq Y \vee B$ and $A \simeq B$, we have

$$(5) \qquad h(X) \oplus h(A) \cong h(Y) \oplus h(B)\,, \qquad h(A) \cong h(B)\,.$$

By hypothesis the isomorphisms (5) are isomorphisms of graded abelian groups which are finitely generated in each degree. Thus Theorem 3 follows from the fundamental cancellation law for finitely generated abelian groups.

COROLLARY 1. *Under the hypotheses of Theorem 1, the ordinary homology groups, the stable homotopy groups and stable cohomotopy groups of X and Y coincide.*

Plainly there exist spaces X and Y of the type described in Theorem 2. For example, the stable $(2p-3)$ stem contains a Z_p summand for any prime p. Thus if $p \neq 2, 3$ we get an example with $m = n + 2p - 2$ and $n \geqslant 2p-2$. Freyd gives the example of $n = 5$, $m = 9$ and α of order 8 in $\pi_8(S^5)$, so that we may take $l = 3$ or $l = 5$. We remark that if X and Y yield an example so do ΣX and ΣY; so indeed do $\Sigma^{-1}X$ and $\Sigma^{-1}Y$ provided we remain in the stable range.

If we grant Theorem 1 we have immediately available spaces which provide an answer to the question of Borsuk. For if X and Y are two spaces satisfying the conditions of Theorem 2, then the homotopy groups of ΣX and ΣY are isomorphic by Theorem 1 and the homology groups of ΣX and ΣY are isomorphic by Theorem 3; but $\Sigma X \not\simeq \Sigma Y$ by Theorem 2. An explicit example, then, is furnished by

$$(6) \qquad \Sigma X = S^6 \cup_\alpha e^{10}\,, \qquad \Sigma Y = S^6 \cup_{3\alpha} e^{10}\,,$$

where $\alpha \in \pi_9(S^6)$ is of order 8.

It remains to prove Theorem 1 and the rest of this paper will be devoted to this purpose. We are working in the category of based compact connected polyhedra. Within this category there is an addition given by

$$(7) \qquad P + Q = P \vee Q$$

and a product defined by

$$(8) \qquad PQ = P \# Q = P \times Q / P \vee Q\,.$$

With respect to these operations we have the following laws:

$$(9) \qquad \begin{cases} P + Q = Q + P\,, & (P+Q)+R = P+(Q+R)\,, \\ PQ = QP\,, & (PQ)R = P(QR)\,, \\ P(Q+R) = PQ + PR\,, & \end{cases}$$

where the equality sign stands for natural topological equivalence. Moreover, the operations (7), (8) are compatible with the based homotopy relation: if $P_1 \simeq P_2$, $Q_1 \simeq Q_2$, then

$$(10) \qquad P_1 + Q_1 \simeq P_2 + Q_2\,, \qquad P_1 Q_1 \simeq P_2 Q_2\,.$$

Finally we have the connectivity relations

$$(11) \qquad \begin{aligned} \operatorname{conn}(P+Q) &= \min(\operatorname{conn} P, \operatorname{conn} Q)\,, \\ \operatorname{conn}(PQ) &\geqslant \operatorname{conn} P + \operatorname{conn} Q + 1\,. \end{aligned}$$

From (9) we infer the lemma,

LEMMA 1. *If* $X+A=K$, *then*

$$X^mA^n+\left(mK^{m-1}A^{n+1}+\binom{m}{3}K^{m-3}A^{n+3}+\ldots\right)=K^mA^n+\binom{m}{2}K^{m-2}A^{n+2}+\ldots$$

Proof. Intuitively we expand $(K-A)^m$, bring the negative terms over to the left and multiply through by A^n. This yields the formula of Lemma 1 but does not give a rigorous proof since we have no subtraction operation. However it is clear from the manner of obtaining the formula that if we substitute $X+A$ for K on both sides of the equality sign we will obtain an identity, using only the laws (9). We could, of course, obtain Lemma 1 also by arguing by induction on m.

