On embedding decomposition spaces of E^n in E^{n+1}

by

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1. Introduction. The following question arises in the study of upper semi-continuous decompositions of E^n:

Is it true that if n is any positive integer and G is an upper semi-continuous decomposition of E^n into point-like compact continua, then the associated decomposition space can be embedded in E^{n+1}?

In [9], [10], Keldyš gives affirmative answers to special cases of this question. In order to state this result due to Keldyš and the main result of this paper, we first introduce some notation.

If n is any positive integer and G is a point-like decomposition of E^n, let H_G denote the union of all the non-degenerate elements of G and let P denote the projection map from E^n onto the associated decomposition space E^n/G.

Keldyš has proved the following theorem: If n is any positive integer and G is a point-like decomposition of E^n such that $P[H_G]$ is contained in a compact 0-dimensional set, then E^n/G can be embedded in E^{n+1}. In this paper, we shall extend this result by proving the following theorem:

If n is any positive integer, G is a point-like decomposition of E^n, and $P[H_G]$ is 0-dimensional, then E^n/G can be embedded in E^{n+1}.

The restriction, in the theorems above, to point-like decompositions of E^n is necessary if $n > 2$. In [4], Bing and Curtis gave an example of a monotone decomposition G of E^3 such that G has only nine non-degenerate elements, each non-degenerate element of G is a simple closed curve, and E^3/G cannot be embedded in E^4. Further, there is a well-known theorem of Hurewicz [8] which states that if X is any compact metric space, there is a *monotone* decomposition G of E^3 such that E^3/G contains a homeomorphic copy of X.

Curtis [6] has proved an embedding theorem for decomposition spaces of certain monotone decompositions of E^n; his result is applicable to some point-like decompositions of E^n.

The embedding theorem that we prove in this paper shows that the embedding of E^n/G into E^{n+1} may be realized as the final stage of a pseudo-isotopy φ from $E^{n+1}\times[0, 1]$ into E^{n+1} such that φ_0 is the identity map. Theorems 1 and 2 of [4] are analogous results for a restricted class of monotone decompositions of E^3.

The main result of this paper is proved in section 6. Sections 3, 4, and 5 contain lemmas to be used in the proof of the main result. In section 4, we apply the lemmas of section 3 to establish some results concerning the existence of homeomorphisms and isotopies having certain properties useful in the study of monotone decompositions of E^n. In addition to their use in establishing the main result, the results of section 4 are of independent interest as well.

2. Notation and terminology. Suppose that G is an upper semi-continuous decomposition of a topological space X. Then X/G denotes the associated decomposition space, P denotes the projection map from X onto X/H, and H_G denotes the union of all the non-degenerate elements of G.

If X is a metric space, the statement that G is a *monotone decomposition of* X means that G is an upper semi-continuous decomposition of X such that each element of G is a compact continuum.

If n is a positive integer and M is a subset of E^n, the statement that M is a point-like subset of E^n means that M is a compact continuum such that for any point p of E^n, $E^n - M$ is homeomorphic to $E^n - \{p\}$. The statement that G is a *point-like* decomposition of E^n means that G is an upper semi-continuous decomposition of E^n into point-like subsets of E^n.

If n is a positive integer and M is a subset of E^n, the statement that M is *cellular* in E^n means that there exists a sequence $C_1, C_2, \ldots$ of n-cells in E^n such that (1) if i is any positive integer, $C_{i+1} \subset \operatorname{Int} C_i$ and (2) $\bigcap_{i=1}^{\infty} C_i = M$. It is known that if M is any subset of E^n, then M is point-like in E^n if and only if M is cellular in E^n; see [13] for $n = 3$.

We use Bd and Cl to denote topological boundary and closure, respectively. The usual metric for E^n is denoted by d, and if $M \subset E^n$, diam M denotes the diameter of M. If $M \subset E^n$ and ε is any positive number, then $V(M, \varepsilon)$ denotes the open ε-neighborhood of M. If K is an n-cell, then $\operatorname{Int} K$ denotes the interior of K.

Throughout this paper, n denotes some definite positive integer.

3. Preliminary lemmas.

LEMMA 1. *Suppose that G is a monotone decomposition of E^n such that $P[H_G]$ is 0-dimensional, and $\mathfrak{U}$ is an open covering in E^n of H_G such that each set of $\mathfrak{U}$ is bounded and is a union of elements of G. Then there is an open covering $\mathfrak{V}$ in E^n of H_G such that*

(1) *the sets of $\mathfrak{V}$ are mutually disjoint,*

(2) *each set of $\mathfrak{V}$ lies in some set of $\mathfrak{U}$, and*

(3) *if $V \in \mathfrak{V}$, $\operatorname{Bd} V$ and H_G are disjoint.*

Proof. If $U \in \mathfrak{U}$, $P[U]$ is open in E^n/G. Since $P[H_G]$ is 0-dimensional, then if $x \in P[H_G]$, there is a set W_x open in E^n/G and such that (1) $x \in W_x$, (2) there is a set U of $\mathfrak{U}$ such that $W_x \subset P[U]$, and (3) Bd W_x and $P[H_G]$ are disjoint. Now $\{W_x\colon x \in P[H_G]\}$ covers $P[H_G]$ and there is a countable subset $\{W_1, W_2, \ldots\}$ of $\{W_x\colon x \in P[H_G]\}$ that covers $P[H_G]$. For each positive integer i, let Y_i denote $P^{-1}[W_i]$.

Now $\{Y_1, Y_2, \ldots\}$ is a countable covering of H_G by sets open in E^n such that for each positive integer i, (1) Y_i lies in some set of $\mathfrak{U}$ and (2) Bd Y_i and H_G are disjoint.

To see that (2) holds, suppose for some positive integer i, Bd Y_i intersects some non-degenerate element g of G. Since Y_i is a union of elements of G, g and Y_i are disjoint. Then g contains a limit point of Y_i. Hence the point g of E^n/G is a limit point in E^n/G of W_i, and is therefore a boundary point of W_i. Since $g \in P[H_G]$, this is a contradiction, and therefore (2) holds.

Let V_1 denote Y_1, and let V_2 denote

$$Y_2 - \operatorname{Cl} V_1 .$$

Suppose that k is a positive integer and that $V_1, V_2, \ldots, V_k$ are defined. Let V_{k+1} denote

$$Y_{k+1} - \operatorname{Cl}\Big[\bigcup_{i=1}^{k} V_i\Big] .$$

Then for each positive integer m, V_m is defined, and let $\mathfrak{V}$ denote the collection $\{V_1, V_2, \ldots\}$.

We shall show now that if i is any positive integer, $\operatorname{Bd} V_i \subset \bigcup_{j=1}^{i} \operatorname{Bd} Y_j$. Clearly $\operatorname{Bd} V_1 \subset \operatorname{Bd} Y_1$. Suppose that i is any positive integer and it is true that if m is any positive integer less than or equal to i, $\operatorname{Bd} V_m \subset \bigcup_{j=1}^{m} \operatorname{Bd} Y_j$. Since

$$V_{j+1} = Y_{i+1} - \operatorname{Cl}\Big[\bigcup_{j=1}^{i} V_j\Big] ,$$

it follows that

$$\operatorname{Bd} V_{i+1} \subset \operatorname{Bd} Y_{i+1} \cup \Big[\bigcup_{j=1}^{i} \operatorname{Bd} V_i\Big] .$$

By the inductive hypothesis, then,

$$\operatorname{Bd} V_{i+1} \subset \bigcup_{j=1}^{i+1} \operatorname{Bd} A_j .$$

Hence the desired result follows by induction.

Suppose now that i is any positive integer. It is clear that $V_i \subset Y_i$ and hence V_i lies in some set of $\mathfrak{U}$. Further, since for each positive integer j, Bd Y_j and H_G are disjoint, it follows from the results of the preceding paragraph that $\mathrm{Bd}\,V_i$ and H_G are disjoint. Further, $V_1, V_2, \ldots$ are mutually disjoint open subsets of E^n by construction.

