On compact metric space sequences, monotonic by r-domination

by

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Introduction. In this paper we shall deal with the notion of r-domination, introduced by K. Borsuk [1].

A mapping $\varphi\colon X \to Y$ which is a homeomorphism of X onto $\varphi(X)$, will be called an *embedding*.

A space Y *r-dominates* over a space X (we shall write $X \leqslant_r Y$, or $Y \geqslant_r X$) if there exist an embedding φ of X into Y and a retraction of Y onto $\varphi(X)$ (see [1], p. 322). The spaces X and Y are *r-equal* ($X =_r Y$) if $X \leqslant_r Y$ and $Y \leqslant_r X$, otherwise X and Y are *r-distinct* ($X \neq_r Y$). The space X is *r-less* than Y ($X <_r Y$ or $Y >_r X$) if $X \leqslant_r Y$ and $X \neq_r Y$; they are *r-uncomparable* if neither $X \leqslant_r Y$ nor $Y \leqslant_r X$.

Definition 1. Let $\{X_n\}$ be an r-increasing sequence of spaces (i.e. $X_1 <_r X_2 <_r X_3 \ldots$). The sequence $\{X_n\}$ *attains a space* X^0 if the following conditions are satisfied:

1. $X_n \leqslant_r X^0$ for $n = 1, 2, \ldots$
2. Each space X such that $X_n \leqslant_r X \leqslant_r X^0$ for $n = 1, 2, \ldots$, is r-equal to X^0.

Definition 2. Let $\{X_n\}$ be an r-decreasing sequence of spaces (i.e. $X_1 >_r X_2 >_r X_3 \ldots$). The sequence $\{X_n\}$ *attains a space* X^0 if the following conditions are satisfied:

1'. $X_n \geqslant_r X^0$ for $n = 1, 2, \ldots$

2'. Each space X such that $X_n \geqslant_r X \geqslant_r X^0$ for $n = 1, 2, \ldots$, is r-equal to X^0.

Remark 1. If a sequence (either r-decreasing or r-increasing) attains two r-distinct spaces, then they are r-uncomparable.

Remark 2. If an r-increasing (or r-decreasing) sequence $\{X_n\}$ attains X^0, then $X^0 >_r X_n$ (or $X^0 <_r X_n$) for $n = 1, 2, \ldots$

So far "a space" has denoted an arbitrary topological space, but from now on by a space we shall mean a compact metric space. This restriction does not change the sense of definitions 1 and 2, because if X

is a compact metric space and $Y \leqslant_r X$, then Y is also compact and metrisable.

The aim of this paper is to give some examples illustrating the case where a space is attained by a sequence of spaces, especially if all the spaces are AR-sets. In § 1 an r-increasing sequence $\{C_n\}$ and a family $\{C^\nu\}_{\nu \in N}$ of spaces are constructed, such that $C^\nu \neq_r C^\mu$ for $\nu \neq \mu$, $\nu, \mu \in N$, and for each $\nu \in N$ C^ν is attained by $\{C_n\}$, where the set N is of power $\mathfrak{c}$. § 2 contains the construction of an r-decreasing sequence and a family of spaces with the same properties.

In § 3 it is shown that the dimension of a space attained by an r-decreasing sequence may be less than the dimension of all spaces of that sequence (¹).

In § 4 it is proved that the n-cube Q^n and the n-sphere S^n $(n = 1, 2, \ldots)$ are not attained by any r-increasing sequence.