With this preparation we return to the proof of Theorem 1. We choose a fixed but arbitrary positive integer i and seek to prove that $\pi_i(\Sigma X)\cong\pi_i(\Sigma Y)$. We observe that the conclusion certainly holds if $\operatorname{conn}A\geqslant i-1$, for then $\operatorname{conn}B\geqslant i-1$ and

$$\pi_i(\Sigma X)\cong\pi_i(\Sigma X\vee\Sigma A)\cong\pi_i(\Sigma Y\vee\Sigma B)\cong\pi_i(\Sigma Y).$$

Thus we may argue by downward induction on $\operatorname{conn}A=\operatorname{conn}B$. Now Lemma 1 may be condensed to

$$X^mA^n+P_{mn}(K,A)=Q_{mn}(K,A); \tag{12}$$

similarly if we write $Y+B=L$ then

$$Y^mB^n+P_{mn}(L,B)=Q_{mn}(L,B). \tag{13}$$

By (10) we have $P_{mn}(K,A)\simeq P_{mn}(L,B)$, $Q_{mn}(K,A)\simeq Q_{mn}(L,B)$. Moreover, it is plain from (11) that if we confine attention to $m,n\geqslant 1$, then

$$\operatorname{conn}P_{mn}(K,A)>\operatorname{conn}A. \tag{14}$$

Thus we may invoke the inductive hypothesis to infer

$$\pi_i(\Sigma X^mA^n)\cong\pi_i(\Sigma Y^mB^n),\quad m,n\geqslant 1. \tag{15}$$

We now invoke the Hilton–Milnor formula (Theorem 4 of [2]). According to this formula we have an expansion

$$\pi_i(\Sigma X\vee\Sigma A)\cong\pi_i(\Sigma X)\oplus\pi_i(\Sigma A)\oplus\bigoplus_j\pi_i(\Sigma T_j), \tag{16}$$

where each T_j has the form

$$T_j=X^{m_j}A^{n_j},\quad m_j,n_j\geqslant 1. \tag{17}$$

The number of factors of a given form is given by the Witt formula; but we need not stop to make this number explicit, since we propose to compare (16) with the corresponding expansion

$$\pi_i(\Sigma Y\vee\Sigma B)\cong\pi_i(\Sigma Y)\oplus\pi_i(\Sigma B)\oplus\bigoplus_j\pi_i(\Sigma U_j), \tag{18}$$

where each U_j has the form

$$U_j=Y^{m_j}B^{n_j},\quad m_j,n_j\geqslant 1. \tag{19}$$

Now $\Sigma X\vee\Sigma A$ is a 1-connected compact polyhedron so that its homotopy groups are finitely generated. Thus (16) is a direct sum decomposition of the finitely generated abelian group G,

$$G\cong\pi_i(\Sigma X)\oplus G_0,$$

and (18) is a direct sum decomposition

$$H\cong\pi_i(\Sigma Y)\oplus H_0.$$

But $G\cong H$ since $\Sigma X\vee\Sigma A\simeq\Sigma Y\vee\Sigma B$, and $G_0\cong H_0$ by (15) and the fact that $\Sigma A\simeq\Sigma B$. Thus the cancellation law for finitely generated abelian groups implies that

$$\pi_i(\Sigma X)\cong\pi_i(\Sigma Y),$$

and the theorem is proved.

Added later: A more refined argument enables us to show that $\pi_i(X)\simeq\pi_i(Y)$ for any X, Y constructed as in Theorem 2. Thus we see that Freyd's complexes

$$X=S^5\cup_\alpha e^9=S^5\cup_{3\alpha}e^9$$

have isomorphic homotopy groups but different homotopy types. The details are contained in a forthcoming paper, *On the Grothendieck ring of compact polyhedra.*

References

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[2] J. Milnor, *The construction FK*, Mimeographed notes (Princeton University).

[3] J. M. James and J. H. C. Whitehead, *The homotopy theory of sphere-bundles over spheres I*, Proc. London Math. Soc. 4 (1954), pp. 196-217.

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