To see that $\mathfrak{V}$ covers H_G, suppose that g is a non-degenerate element of G. Now g is contained in some one of $Y_1, Y_2, \ldots$, and let i be the least positive integer j such that $g \subset Y_i$. It is clear th t $g \subset V_i$, and hence $\mathfrak{V}$ covers H_G. Hence Lemma 1 is proved.

LEMMA 2. *Suppose that the hypothesis of Lemma* 1 *holds. Then there is a covering* $\mathfrak{D}$ *of* H_G *by sets open in* E^n *such that*

(1) *each set of* $\mathfrak{D}$ *lies in some set of* $\mathfrak{U}$,

(2) *the sets of* $\mathfrak{D}$ *are mutually disjoint,*

(3) *if* $D \in \mathfrak{D}$, $\mathrm{Bd}\,D$ *and* H_G *are disjoint, and*

(4) *if* $D \in \mathfrak{D}$, *there is a set* g *of* G *such that* $g \subset D$ *and* $D \subset V(g, \mathrm{diam}\, g)$.

Proof. Let $\{V_1, V_2, \ldots\}$ denote an open covering of H_G satisfying the conclusion of Lemma 1. Suppose that i is some positive integer such that V_i exists. $\bar{V}_i$ is compact, and $\mathrm{Bd}\,V_i$ and H_G are disjoint.

Let M_{i1} be the union of all the sets of G lying in V_i and having diameter greater than or equal to 1. By upper semi-continuity of G, M_{i1} is compact. If $g \in G$ and $g \subset M_{i1}$, there is an open set T_g in E^n such that (1) $g \subset T_g$ and $T_g \subset V_i$, (2) $\mathrm{Bd}\,T_g$ and H_G are disjoint, and (3) $T_g \subset V(g, \mathrm{diam}\, g)$. Since M_{i1} is compact and $\{T_g\colon\ g \in G \text{ and } g \subset M_{i1}\}$ covers M_{i1}, there exist finitely many distinct sets $g_{11}, g_{12}, \ldots, g_{1m_1}$ of G lying in M_{i1} such that $\{T_{g_{11}}, T_{g_{12}}, \ldots, T_{g_{1m_1}}\}$ covers M_{i1}. If $j = 1, 2, \ldots, m_1$, let T_{1j} denote $T_{g_{1j}}$.

The sets $g_{11}, g_{12}, \ldots, g_{1m_1}$ are mutually disjoint compact sets. There exist mutually disjoint open sets $L_{11}, L_{12}, \ldots, L_{1m_1}$ such that if $j = 1, 2, \ldots, m_1$, $g_{1j} \subset L_{1j}$, $L_{1j} \subset T_{1j}$, and $\mathrm{Bd}\,L_{1j}$ and H_G are disjoint. Define sets $X_{11}, X_{12}, \ldots$, and X_{1m_1} as follows:

$$X_{11} = T_{11} - (\mathrm{Cl}[\bigcup_{j=2}^{m_1} L_{1j}]) ,$$

$$X_{12} = [(T_{12} - \mathrm{Cl}\,X_{11}) - (\mathrm{Cl}[\bigcup_{j=3}^{m_1} L_{1j}])] \cup L_{12} ,$$

. . . .

$$X_{1,k+1} = [(T_{1,k+1} - \mathrm{Cl}[\bigcup_{j=1}^{k} X_{1j}]) - (\mathrm{Cl}[\bigcup_{j=k+2}^{m_1} L_{1j}])] \cup L_{1,k+1} ,$$

. . . .

$$X_{1m_1} = (T_{1m_1} - \mathrm{Cl}[\bigcup_{j=1}^{m_1-1} X_{1j}]) \cup L_{1m_1} .$$

The sets $X_{11}, X_{12}, \ldots, X_{1m_1}$ have the following properties:

(1) $X_{11}, X_{12}, \ldots, X_{1m_1}$ *are mutually disjoint open sets.*

(2) *If* $j = 1, 2, \ldots, m_1$, $X_{1j} \subset T_{1j}$, $g_{1j} \subset X_{1j}$, *and* $X_{1j} \subset V(g_{1j}, \mathrm{diam}\, g_{1j})$.

(3) *If* $j = 1, 2, \ldots, m_1$, $\mathrm{Bd}\,X_{1j}$ *and* H_G *are disjoint.*

(4) $\{X_{11}, X_{12}, \ldots, X_{1m_1}\}$ *covers* M_{i1}.

Properties (1) and (2) are easily established, using the definitions of $X_{11}, X_{12}, \ldots$, and X_{1m_1}.

Proof of (3). Since

$$\mathrm{Bd}\,X_{11} \subset (\mathrm{Bd}\,T_{11}) \cup (\bigcup_{r=1}^{m_1} \mathrm{Bd}\,L_{1r}) ,$$

it follows that $\mathrm{Bd}\,X_{11}$ and H_G are disjoint. If $j = 1, 2, \ldots, m_1-1$,

$$\mathrm{Bd}\,X_{1,j+1} \subset (\mathrm{Bd}\,T_{1,j+1}) \cup (\bigcup_{r=1}^{j} \mathrm{Bd}\,X_{1r}) \cup (\bigcup_{r=j}^{m_1} \mathrm{Bd}\,L_{1r}) ,$$

and an inductive proof shows that if $j = 1, 2, \ldots, m_1$, $\mathrm{Bd}\,X_{1j}$ and H_G are disjoint.

Proof of (4). Suppose that $g \in G$ and $g \subset M_{i1}$. Let j be the integer such that $g \subset T_{1j}$. Now if $k = 1, 2, \ldots, j-1$, $\mathrm{Bd}\,T_{1k}$ and g are disjoint, and hence g is disjoint from $\bigcup_{k=1}^{j-1} T_{1k}$.

Suppose that g is not contained in any one of $L_{11}, L_{12}, \ldots, L_{1m_1}$. Then by an argument similar to one used in the preceding paragraph, g is disjoint from $\bigcup_{k=1}^{m_1} L_{1k}$. Now if $k = 1, 2, \ldots, j-1$, $\mathrm{Cl}\,X_{1k} \subset \mathrm{Cl}\,T_{1k}$, and hence

$$g \subset T_{1j} - (\mathrm{Cl}[\bigcup_{k=1}^{j-1} X_{1k}]) - (\mathrm{Cl}[\bigcup_{k=j+1}^{m_1} L_{1k}]) .$$

But this implies that $g \subset X_{1j}$.

Clearly if $k = 1, 2, \ldots, m_1$ and $g \subset L_{1k}$, then $g \subset X_{1k}$.

Suppose now that $j = 1$. It is necessary to consider only the case where g and $\bigcup_{k=1}^{m_1} L_{1k}$ are disjoint. In that case,

$$g \subset T_{11} - (\mathrm{Cl}[\bigcup_{k=1}^{m_1} L_{1k}]);$$

it follows that $g \subset X_{11}$. This establishes property (4).

Now $\bigcup_{k=1}^{m_1} X_{1k}$ is open, and $\mathrm{Bd}(\bigcup_{k=1}^{m_1} X_{1k})$ and H_G are disjoint. Let V_{i1} denote

$$V_i - \mathrm{Cl}[\bigcup_{k=1}^{m_1} X_{1k}] .$$

Then V_{i1} is open, lies in V_i, and has the property that $\mathrm{Bd}\,V_{i1}$ and H_G are disjoint. In addition if $g \in G$ and $g \subset V_{i1}$, $(\mathrm{diam}\, g) < 1$.

Let M_{i2} denote

$$\bigcup \{g = g \in G,\ g \subset V_{i1}\ ,\ \textit{and}\ (\mathrm{diam}\, g) \geqslant 1/2\} .$$

There exist mutually disjoint open sets $X_{21}, X_{22}, \ldots, X_{2m_2}$ having properties, relative to M_{i2} and V_{i1}, analogous to properties (1), (2), (3), and (4) stated above for $X_{11}, X_{12}, \ldots, X_{1m_1}$.