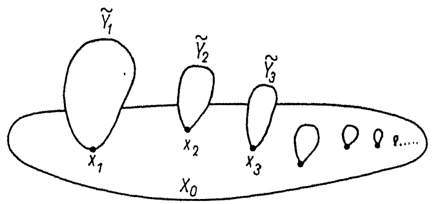

Fig. 1

The construction given by the following scheme is frequently used in this paper. Given: a space X_0, a sequence of spaces $\{Y_i\}$, points $x_i \in X_0$ $(x_i \neq x_j$ for $i \neq j)$ and $y_i \in Y_i$ for $i = 1, 2, \ldots$ There exists a space (topologically—exactly one) $\Sigma = X_0 \cup \bigcup_{i=1}^{\infty} \widetilde{Y}_i$ such that (see Fig. 1):

(1) $\lim_{i\to\infty} \operatorname{diam} \widetilde{Y}_i = 0$;

(2) $\widetilde{Y}_i \cap X_0 = \{x_i\}$ *and* $\widetilde{Y}_i \cap \widetilde{Y}_j = 0$ *for* $i \neq j$, $i, j = 1, 2, \ldots$;

(3) *There exists a homeomorphism of* Y_i *onto* $\widetilde{Y}_i$ *which sends* y_i *onto* x_i $(i = 1, 2, \ldots)$.

The construction which gives such a space Σ will be called the *construction* (σ). If X_0 and Y_i $(i = 1, 2, \ldots)$ are all AR-sets, then by construction (σ) we also obtain an AR-set. To prove this, it is sufficient to do it for $X_0 = Y_i = Q^\omega$, where Q^ω is the Hilbert cube.

§ 1. A family of power $\mathfrak{c}$ consisting of dendrites attained by one r-increasing sequence.

Let

$$A = \{(x, y)\colon\ -1 \leqslant x \leqslant 1;\ y = 0\,,\ \text{or}\ x = 0;\ -1 \leqslant y \leqslant 1\};$$

(¹) A. Trybulec gave an example of an r-increasing sequence which attains a finite-dimensional space of dimension greater than the dimension of all spaces of that sequence.

and

$$A_n = A \cup \bigcup_{k=1}^{n} \left\{(x, y)\colon\ x = \frac{k}{n+2};\ 0 \leqslant y \leqslant 1\right\}, \qquad n = 1, 2, \ldots$$

A_n is a dendrite and $A_n <_r A_{n+1}$. It is easy to see that the following lemma is true:

Lemma 1. *There exist two r-distinct dendrites attained by $\{A_n\}$, namely*

$$A^0 = A \cup \bigcup_{k=1}^{\infty} \left\{(x, y)\colon\ x = 1 - \frac{1}{k};\ 0 \leqslant y \leqslant \frac{1}{k}\right\}$$

and

$$A^1 = A \cup \bigcup_{k=2}^{\infty} \left\{(x, y)\colon\ x = \frac{1}{k};\ 0 \leqslant y \leqslant \frac{1}{k}\right\}$$

(see Fig. 2).

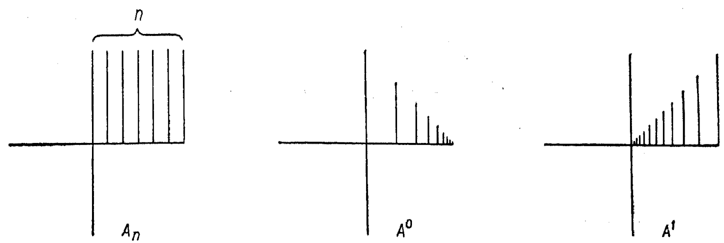

Fig. 2

Let us notice the following properties of A^i and A_n $(i = 0, 1;\ n = 1, 2, 3, \ldots)$.

(1a) *For each* $p \in A_n$, $\operatorname{Ord}_p A_n \leqslant 4$, *and similarly* $\operatorname{Ord}_p A^i \leqslant 4$.

(2a) A_n *is a tree;* A^i *contains only one point of order* 4.

(3a) *The sequence* $\{A_n\}$ *is* r-*increasing*.

Now let $\{B_n\}$ be the r-decreasing sequence of dendrites constructed by K. Sieklucki in [2], p. 331. Let us notice the following properties of dendrites B_n $(n = 1, 2, 3, \ldots)$:

(1b) *For each* $p \in B_n$, $\operatorname{Ord}_p B_n \leqslant 5$.