Let this process be continued; either it terminates after finitely many steps or it continues idenfinitely. Let $\mathfrak{D}_i$ denote the countable collection

$$\{X_{11}, X_{12}, \ldots, X_{1m_1}, X_{21}, X_{22}, \ldots, X_{2m_2}, \ldots\};$$

$\mathfrak{D}_i$ is a collection of mutually disjoint open subsets of E^n covering $V_i \cap H_G$ and such that

(1) each set of $\mathfrak{D}_i$ lies in V_i,

(2) if $D \in \mathfrak{D}_i \mathrm{Bd} D$ and H_G are disjoint, and

(3) if $D \in \mathfrak{D}_i$, there is a set g of G such that $g \subset D$ and $D \subset V(g, \operatorname{diam} g)$.

Let $\mathfrak{D}$ denote $\bigcup_{i=1}^{\infty} \mathfrak{D}_i$. It may be shown that $\mathfrak{D}$ satisfies the conclusion of Lemma 2.

Suppose that $\mathcal{A}$ is a collection of subsets of a metric space. The statement that $\mathcal{A}$ is a *null collection* means that if ε is any positive number, there exist at most finitely many sets of $\mathcal{A}$ having diameters greater than ε.

LEMMA 3. *Suppose that the hypothesis of Lemma 1 holds and that, in addition, if B is any bounded subset of E^n,*

$$\bigcup \{U\colon\ U \in \mathfrak{U} \text{ and } U \text{ intersects } B\}$$

is bounded. Then the collection $\mathfrak{D}$ constructed in the proof of Lemma 2 has the following property: If B is any bounded subset of E^n,

$$\{D\colon\ D \in \mathfrak{D} \text{ and } D \text{ intersects } B\}$$

is a null collection.

Proof. Suppose that there is a bounded subset B of E^n such that

$$\{D\colon\ D \in \mathfrak{D} \text{ and } D \text{ intersects } B\}$$

is not a null collection. Then there is some positive number ε and a sequence $D_1, D_2, \ldots$ of distinct sets of $\mathfrak{D}$ such that for each positive integer i, $\operatorname{diam} D_i \geqslant \varepsilon$. Since

$$\bigcup \{U\colon\ U \in \mathfrak{U} \text{ and } U \text{ intersects } B\}$$

is bounded, $\bigcup_{i=1}^{\infty} D_i$ is bounded.

For each positive integer i, there is a set g_i of G such that $g_i \subset D_i$ and $D_i \subset V(g_i, \operatorname{diam} g_i)$. By [12], Chapter I, Theorem 59, the sequence $g_1, g_2, \ldots$ has a convergent subsequence $g_{n_1}, g_{n_2}, \ldots$. For each positive integer i, $(\operatorname{diam} g_{n_i}) \geqslant \varepsilon/4$. This may be proved since for each positive integer i, $(\operatorname{diam} D_{n_i}) \geqslant \varepsilon$ and $D_{n_i} \subset V(g_{n_i}, \operatorname{diam} g_{n_i})$.

It follows, by upper semi-continuity of G, that $g_{n_1}, g_{n_2}, \ldots$ converges to a subset of a non-degenerate element g_0 of G. Since g_0 is non-degenerate, there is a set D_0 of $\mathfrak{D}$ such that $g_0 \subset D_0$. Since there is at most one positive integer k such that $D_0 = D_k$, it follows that D_0 contains at most one of the sets $g_{n_1}, g_{n_2}, \ldots$ This is contrary to the fact that $g_{n_1}, g_{n_2}, \ldots$ converges to g_0. This contradiction establishes Lemma 3.

4. Results on homeomorphisms and isotopies. In this section we shall use the results of section 3 to construct homeomorphisms and isotopies having certain properties. We shall establish, in Theorems 1 and 2, the equivalence of certain pairs of conditions that are useful in the study of decompositions.

LEMMA 4. *Suppose that $\{V_1, V_2, \ldots\}$ is a countable collection of mutually disjoint bounded open sets in E^n such that if B is any bounded subset of E^n, $\{V_i\colon\ i$ is a positive integer and V_i intersects $B\}$ is a null collection. Suppose that h_0 is a homeomorphism from E^n into E^n and for each positive integer i, h_i is a homeomorphism from $\overline{V}_i$ into $h_0[\overline{V}_i]$ such that $h_i|\mathrm{Bd} V_i = h_0|\mathrm{Bd} V_i$. Let f be the function with domain E^n and such that*

(1) *if $x \notin \bigcup_{i=1}^{\infty} V_i$, $f(x) = h_0(x)$, and*

(2) *if i is a positive integer and $x \in V_i$, then $f(x) = h_i(x)$.*

Then f is a homeomorphism from E^n into E^n. If h_0 is onto E^n and for each positive integer i, h_i is onto $h_0[\overline{V}_i]$, then f is onto E^n.

Proof. It is clear that f is well-defined, from E^n into E^n, and one-to-one. We shall show now that f is continuous. Let V_0 denote $E^n - (\mathrm{Cl} \bigcup_{i=1}^{\infty} V_i)$.

Suppose that $x \in E^n$. If there is a non-negative integer i such that $x \in V_i$, then since $f|V_i = h_i|V_i$, it is clear that f is continuous at x. Suppose then that $x \in \mathrm{Cl} \bigcup_{i=0}^{\infty} \mathrm{Bd} V_i$ and that $x_1, x_2, \ldots$ is a sequence of points of $E^n - \{x\}$ converging to x. If there exists a finite subset $\{V_{i_1}, V_{i_2}, \ldots, V_{i_m}\}$ of $\{V_0, V_1, V_2, \ldots\}$ such that for each positive integer j, x_j belongs to one of $\overline{V}_{i_1}, \overline{V}_{i_2}, \ldots, \overline{V}_{i_m}$, then it is easy to see that $x \in \bigcap_{k=1}^{m} \mathrm{Bd} V_{i_k}$ and that $f(x_1), f(x_2), \ldots$ converges to $f(x)$. Now suppose that there exist infinitely many distinct sets $V_{i_1}, V_{i_2}, \ldots$ and a subsequence $x_{j_1}, x_{j_2}, \ldots$ of $x_1, x_2, \ldots$ such that for each positive integer k, $x_{j_k} \in \overline{V}_{i_k}$. Suppose that U is any bounded neighborhood of $f(x)$.

Now $\{V_i\colon\ i$ *is a positive integer and* V_i *intersects* $h^{-1}[U]\}$ is a null collection. Since $x_1, x_2, \ldots$ converges to x, then all but finitely many of $V_{i_1}, V_{i_2}, \ldots$ intersect $h^{-1}[U]$, and hence $\{V_{i_1}, V_{i_2}, \ldots\}$ is a null collection.

It follows that all but finitely many of $h_0[\bar{V}_{i_1}]$, $h_0[\bar{V}_{i_2}]$, ... lie in U. It is easy now to show that $f(x_1), f(x_2), \ldots$ converges to $f(x)$. Hence f is continuous at x.

For each non-negative integer j, $h_j[V_j]$ is an open subset of E^n. Further, if B is any bounded subset of E^n, $\{h_i[V_i]$: *i is a positive integer and* $h_i[V_i]$ *intersects* $B\}$ is a null collection. Hence an argument similar to that used to show that f is continuous may be used to show that f^{-1} is continuous. Therefore f is a homeomorphism. It is clear that if h_0 is onto E^n and for each positive integer i, h_i is onto $h_0[\bar{V}_i]$, then f is onto E^n.

We shall now apply Lemmas 3 and 4 to establish the equivalence, for a certain class of decompositions, of two conditions which insure the existence of homeomorphisms that shrink certain sets to small size.