(2b) *The set of points of order* 4 *in* B_n *is infinite.*

(3b) *The sequence* $\{B_n\}$ *is* r-*decreasing*.

Let C be union of the following segments on the plane (see Fig. 3): $\overline{(-1, 0), (3, 0)}$, $\overline{(0, 1), (0, -1)}$, $\overline{(-1, -1), (0, 0)}$, $\overline{(-1, 1), (0, 0)}$, $\overline{(2, 1), (2, -1)}$, $\overline{(3, 1), (2, 0)}$, $\overline{(3, -1), (2, 0)}$.

Now let us make the construction (σ) substituting $X_0 = C$, $Y_i = B_i$, $x_i = \big(1/(2i-1), 0\big)$, y_i-being an arbitrary point of order 1 in B_i.

Let B denote the dendrite Σ obtained by this construction. For simplicity we may assume that $B_i = \widetilde{Y}_i$.

For an arbitrary positive integer n let C_n denote the dendrite obtained by the construction (σ) if we substitute $X_0 = B$, $Y_i = A_n$, $x_i = (1/2i, 0)$, $y_i = (-1, 0)$ (y_i is an end-point in A_n). The subset $\widetilde{Y}_i$ of C_n will be denoted by $A_{n,i}$. Let us remark that the sequence $\{C_n\}$ is r-increasing. Naturally, $C_n \leqslant_r C_{n+1}$; to prove that $C_n \neq_r C_{n+1}$ let us suppose, on the contrary, that φ is an embedding of C_{n+1} into C_n. It follows from (1a) and (1b), that the points $c_1 = (0, 0)$ and $c_2 = (2, 0)$ are the only points of order 6 in C_n and in C_{n+1}; therefore we must have $\varphi(\{c_1, c_2\}) = \{c_1, c_2\}$; further $\varphi(\overline{c_1, c_2}) = \overline{c_1, c_2}$ and $\varphi(1/k, 0) = (1/i, 0)$, and in particular $\varphi(1, 0) = (1/m, 0)$

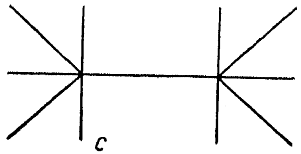

Fig. 3

where m is a positive integer. But by (2a), (2b), and (3b) there is no embedding of B_1 into A_k (for $k = 1, 2, \ldots$) and into B_k (for $k = 2, 3, \ldots$), whence $\varphi(1, 0) = (1, 0)$. Similarly $\varphi(\frac{1}{3}, 0) = (\frac{1}{3}, 0)$ and consequently $\varphi(\frac{1}{2}, 0) = (\frac{1}{2}, 0)$, which implies $\varphi(A_{n+1,1}) \subset A_{n,1}$ and we get a contradition of (3a).

Theorem 1. *There exists a family of power* $\mathfrak{c}$ *consisting of mutually* r-*uncomparable dendrites which are attained by the sequence* $\{C_n\}$ *(constructed above).*

Proof. Let N be the set of all sequences $\{\nu_i\}$ such that $\nu_i = 0$ or 1 for $i = 1, 2, \ldots$ For each $\nu \in N$ let C^ν denote the dendrite obtained by the construction (σ) if we substitute $X_0 = B$, $Y_i = A^{\nu_i}$, $x_i = (1/2i, 0)$, $y_i = (-1, 0)$, where B, A^0, A^1 are the dendrites described at the beginning of this section. Let $A^{\nu_i, i}$ be the subset $\widetilde{Y}_i$ of C^ν. We shall prove that $\{C^\nu\}_{\nu \in N}$ is the required family. First we prove that $\{C_n\}$ attains C^ν for each $\nu \in N$. The inequality $C_n \leqslant_r C^\nu$ (for $n = 1, 2, \ldots$) is a consequence of the fact that A_n may be embedded into A^i in such a way that the image of the point $(-1, 0)$ is $(-1, 0)$ (for $i = 0, 1$; $n = 1, 2, \ldots$).

Now let X be a space such that $C_n \leqslant_r X \leqslant_r C^\nu$ for $n = 1, 2, \ldots$

X must be a dendrite, and we may suppose that $X \subset C^\nu$. Let $\varphi_n\colon C_n \to X$ be an embedding. Then φ_n is simultaneously an embedding of C_n into C^ν, which implies $\varphi_n(0, 0) = (0, 0)$, $\varphi_n(2, 0) = (2, 0)$, $\varphi_n\big(\overline{(0, 0), (2, 0)}\big) = \overline{(0, 0), (2, 0)}$; further $\varphi_n(1, 0) = (1, 0)$, since by (2a), (2b) and (3b) the set B_1 cannot be embedded into A^0, A^1, and B_i (for $i = 2, 3, \ldots$); similarly $\varphi_n(\frac{1}{3}, 0) = (\frac{1}{3}, 0)$ and by induction

$$\varphi_n\left(\frac{1}{2k-1}, 0\right) = \left(\frac{1}{2k-1}, 0\right) \quad \text{for} \quad k = 1, 2, 3, \ldots$$

This yields

$$\varphi_n\left(\frac{1}{2k}, 0\right) = \left(\frac{1}{2k}, 0\right) \quad \text{and} \quad \varphi_n(A_{n,k}) \subset A^{\nu_k, k},$$

whence

$$\varphi_n(A_{n,k}) \subset X \cap A^{\nu_k, k}.$$

But the sequence $\{A_n\}$ attains A^{ν_k}, and therefore $X \cap A^{\nu_k, k} =_r A^{\nu_k}$ and there exist an embedding φ^k of $A^{\nu_k, k}$ into $X \cap A^{\nu_k, k}$; we may suppose that $\varphi^k(1/2k, 0) = (1/2k, 0)$, because $(1/2k, 0) \in X$. The existence of the embedding φ of C^ν into X defined by $\varphi|B = \varphi_1|B$ and $\varphi|A^{\nu_k, k} = \varphi^k$ (for $k = 1, 2, \ldots$) proves that $C^\nu \leqslant_r X$.

Finally, we shall prove that $C^\nu \leqslant_r C^\mu$ is possible only if $\nu = \mu$, for $\nu, \mu \in N$. Let ψ be an embedding of C^ν into C^μ, where $\nu = \{\nu_k\}$ and $\mu = \{\mu_k\}$. As before, we must have $\psi(1, 0) = (1, 0)$, $\psi(\frac{1}{3}, 0) = (\frac{1}{3}, 0)$ and so on, i.e. $\psi\big(1/(2k-1), 0\big) = \big(1/(2k-1), 0\big)$, which yields $\psi(1/2k, 0) = (1/2k, 0)$ (for $k = 1, 2, \ldots$). Hence $\psi(A^{\nu_k, k}) \subset A^{\mu_k, k}$, but by the r-incomparability of A^0 and A^1, this is possible only if $\nu_k = \mu_k$, which ends the proof.

§ 2. A family of power $\mathfrak{c}$ consisting of 2-dimensional AR-sets attained by one r-decreasing sequence of 2-dimensional AR-sets. As in § 1 we shall first construct an r-decreasing sequence which attains (at least) two r-distinct spaces.

Let

$$P = \{(x, y, z)\colon z = 0\,,\ x^2 + y^2 \leqslant 1\} \cup \{(x, y, z)\colon x = 0\,,\ y = \tfrac{1}{2},\ -1 \leqslant z \leqslant 0\},$$

and

$$Q = \{(x, y, z)\colon z = 1, x^2 + y^2 \leqslant 1\}.$$

Let us remark that P cannot be embedded into Q.