THEOREM 1. *Suppose that G is a monotone decomposition of E^n such that $P[H_G]$ is 0-dimensional. Then the following two statements are equivalent*:

(1) *If U is any open set containing H_G and ε is any positive number, there exists a homeomorphism h from E^n onto E^n such that*

(a) *if $x \in E^n - U$, $h(x) = x$, and*

(b) *if $g \in G$, $(\operatorname{diam} h[g]) < \varepsilon$.*

(2) *If U is any open set containing H_G, ε is any positive number, and f is any homeomorphism from E^n onto E^n, then there exists a homeomorphism h from E^n onto E^n such that*

(a) *if $x \in E^n - U$, $h(x) = f(x)$, and*

(b) *if $g \in G$, $(\operatorname{diam} h[g]) < \varepsilon$.*

Proof. It is clear that (2) implies (1). To show that (1) implies (2), let U be an open set containing H_G, let ε be a positive number, and let f be a homeomorphism from E^n onto E^n.

If $g \in G$, let γ_g be $\min\{1, \operatorname{diam} g\}$. If $g \in G$, there is an open subset W_g of E^n such that W_g is a union of sets of G, $g \subset W_g$, $W_g \subset U$, and $W_g \subset V(g, \gamma_g)$. Let $\mathfrak{W}$ be $\{W_g\colon g \in G\}$. It is clear that $\mathfrak{W}$ is an open covering in E^n of H_G such that (1) each set of $\mathfrak{W}$ is bounded and is a union of sets of G, and (2) $(\bigcup\{W\colon W \in \mathfrak{W}\}) \subset U$. We shall show that if B is any bounded subset of E^n,

$$\bigcup\{W\colon W \in \mathfrak{W} \text{ and } W \text{ intersects } B\}$$

is bounded.

Suppose that B is a bounded subset of E^n. Let B' denote $V(B, 2)$. It follows from the way in which the sets of $\mathfrak{W}$ are constructed that if $g \in G$ and W_g intersects B, then g intersects B'. Now

$$\bigcup\{g\colon g \text{ intersects } \operatorname{Cl} B'\}$$

is a compact set ([12], Chapter V, Theorem 2). It follows that

$$\bigcup\{W\colon W \in \mathfrak{W} \text{ and } W \text{ intersects } B\} \subset \bigcup\{W_g\colon g \in G \text{ and } g \text{ intersects } B'\};$$

further,

$$(\bigcup\{W_g\colon g \in G \text{ and } g \text{ intersects } B'\}) \subset V([\bigcup\{g\colon g \in G \text{ and } g \text{ intersects } B'\}], 1).$$

Hence $\bigcup\{W\colon W \in \mathfrak{W}$ *and* W *intersects* $B\}$ is bounded.

By Lemma 3, there is a covering $\mathfrak{D}$ of H_G by bounded open subsets of E^n such that

(1) each set of $\mathfrak{D}$ lies in some set of $\mathfrak{W}$,

(2) the sets of $\mathfrak{D}$ are mutually disjoint,

(3) if B is any bounded subset of E^n, then

$$\{D\colon D \in \mathfrak{D} \text{ and } D \text{ intersects } B\}$$

is a null collection, and

(4) if $D \in \mathfrak{D}$, $\operatorname{Bd} D$ and H_G are disjoint.

Let $D_1, D_2, \ldots$ denote the distinct sets of $\mathfrak{D}$ and let V denote $\bigcup_{i=1}^{\infty} D_i$.

Suppose that i is a positive integer. Since f is a homeomorphism and D_i is bounded, then $f|\bar{D}_i$ is uniformly continuous. Thus there exists a positive number δ_i such that if X is any subset of $\bar{D}_i$ such that $\operatorname{diam} X < \delta_i$, then $\operatorname{diam} f[X] < \varepsilon$. By hypothesis, there exists a homeomorphism k_i from E^n onto E^n such that

(1) if $x \in E^n - V$, then $k_i(x) = x$, and

(2) if $g \in G$, then $(\operatorname{diam} k_i[g]) < \delta_i$.

Let h_i denote $(f \circ k_i)|\bar{D}_i$.

For each positive integer i, h_i is a homeomorphism from $\bar{D}_i$ onto $f[\bar{D}_i]$ such that $h_i|\operatorname{Bd} D_i = f|\operatorname{Bd} D_i$. Let h be the function with domain E^n and such that

(1) if $x \in E^n - V$, $h(x) = f(x)$, and

(2) if i is a positive integer and $x \in D_i$, then $h(x) = h_i(x)$.

By Lemma 4, h is a homeomorphism from E^n onto E^n. Since $V \subset U$, it is clear that if $x \in E^n - U$, $h(x) = f(x)$. Now suppose that g is any non-degenerate element of G. By construction, there is a positive integer i such that $g \subset D_i$. Then $(\operatorname{diam} k_i[g]) < \delta_i$ and hence $(\operatorname{diam} h_i[g]) < \varepsilon$. Therefore $(\operatorname{diam} h[g]) < \varepsilon$. Hence (1) implies (2), and Theorem 1 is proved.

It is known that if G is a point-like decomposition of E^3 such that $P[H_G]$ is either a countable set or a compact 0-dimensional set, then E^3/G is homeomorphic to E^3 if and only if condition (1) of Theorem 1 is satisfied ([1], [2]).

Suppose that H is a homotopy from $E^n \times [0, 1]$ into E^n. If $t \in [0, 1]$, then H_t denotes the function from E^n into E^n such that if $x \in E^n$, $H_t(x) = H(x, t)$. The statement that φ is an *isotopy* from $E^n \times [0, 1]$ into E^n means that φ is a homotopy from $E^n\times[0,1]$ into E^n such that if $t\in[0,1]$, φ_t is a homeomorphism.

By using methods similar to those used in the proofs of Lemma 4 and Theorem 1, the lemma and theorem below may be proved.

LEMMA 5. *Suppose that $\{V_1, V_2, \ldots\}$ is a countable collection of mutually disjoint bounded open sets in E^n such that if B is any bounded subset of E^n,*

$$\{V_i\colon i \text{ is a positive integer and } V_i \text{ intersects } B\}$$

is a null collection. Suppose that h is a homeomorphism from E^n into E^n, and for each positive integer i, h^i is an isotopy from $\overline{V}_i\times[0,1]$ into $h[\overline{V}_i]$ such that if $t\in[0,1]$, $h^i_t|\mathrm{Bd}V_i = h|\mathrm{Bd}V_i$. Suppose that φ is the function with domain $E^n\times[0,1]$ and such that

(1) *if $x\in E^n-\bigcup_{i=1}^{\infty} V_i$ and $t\in[0,1]$, $\varphi(x,t)=h(x)$, and*

(2) *if i is a positive integer, $x\in V_i$, and $t\in[0,1]$, then $\varphi(x,t)=h^i_t(x)$.*

Then φ is an isotopy from $E^n\times[0,1]$ into E^n. If h is onto E^n and for each positive integer i and each number t of $[0,1]$, h^i_t is onto $h[\overline{V}_i]$, then for each number t of $[0,1]$, φ_t is onto E^n.

THEOREM 2. *Suppose that G is a monotone decomposition of E^n such that $P[H_G]$ is 0-dimensional. Then the following two statements are equivalent:*

(1) *If U is any open set containing H_G and ε is any positive number, there is an isotopy φ from $E^n\times[0,1]$ into E^n such that*

(a) *φ_0 is the identity from E^n onto E^n,*

(b) *if $t\in[0,1]$, φ_t is onto E^n,*

(c) *if $x\in E^n-U$ and $t\in[0,1]$, $\varphi_t(x)=x$, and*

(d) *if $g\in G$, $(\mathrm{diam}\,\varphi_1[g])<\varepsilon$.*

(2) *If U is any open set containing H_G, ε is any positive number, and f is any homeomorphism from E^n onto E^n, there exists an isotopy φ from $E^n\times[0,1]$ into E^n such that*

(a) *$\varphi_0=f$,*

(b) *if $t\in[0,1]$, φ_t is onto E^n,*

(c) *if $x\in E^n-U$ and $t\in[0,1]$, $\varphi_t(x)=f(x)$, and*

(d) *if $g\in G$, $(\mathrm{diam}\,\varphi_1[g])<\varepsilon$.*

5. Construction of isotopies. This section is devoted to the construction of isotopies to be used in establishing the embedding theorem of section 6. We first introduce some additional notation.