Now let

$$C_{n,k} = \left\{(x, y, z)\colon y = 0\,,\ x^2 + \left(z - \frac{2k-1}{2n}\right)^2 \leqslant \left(\frac{1}{2n}\right)^2\right\}$$

and let

$$F_n = P \cup Q \cup \bigcup_{k=1}^{n} C_{n,k} \quad (n = 1, 2, 3, \ldots) \quad \text{(see Fig. 4)}.$$

The sequence $\{F_n\}$ obtained is r-decreasing, and all F_n are 2-dimensional AR-sets (compare with [1], p. 325).

LEMMA 2. *The sequence* $\{F_n\}$ *constructed above attains two* (*r-distinct*) *2-dimensional* AR-*sets, namely*

$$F^0 = P \cup Q \cup R \cup \bigcup_{n=1}^{\infty} C_{2^n,2} \quad \text{and} \quad F^1 = P \cup Q \cup S \cup \bigcup_{n=1}^{\infty} C_{2^n,2^n-1},$$

where

$$R = \{(x, y, z)\colon\ y = 0,\ x^2+(z+1)^2 \leqslant 1\},$$

$$S = \{(x, y, z)\colon\ y = 0,\ x^2+(z-2)^2 \leqslant 1\} \quad \text{(see Fig. 4)}.$$

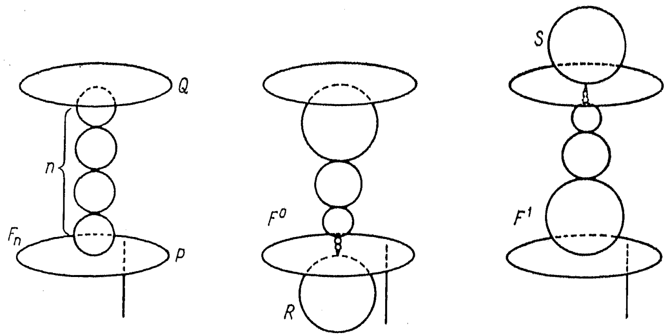

Fig. 4

Proof. First we prove that $F^0 \neq_r F^1$. On the contrary, suppose that φ is an embedding of F^0 into F^1. The points $p = (0, 0, 0)$ and $q = (0, 0, 1)$ are the only points in F^0 and F^1 which have neighbourhoods arbitrarily small, homogeneously 2-dimensional and unflat (i.e. such as cannot be embedded into the plane); therefore $\varphi(\{p, q\}) = \{p, q\}$. If $\varphi(p) = q$, then $\varphi(P) \subset Q$, but this is impossible.

If $\varphi(p) = p$, then $\varphi(q) = q$, $\varphi(Q) \subset Q$ and $\varphi(F^0 - Q) \subset S$, because q has in F^0 a neighbourhood which is not separated by any point except q. In particular, then $\varphi(P) \subset S$, but this is also impossible, because S is a disk, like Q. This contradiction proves that $F^0 \neq_r F^1$. Next we prove that the sequence $\{F_n\}$ attains F^0; the proof for F^1 is analoguous. Evidently, $F^0 \leqslant_r F_n$ for $n = 1, 2, \ldots$; let F be a space such that $F^0 \leqslant_r F \leqslant_r F_n$ for $n = 1, 2, \ldots$. We may assume that $F^0 \subset F$. From $F \leqslant_r F_1$ it follows that F is an AR-set, whence to prove that $F^0 =_r F$ it is sufficient to find an embedding of F into F^0. For each k we have an embedding φ_k of F into F_k; the restriction of φ_k to F^0 is also embedding which gives succesively: $\varphi_k(p) = p$, $\varphi_k(q) = q$, $\varphi_k(P) \subset P$, $\varphi_k(Q) \subset Q$.

Since there are in F_k exactly $k-1$ points separating F_k between P and Q, F contains an infinite number of points separating F between P and Q. Each point separating F between P and Q also separates F^0

between P and Q, because $P \cup Q \subset F^0 \subset F$ and F^0 is connected. But the only points separatnig F^0 between P and Q are the points $p_i = (0, 0, 1/2^i)$, whence there is a subsequence $\{q_i\}$ of $\{p_i\}$ each point of which separates F between P and Q.