Throughout the remainder of this paper, we shall regard E^n as a subset of E^{n+1}. If x and y are two distinct points of some Euclidean space, $\langle xy\rangle$ denotes the closed straight segment from x to y. If M is any subset of E^n and p is a point of $E^{n+1}-E^n$, then $C(p,M)$ denotes the cone from p over M; hence

$$C(p,M)=\bigcup\{\langle px\rangle\colon x\in M\}.$$

We shall use E^{n+1}_+ to denote the set of all points of E^{n+1} whose $(n+1)$st coordinates are positive, and E^{n+1}_- to denote the set of all points of E^{n+1} whose $(n+1)$st coordinates are negative.

In the construction of isotopies to be used in section 6, we first construct isotopies on bounded open sets whose boundaries are disjoint from H_G and which shrink the non-degenerate elements of G to small size.

LEMMA 6. *Suppose that G is a monotone decomposition of E^n such that $P[H_G]$ is 0-dimensional, V is a bounded open subset of E^n such that $\mathrm{Bd}V$ and H_G are disjoint, $p\in E^{n+1}_+$, $q\in E^{n+1}_-$, and δ is a positive number. Then there exists an isotopy φ from $E^{n+1}\times[0,1]$ into E^{n+1} such that*

(1) *φ_0 is the identity,*

(2) *if $t\in[0,1]$, φ_t is onto E^{n+1},*

(3) *if $t\in[0,1]$ and $x\in E^{n+1}-[C(p,V)\cup C(q,V)]$, then $\varphi_t(x)=x$, and*

(4) *if $g\in G$ and $g\subset V$, then $(\mathrm{diam}\,\varphi_1[g])<\delta$.*

Proof. Let K denote

$$\bigcup\{g\colon g\in G,\ g\subset V,\ \text{and } (\mathrm{diam}\,g)\geqslant\delta/4\}.$$

K is compact since G is upper semi-continuous, V is bounded, and $\mathrm{Bd}V$ and H_G are disjoint.

Now we shall show that there is a sequence $U_1, U_2, \ldots$ of open subsets of V such that

(1) $K\subset U_1$, and

(2) if i is any positive integer, $\overline{U}_i\subset U_{i+1}$, and $\mathrm{Bd}\,U_i$ and H_G are disjoint.

Since $\mathrm{Bd}V$ and H_G are disjoint, $P[V]$ is open in E^n/G. Further, $P[K]$ is a closed set contained in $P[V]$. Since $P[H_G]$ is 0-dimensional, there exists, by [9], p. 15, an open set W_1 in E^n/G such that $P[K]\subset W_1$, $\overline{W}_1\subset P[V]$, and the boundary in E^n/G of W_1 is disjoint from $P[H_G]$. Let U_1 denote $P^{-1}[W_1]$; U_1 is an open subset of V such that $K\subset U_1$, $\overline{U}_1\subset V$, and $\mathrm{Bd}\,U_1$ and H_G are disjoint.

By an analogous argument, it may be shown that there is an open subset U_2 of V such that $\overline{U}_1\subset\overline{U}_2$, $U_2\subset V$, and $\mathrm{Bd}\,U_2$ and H_G are disjoint. By a continuation of this process, there can be constructed a sequence $U_1, U_2, \ldots$ of open subsets of V having the properties stated previously.

Let r be a point of $\overline{V}$ such that if $x\in\overline{V}$, $d(x,p)\leqslant d(r,p)$. There is a finite set

$$\{\pi_0, \pi_1, \pi_2, \ldots, \pi_m\}$$

of n-hyperplanes of E^{n+1}, each parallel to E^n, such that

(1) $p \in \pi_0$, $\pi_m = E^n$, and if $i = 1, 2, \ldots, m$, π_i is below π_{i-1},

(2) $(\operatorname{diam}[C(p, \overline{V}) \cap \pi_1]) < \delta$, and

(3) if when $i = 0, 1, 2, \ldots, m$, r_i is the point common to $\langle pr\rangle$ and π_i, then if $i = 1, 2, \ldots, m$, $(\operatorname{diam}\langle r_{i-1}r_i\rangle) < \delta/4$.

If $i = 1, 2, \ldots, m$, let S_i denote the set of all points of E^{n+1} that are either on π_i, on π_{i-1}, or between π_{i-1} and π_i.

There exists a homeomorphism f from $\overline{V}$ into $C(p, \overline{V})$ such that

(1) if $x \in V$, $f(x)$ is an interior point of $\langle px\rangle$,

(2) if $x \in \operatorname{Bd} V$, $f(x) = x$,

(3) $f[\overline{U}_1] \subset \pi_1$,

(4) if $i = 1, 2, \ldots, m$, $f[\operatorname{Bd} U_i] \subset \pi_i$,

(5) if $i = 2, 3, \ldots, m$,

$$f[U_i - \overline{U}_{i-1}] \subset S_i \cap C(p, V),$$

and (6) if $x \in V - \overline{U}_m$, then $f(x) = x$.

Such a homeomorphism may be constructed using a coordinate system for $C(p, \overline{V}) - \{p\}$ which we shall describe now. Suppose that if $x \in E^{n+1}$, its usual coordinates are denoted by $(x_1, x_2, \ldots, x_n, x_{n+1})$. If $x \in C(p, \overline{V}) - \{p\}$, let $(y_1, y_2, \ldots, y_n, y_{n+1})$ denote coordinates for x obtained as follows: Let x' denote the point of $\overline{V}$ such that $x \in \langle px'\rangle$. Then let $y_1, y_2, \ldots, y_n$ be $x_1', x_2', \ldots, x_n'$, respectively, and let y_{n+1} be x_{n+1}.

To construct f, first define f on $\operatorname{Bd} V$, $\overline{U}_1$, $V - \overline{U}_m$, and each of $\operatorname{Bd} U_2, \operatorname{Bd} U_3, \ldots, \operatorname{Bd} U_m$ so that (1), (2), (3), (4), and (6) above are satisfied. By using the coordinate system described above and Urysohn's lemma, it is easy to define f on the remaining part of $\overline{V}$ so that (1) and (5) above are satisfied.

Now it is easy to construct an isotopy φ satisfying the conditions of the conclusion of Lemma 6. We define φ so that the following conditions hold:

(1) φ_0 is the identity.

(2) If $x \in E^{n+1} - [C(p, V) \cup C(q, V)]$ and $t \in [0, 1]$, $\varphi_t(x) = x$.

(3) If $x \in \overline{V}$, $\varphi_1(x) = f(x)$.

(4) If $x \in \overline{V}$ and $t \in [0, 1]$, then

(a) if $y \in \langle px\rangle$, $\varphi_t(y) \in \langle py\rangle$, and

(b) if $y \in \langle qx\rangle$, $\varphi_t(y) \in \langle qx\rangle \cup \langle xf(x)\rangle$.

(5) If $x \in V$ and $y \in \langle px\rangle$, $\varphi_1(y) \in \langle pf(x)\rangle$.

It is clear that conditions (1), (2), and (3) of the conclusion of Lemma 6 hold. Now suppose that $g \in G$ and $g \subset V$. If $g \subset U_1$, then since $\varphi_1[U_1] \subset \pi_1$ and

$$\big(\operatorname{diam}[\pi_1 \cap C(p, \overline{V})]\big) < \delta,$$

it follows that $(\operatorname{diam}\varphi_1[g]) < \delta$.

Suppose that $i = 2, 3, \ldots, m$, and $g \subset (U_i - \overline{U}_{i-1})$. Then $(\operatorname{diam} g) < \delta/4$ by definition of $K, U_1, U_2, \ldots$. Suppose that x and y are any two points of g. Let x' and y' be the points of $\pi_{i-1} \cap \langle px\rangle$ and $\pi_{i-1} \cap \langle py\rangle$, respectively. It is clear that

$$d(x', y') \leqslant d(x, y).$$

Now $\varphi_1(x)$ and $\varphi_1(y)$ lie in S_i, and it follows from the construction of the hyperplanes $\pi_0, \pi_1, \ldots, \pi_m$ that

$$d\big(x', \varphi_1(x)\big) < \delta/4 \quad \text{and} \quad d\big(y', \varphi_1(y)\big) < \delta/4 .$$

Hence $d\big(\varphi_1(x), \varphi_1(y)\big) < 3\delta/4$, and it follows that

$$(\operatorname{diam}\varphi_1[g]) < \delta .$$

If $g \subset V -_m \overline{U}$, then $(\operatorname{diam} g) < \delta/4$, $\varphi_1|g$ is the identity, and hence $(\operatorname{diam}\varphi_1[g]) < \delta$.