Evidently, $F = \varphi_1^{-1}(P) \cup \varphi_1^{-1}(Q) \cup \varphi_1^{-1}(C_{1,1})$. The point q_1 separates the set $\hat{F} = \varphi_1^{-1}(C_{1,1})$ between p and q; let $\hat{F}_1$ denote the closure of the union of these components of $\hat{F} - \{q_1\}$ which do not contain p. $\hat{F}_1$ contains q and q_1 and there is an embedding ψ_1 of $\hat{F}_1$ into $C_{2,2}$, which sends q onto q and q_1 onto p_1. Let $\hat{F}_2$ denote the closure of the union of those components of $(\hat{F} - \hat{F}_1) - \{q_2\}$ which do not contain p. $\hat{F}_2$ contains q_1 and q_2 and there is an embedding ψ_2 of $\hat{F}_2$ into $C_{4,2}$, which sends q_1 onto p_1 and q_2 onto p_2. Let us suppose that we have defined the sets $\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_n$ and the embeddings ψ_i of $\hat{F}_i$ into $C_{2^i,2}$ such that

$$\psi_i(q_{i-1}) = p_{i-1}, \quad \psi_i(q_i) = p_i \quad \text{for } i = 1, 2, 3, \ldots, n \geqslant 2.$$

Let $\hat{F}_{n+1}$ denote the closure of the union of those components of $(F - \bigcup_{i=1}^{n} F_i) - \{q_{n+1}\}$ which do not contain p. $\hat{F}_{n+1}$ contains q_n and q_{n+1} and there is an embedding ψ_{n+1} of $\hat{F}_{n+1}$ into $C_{2^{n+1},2}$ which sends q_n onto p_n and q_{n+1} onto p_{n+1}. Finally, the set $\hat{F}_0 = \hat{F} - \bigcup_{n=1}^{\infty} \hat{F}_n$ is the closure of the union of those components of the set $\hat{F} - \{p\}$ which do not contain q; there is an embedding ψ_0 of $\hat{F}^0$ into R which sends p onto p. The required embedding φ of F into F^0 is defined as follows:

$$\varphi|\varphi_1^{-1}(P \cup Q) = \varphi_1|\varphi_1^{-1}(P \cup Q), \quad \varphi|_{\hat{F}_i} = \psi_i \quad \text{for } i = 0, 1, 2, \ldots,$$

and the proof is finished.

Let D_1, D_2, D_3 be three mutually disjoint disks; in the interior of D_i, let there be a disk D_i' $(i = 1, 2, 3)$. Now let D be the set obtained from $D_1 \cup D_2 \cup D_3$ after the identification of the disks D_1', D_2', D_3' onto a new disk D'. Let us assume that D is a subset of the Euclidean 3-space such that D' is a triangle with vertices $(0, 0, 0)$, $(2, 2, 0)$, $(2, -2, 0)$ and the set $D - D'$ lies in the half-space $z < 0$ (see Fig. 5). The boundary of D' will be denoted by $D'^{\bullet}$. Let J_n be the segment with endpoints $\left(\frac{1}{2n-1}, \frac{1}{2n-1}, 0\right), \left(\frac{1}{2n-1}, \frac{-1}{2n-1}, 0\right)$ and let $v_n = \left(\frac{1}{2n}\,0, 0\right)$ for $n = 1, 2, \ldots$ Let us remark that the ends of J_n lie in D'; between J_n and J_{n+1} there lies exactly one point from the sequence $\{v_k\}$, namely v_n.

In the set F_n (see Lemma 2) we choose a closed arc J_n' lying in the boundary of the disk Q.