By construction of $U_1, U_2, \ldots, U_m$, if $g \in G$ and $g \subset V$, then either $g \subset U_1$, $g \subset V - \overline{U}_m$, or there is an integer i such that $i = 2, 3, \ldots, m$ and $g \subset (U_i - \overline{U}_{i-1})$. Hence if $g \in G$ and $g \subset V$, $(\operatorname{diam}\varphi_1[g]) < \delta$. Therefore condition (4) of the conclusion of Lemma 6 holds, and Lemma 6 is proved.

LEMMA 7. *Suppose that G is a monotone decomposition of E^n, U is a bounded open subset of E^{n+1}, V is an open subset of E^n such that $\overline{V} \subset U$ and $\operatorname{Bd} V$ and H_G are disjoint, and K is an n-cell lying in U and such that $\overline{V} \subset \operatorname{Int} K$. If δ is a positive number, there exist an isotopy φ from $E^{n+1} \times [0, 1]$ into E^{n+1} and a compact set S such that*

(1) *φ_0 is the identity,*

(2) *if $t \in [0, 1]$, φ_t is onto E^{n+1},*

(3) *$S \subset U$, $S \cap E^n = \overline{V}$, and if $x \in E^{n+1} - S$ and $t \in [0, 1]$, $\varphi_t(x) = x$, and*

(4) *if $g \in G$ and $g \subset V$, then*

$$(\operatorname{diam}\varphi_1[g]) < \delta .$$

Proof. There is an n-cell M such that $M \subset \operatorname{Int} K$, $\overline{V} \subset \operatorname{Int} M$, and $\operatorname{Bd} M$ is bi-collared (or, has a cartesian product neighborhood). By [5], there is a homeomorphism h from E^n onto E^n such that $h[M]$ is B, where

$$B = \{x\colon\ x \in E^n \text{ and } d(x, 0) \leqslant 1\} .$$

Let k be the extension of h to E^{n+1} defined as follows: If $x \in E^n$ and $s \in E^1$, then $k(x, s) = \big(h(x), s\big)$.

Now $k[U]$ is open in E^{n+1} and B is a compact subset of $k[U]$. There is a positive number b such that $(B \times [-b, b]) \subset k[U]$. Note that $B \times [-b, b]$

is a convex $(n+1)$-cell B' containing $k[\overline{V}]$. Let p and q be points of $E_+^{n+1} \cap (\operatorname{Int} B')$ and $E_-^{n+1} \cap (\operatorname{Int} B')$, respectively. Then

$$[C(p, k[\overline{V}]) \cup C(q, k[\overline{V}])] \subset B' .$$

Since k is a homeomorphism, there is a positive number ε such that if X is any subset of B' of diameter less than ε, then $(\operatorname{diam} k^{-1}[X]) < \delta$. By Lemma 6, there is an isotopy γ from $E^{n+1} \times [0,1]$ into E^{n+1} such that

(1) γ_0 is the identity,

(2) if $t \in [0,1]$, γ_t is onto E^{n+1},

(3) if $t \in [0,1]$ and $x \in E^{n+1} - [C(p, k[\overline{V}] \cup C(q, k[\overline{V}])]$, then $\gamma_t(x) = x$, and

(4) if $g \in G$ and $g \subset V$, then

$$(\operatorname{diam} \gamma_1[k[g]]) < \varepsilon .$$

Define a function φ from $E^{n+1} \times [0,1]$ into E^{n+1} as follows: If $t \in [0,1]$, let φ_t be $k^{-1}\gamma_t k$. It is clear that φ is an isotopy from $E^{n+1} \times [0,1]$ into E^n satisfying conditions (1) and (2) of the conclusion of Lemma 7.

Let S denote $k^{-1}[C(p, k[\overline{V}]) \cup C(q, k[\overline{V}])]$. It is easy to see that condition (3) of the conclusion of Lemma 7 holds.

Suppose that $g \in G$ and $g \subset V$. Then by construction of γ, $(\operatorname{diam} \gamma_1 k[g]) < \varepsilon$ and hence $(\operatorname{diam} k^{-1}\gamma_1 k[g]) < \delta$. Hence $(\operatorname{diam} \varphi_1[g]) < \delta$, condition (4) of the conclusion of Lemma 7 holds, and Lemma 7 is proved.

Each of the preceding lemmas and theorems is valid for *monotone* decompositions of E^n. In the following lemma, we restrict our attention to *point-like* decompositions.

LEMMA 8. *Suppose that G is a point-like decomposition of E^n such that $P[H_G]$ is 0-dimensional. Suppose that V is a bounded open subset of E^n such that* $\operatorname{Bd} V$ *and H_G are disjoint. Suppose that ε and b are positive numbers. Then there is an isotopy φ from $E^{n+1} \times [0,1]$ into E^{n+1} such that*

(1) *φ_0 is the identity,*

(2) *if $t \in [0,1]$, φ_t is onto E^{n+1},*

(3) *if $x \in E^{n+1} - [V \times (-b, b)]$ and $t \in [0,1]$, $\varphi_t(x) = x$, and*

(4) *if $g \in G$ and $g \subset V$, then* $(\operatorname{diam} \varphi_1[g]) < \varepsilon$.

Proof. If for each set g of G contained in V, $(\operatorname{diam} g) < \varepsilon$, then define φ so that if $t \in [0,1]$, φ_t is the identity from E^{n+1} onto E^{n+1}. Suppose now that there is a set g of G such that $g \subset V$ and $(\operatorname{diam} g) \geqslant \varepsilon$. Let K be

$$\bigcup \{g\colon\ g \in G,\ g \subset V,\ \textit{and}\ (\operatorname{diam} g) \geqslant \varepsilon\};$$

K is a compact subset of V.

Suppose $g \in G$ and $g \subset V$. Since G is a point-like decomposition of E^n, there is an n-cell M_g such that $g \subset \operatorname{Int} M_g$ and $M_g \subset V$. Since G is upper

semi-continuous and $P[H_G]$ is 0-dimensional, there is an open subset W_g of E^n such that $g \subset W_g$, $\overline{W}_g \subset \operatorname{Int} M_g$, and $\operatorname{Bd} W_g$ and H_G are disjoint. Then $\{W_g\colon g \in G$ *and* $g \subset K\}$ covers K and hence there exist finitely many sets $g_1, g_2, \ldots, g_m$ of G contained in K such that $\{W_{g_1}, W_{g_2}, \ldots, W_{g_m}\}$ covers K. If $i = 1, 2, \ldots, m$, let M_i and W_i denote M_{g_i} and W_{g_i}, respectively.

Let R_1 denote W_1, let R_2 denote $W_2 - \overline{R}_1$, and if $i = 2, 3, \ldots, m-1$, let R_{i+1} denote $W_{i+1} - \bigcup\limits_{j=1}^{i} \overline{R}_i$. It is clear that $\{R_1, R_2, \ldots, R_m\}$ is a set of mutually disjoint open subsets of E^n covering K such that if $i = 1, 2, \ldots$ $\ldots, m$, $\overline{R}_i \subset \operatorname{Int} M_i$, and $\operatorname{Bd} R_i$ and H_G are disjoint.

Let U denote $V \times (-b, b)$.

By Lemma 7, there exist an isotopy φ^1 from $E^{n+1} \times [0,1]$ into E^{n+1} and a compact subset S_1 of E^{n+1} such that

(1) φ_0^1 is the identity,

(2) if $t \in [0,1]$, φ_t^1 is onto E^n,

(3) $S_1 \subset U$, $S_1 \cap E^n = \overline{R}_1$, and if $x \in E^{n+1} - S_1$ and $t \in [0,1]$, $\varphi_t^1(x) = x$, and

(4) if $g \in G$ and $g \subset R_1$, then

$$(\operatorname{diam} \varphi_1^1[g]) < \varepsilon .$$

Since S_1 is compact, there is a positive number ε_2 such that if X is any subset of E^{n+1} of diameter less than δ_2, then

$$(\operatorname{diam} \varphi_1^1[X]) < \varepsilon .$$

Observe that if $t \in [0,1]$, $\varphi_t^1|(E^n - R_1)$ is the identity on $E^n - R_1$.

There exist an isotopy φ^2 from $E^{n+1} \times [0,1]$ into E^{n+1} and a compact subset S_2 of E^{n+1} such that

(1) φ_0^2 is the identity,

(2) if $t \in [0,1]$, φ_t^2 is into E^{n+1},

(3) $S_2 \subset U$, $S_2 \cap E^n = \overline{R}_2$, and if $x \in E^{n+1} - S_2$ and $t \in [0,1]$, $\varphi_t^2(x) = x$, and

(4) if $g \in G$ and $g \subset R_2$, then

$$(\operatorname{diam} \varphi_1^2[g]) < \delta_2 .$$

There is a positive number δ_3 such that if X is any subset of E^{n+1} and $(\operatorname{diam} X) < \delta_3$, then $(\operatorname{diam} \varphi_1^2[X]) < \delta_2$. Observe that if $t \in [0,1]$, $\varphi_t^2|(E^n - R_2)$ is the identity on $E^n - R_2$.

Continue this process; there exist isotopies $\varphi^3, \varphi^4, \ldots, \varphi^m$, each from $E^{n+1} \times [0,1]$ into E^{n+1}, compact subsets $S_3, S_4, \ldots, S_m$, and positive numbers $\delta_4, \delta_5, \ldots, \delta_m$ such that

(1) if $i = 3, 4, \ldots, m$,

(a) φ_0^i is the identity,

(b) if $t \in [0, 1]$, φ_t^i is onto E^{n+1},

(c) $S_i \subset U$, $S_i \cap E^n = \bar{R}_i$, and if $x \in E^{n+1} - S_i$ and $t \in [0, 1]$, $\varphi_t^i(x) = x$, and

(d) if $g \in G$ and $g \subset R_i$, then

$$(\operatorname{diam} \varphi_1^i[g]) < \delta_i ,$$

and (2) if $i = 3, 4, \ldots, m-1$ and X is any subset of E^{n+1} such that $(\operatorname{diam} X) < \delta_{i+1}$, then $(\operatorname{diam} \varphi_1^i[X]) < \delta_i$.

Observe that if $i = 3, 4, \ldots, m$, and $t \in [0, 1]$, $\varphi_t^i | E^n - R_i$ is the identity on $E^n - R_i$.

Define a homotopy φ in the following way: Let

$$0 = t_0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = 1$$

be a partition of $[0, 1]$ into m subintervals of equal length. Then define φ as follows:

$$\text{If } t_0 \leqslant t \leqslant t_1 , \quad \varphi(x, t) = \varphi^m\big(x, m(t - t_0)\big) .$$

$$\text{If } t_1 \leqslant t \leqslant t_2 , \quad \varphi(x, t) = \varphi^{m-1}\big(\varphi_1^m(x), m(t - t_1)\big) .$$

. . . .

$$\text{If } t_i \leqslant t \leqslant t_{i+1} , \quad \varphi(x, t) = \varphi^{m-i}\big(\varphi_1^{m-(i-1)}\varphi_i^{m-(i-2)} \ldots \varphi_1^{m-1}\varphi_1^m(x), m(t - t_i)\big) .$$

. . . .

$$\text{If } t_{m-1} \leqslant t \leqslant t_m , \quad \varphi(x, t) = \varphi^1\big(\varphi_1^2\varphi_1^3 \ldots \varphi_1^{m-1}\varphi_1^m(x), m(t - t_{m-1})\big) .$$

It is clear that φ is well-defined and is an isotopy from $E^{n+1} \times [0, 1]$ into E^{n+1}. Clearly conditions (1) and (2) of the conclusion of Lemma 8 are satisfied, and since each of $S_1, S_2, \ldots, S_m$ lies in U, condition (3) also holds.

Suppose that $g \in G$ and $g \subset V$. If g does not lie in any one of $R_1, R_2, \ldots, R_m$, then $(\operatorname{diam} g) < \varepsilon$ and $\varphi[g] = g$, and hence $(\operatorname{diam} \varphi_1[g]) < \varepsilon$.

Suppose now that there is a positive integer j such that $j = 1, 2, \ldots, m$ and $g \subset R_j$. Then

$$\varphi_1[g] = \varphi_1^1\varphi_1^2 \ldots \varphi_1^{m-1}\varphi_1^m[g] .$$

If $i = j+1, j+2, \ldots, m$, and $t \in [0, 1]$, then $\varphi_t^i | g$ is the identity on g, and hence

$$\varphi_1[g] = \varphi_1^1\varphi_1^2 \ldots \varphi_1^j[g] .$$

Now $(\operatorname{diam} \varphi_1^j[g]) < \delta_j$, and hence

$$(\operatorname{diam} \varphi_1^{j-1}\varphi_1^j[g]) < \delta_{j-1} .$$

Similarly

$$(\operatorname{diam} \varphi_1^{j-2}\varphi_1^{j-1}\varphi_1^j[g]) < \delta_{j-2} ,$$

. . . .

$$(\operatorname{diam} \varphi_1^2\varphi_1^3 \ldots \varphi_1^j[g]) < \delta_2 ,$$

and hence

$$(\operatorname{diam} \varphi_1^1\varphi_1^2 \ldots \varphi_1^j[g]) < \varepsilon .$$

Therefore $(\operatorname{diam} \varphi_1[g]) < \varepsilon$. Hence condition (4) of the conclusion of Lemma 8 holds, and the proof of Lemma 8 is completed.

We are now ready to prove the main result of this section. In Lemma 9, we construct an isotopy by pasting together isotopies of the sort constructed in the proof of Lemma 8.

LEMMA 9. *If G is a point-like decomposition of E^n such that $P[H_G]$ is 0-dimensional, U is an open subset of E^{n+1} containing H_G, and ε is a positive number, there is an isotopy φ from $E^{n+1} \times [0, 1]$ into E^{n+1} such that*

(1) *φ_0 is the identity,*

(2) *if $t \in [0, 1]$, φ_t is onto E^{n+1},*

(3) *if $x \in E^{n+1} - U$ and $t \in [0, 1]$, $\varphi_t(x) = x$, and*

(4) *if $g \in G$, then $(\operatorname{diam} \varphi_1[g]) < \varepsilon$.*

Proof. By an argument similar to one used in the proof of Theorem 1, it may be proved that there is a sequence $V_1, V_2, V_3, \ldots$ of mutually disjoint bounded open subsets of E^n such that

(1) $\{V_1, V_2, \ldots\}$ covers H_G,

(2) if $i = 1, 2, \ldots$, $\bar{V}_i \subset U$, and $\operatorname{Bd} V_i$ and H_G are disjoint, and

(3) if B is any bounded subset of E^n,

$$\{V_i\text{: } i \text{ is a positive integer and } V_i \text{ intersects } B\}$$

is a null collection.

There is a sequence $b_1, b_2, \ldots$ of positive numbers, converging to 0, and such that if $i = 1, 2, \ldots$,

$$\big(V_i \times (-b_i, b_i)\big) \subset U .$$

If i is any positive integer, let U_i denote $V_i \times (-b_i, b_i)$.

It is easy to see that (1) if i and j are distinct positive integers, U_i and U_j are disjoint and (2) if B' is any bounded subset of E^{n+1},

$$\{U_i\text{: } i \text{ is a positive integer and } U_i \text{ intersects } B'\}$$

is a null collection.

If i is any positive integer, there is, by Lemma 8, an isotopy φ^i from $E^{n+1} \times [0, 1]$ into E^{n+1} such that

(1) φ_0^i is the identity,

(2) if $t \in [0, 1]$, φ_t^i is onto E^{n+1},

(3) if $x \in E^{n+1} - U_i$ and $t \in [0,1]$, $\varphi_t^i(x) = x$, and

(4) if $g \in G$ and $g \subset V_i$, then

$$(\operatorname{diam} \varphi_1^i[g]) < \varepsilon .$$

Let φ be the homotopy defined as follows:

(1) If $x \in E^{n+1} - \bigcup_{i=1}^{\infty} U_i$ and $t \in [0,1]$, then $\varphi_t(x) = x$.

(2) If $i = 1, 2, \ldots$, $x \in U_i$, and $t \in [0,1]$, then $\varphi_t(x) = \varphi_t^i(x)$.

By Lemma 5, φ is an isotopy from $E^{n+1} \times [0,1]$ into E^{n+1}. It is easily seen that conditions (1), (2), and (3) of the conclusion of Lemma 9 hold. If g is a non-degenerate element of G, there is a positive integer i such that $g \subset V_i$, and hence

$$(\operatorname{diam} \varphi_1^i[g]) < \varepsilon .$$

Since $\varphi_1 | V_i = \varphi^i | V_i$, it follows that

$$(\operatorname{diam} \varphi_1[g]) < \varepsilon .$$

Hence condition (4) of the conclusion of Lemma 9 holds, and Lemma 9 is proved.

6. An embedding theorem. In the embedding theorem that we prove, the embedding is realized as the final stage of a pseudo-isotopy from $E^{n+1} \times [0,1]$ into E^{n+1} that starts at the identity. The statement that φ is a pseudo-isotopy from $E^{n+1} \times [0,1]$ into E^{n+1} means that φ is a homotopy from $E^{n+1} \times [0,1]$ into E^{n+1} such that if $t \in [0,1)$, φ_t is a homeomorphism.

We state without proof a result due to Bing. In [3], Bing points out that his argument for Theorem 1 of [3] may, in certain cases, be modified to show the existence of a pseudo-isotopy with certain properties. By using Theorem 2 and a construction patterned after Bing's proof of Theorem 1 of [3], the following theorem can be proved.

THEOREM 3. *Suppose that G is a monotone decomposition of E^n such that $P[H_G]$ is 0-dimensional. Suppose that if U is any open subset of E^n containing H_G and ε is any positive number, there is an isotopy μ from $E^n \times [0,1]$ into E^n such that*

(1) *μ_0 is the identity,*

(2) *if $t \in [0,1]$, μ_t is onto E^n,*

(3) *if $x \in E^n - U$ and $t \in [0,1]$, $\mu_t(x) = x$, and*

(4) *if $g \in G$, $(\operatorname{diam} \mu_1[g]) < \varepsilon$.*

Then if W is any open set in E^n containing H_G, there is a pseudo-isotopy φ from $E^n \times [0,1]$ into E^n such that

(1) *φ_0 is the identity,*

(2) *if $t \in [0,1]$, φ_t is onto E^n,*

(3) *if $t \in [0,1]$ and $x \in E^n - W$, $\varphi_t(x) = x$,*

(4) *$G = \{\varphi_1^{-1}[y]\colon y \in E^n\}$, and*

(5) *$\varphi_1 | (E^n - H_G)$ is a homeomorphism from $E^n - H_G$ onto $E^n - \varphi_1[H_G]$.*

We are ready now to prove the main result of this paper.

THEOREM 4. *Suppose that G is a point-like decomposition of E^n such that $P[H_G]$ is 0-dimensional. If W is any open set in E^{n+1} containing H_G, there is a pseudo-isotopy φ from $E^{n+1} \times [0,1]$ into E^{n+1} such that*

(1) *φ_0 is the identity,*

(2) *if $t \in [0,1]$, φ_t is onto E^{n+1},*

(3) *if $x \in E^{n+1} - W$ and $t \in [0,1]$, $\varphi_t(x) = x$,*

(4) *$G = \{\varphi_1^{-1}[y]\colon y \in \varphi_1[E^n]\}$, and*

(5) *$\varphi_1 | (E^n - H_G)$ is a homeomorphism from $E^n - H_n$ onto $\varphi_1[E^n - H_G]$.*

Proof. Let F be the decomposition of E^{n+1} such that $f \in F$ if and only if either $f \in G$ or for some point p of $E^{n+1} - E^n$, $f = \{p\}$. Then F is a monotone decomposition of E^{n+1} and $H_F = H_G$.

If U is any open subset of E^{n+1} containing H_F and ε is any positive number, then by Lemma 9, there is an isotopy μ from $E^{n+1} \times [0,1]$ into E^{n+1} such that

(1) μ_0 is the identity,

(2) if $t \in [0,1]$, μ_t is onto E^{n+1},

(3) if $x \in E^{n+1} - U$ and $t \in [0,1]$, $\mu_t(x) = x$, and

(4) if $g \in G$, $(\operatorname{diam} \mu_1[g]) < \varepsilon$.

Then the existence of a pseudo-isotopy φ satisfying the conclusion of Theorem 4 follows from Theorem 3.

COROLLARY 1. *If G is a point-like decomposition of E^n and $P[H_G]$ is 0-dimensional, then E^n/G can be embedded in E^{n+1}.*

Proof. By an argument similar to one used in the proof of Theorem 1, it may be shown that there is a covering $\{V_1, V_2, \ldots\}$ of H_G by mutually disjoint bounded open subsets of E^{n+1} such that if B is any bounded subset of E^{n+1},

$$\bigcup \{V_i\colon i \text{ is a positive integer and } V_i \text{ intersects } B\}$$

is bounded. Let U denote $\bigcup_{i=1}^{\infty} V_i$.

By Theorem 4, there is a pseudo-isotopy φ from $E^{n+1} \times [0,1]$ into E^{n+1} such that

(1) φ_0 is the identity,

(2) if $t \in [0,1]$, φ_t is onto E^n,

(3) if $x \in E^{n+1} - U$ and $t \in [0,1]$, $\varphi_t(x) = x$,

(4) $G = \{\varphi_1^{-1}[y]\colon y \in \varphi_1[E^n]\}$, and

(5) $\varphi_1|(E^n - W)$ is a homeomorphism from $(E^n - W)$ onto $\varphi_1[E^n - H_G]$.

Let f denote $\varphi_1|E^n$ and let h denote the function fP^{-1} from E^n/G onto $\varphi_1[E^n]$.

It is well known ([7], p. 136) that h is one-to-one and continuous. In order to show that h is a homeomorphism, it is sufficient to show that f is a compact map (i.e., if A is a compact subset of $f[E^n]$, then $f^{-1}[A]$ is compact).

Suppose that A is a compact subset of $f[E^n]$. Then A is a compact set in E^{n+1} and thus

$$\bigcup \{V_i\colon i \text{ is a positive integer and } V_i \text{ intersects } A\}$$

is bounded. Now if i is any positive integer $\varphi_1[V_i] = V_i$ and hence $\varphi_1[A \cap V_i] \subset V_i$. It follows that

$$\varphi_1[A] \subset \bigcup \{V_i\colon i \text{ is a positive integer and } V_i \text{ intersects } A\}.$$

Hence $f[A]$ is compact. Therefore h is a homeomorphism from E^n/G into E^{n+1}, and Corollary 1 is proved.

COROLLARY 2. *If G is a point-like decomposition of E^n such that G has only countably many non-degenerate elements, then E^n/G can be embedded in E^{n+1}.*

COROLLARY 3. *Suppose that G is a monotone decomposition of E^{n+1} such that*

(1) *$P[H_G]$ is 0-dimensional, and*

(2) *each non-degenerate element of G lies in E^n and is a point-like subset of E^n.*

Then E^{n+1}/G is homeomorphic to E^{n+1}.

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