There is a sequence $\{\zeta_n\}$ of embeddings such that ζ_n maps F_n into the half-space $z \geqslant 0$ and such that the following conditions are satisfied:

(1) $\zeta_n(J'_n) = J_n$; (2) $\zeta_n(F_n) \cap D' = J_n$; (3) $\zeta_n(F_n) \cap \zeta_m(F_m) = 0$ for $n \neq m$; (4) $\operatorname{Lim}_{n\to\infty} \operatorname{diam} \zeta_n(F_n) = 0$. The set $E = D \cup \bigcup_{n=1}^{\infty} \zeta_n(F_n)$ is a 2-dimensional AR-set. Now for each positive integer k let H_k denote the 2-dimensional AR-set obtained by construction (σ) if we substitute $X_0 = E$, $Y_i = F_k$, $x_i = v_i$, $y_i = (0, \frac{1}{2}, -1)$. The subset $\widetilde{Y}_i$ of H_k will be called $F_{k,i}$.

The sequence $\{H_k\}$ is r-decreasing. To prove this, let us suppose that φ is an embedding of H_n into H_{n+1}. Then $\varphi(D'^{\bullet}) \subset D'^{\bullet}$, which implies $\varphi(D'^{\bullet}) = D'^{\bullet}$; furthermore $\varphi(D') = D'$, $\varphi(J_1) = J_1$, $\varphi(J_2) = J_2$, $\varphi(v_1) = v_1$ and finally $\varphi(F_{n,1}) \subset F_{n+1,1}$, which is impossible. This proves that $H_n \neq_r H_{n+1}$. On the other hand, there is an embedding of F_{n+1} into F_n, which sends y_{n+1} onto y_n; thus $H_{n+1} \leqslant_r H_n$.

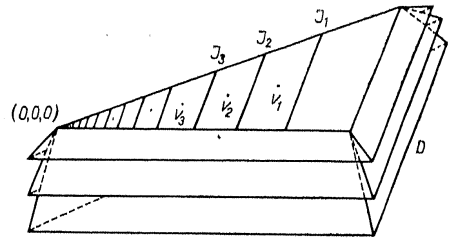

Fig. 5

THEOREM 2. *There exists a family of power* $\mathfrak{c}$ *consisting of mutually* r*-distinct 2-dimensional* AR*-sets which are attained by the* r*-decreasing sequence* $\{H_n\}$ *constructed above.*

Outline of the proof. Let N denote the set of all sequences $\nu = \{\nu_i\}$ such that $\nu_i = 0$ or 1, for $i = 1, 2, \ldots$; the power of this set is equal to $\mathfrak{c}$.

For each $\nu \in N$ let H^{ν} denote the 2-dimensional AR-set obtained by construction (σ) if we substitute $X_0 = E$, $Y_i = F^{\nu_i}$, $y_i = (0, \frac{1}{2}, -1)$, $x_i = v_i$, where E, F^0, F^1 are the sets described at the beginning of this section. $\{H^{\nu}\}_{\nu \in N}$ is the required family. The continuation of this proof is almost exactly a repetition of the proof of Theorem 1; therefore we omit it.

§ 3. r-decreasing sequence of 3-dimensional AR-sets attaining a 2-dimensional AR-set.

In this section let E denote the set E from § 2; let T^2 be the Cartesian product $T \times T$, where T is the union of three segments disjoint except a common endpoint, and let Q be a 3-cube. It is well known that T^2 cannot be embedded into Q; let us remark, moreover, that T^2 is not separated by any point.

Let T_1 denote the 3-dimensional AR-set obtained by construction (σ) if we substitute $X_0 = E$, $x_i = v_i$, $Y_1 = T^2$, $Y_i = Q$ for $i > 1$, y_i being an arbitrarily chosen point in Y_i.

The subset $\widetilde{Y}_1$ of T_1 may be identified with T^2; the subset $\widetilde{Y}_i$ of T_1, for $i > 1$, will be called Q_i. Thus $T_1 = E \cup T^2 \cup \bigcup_{i=2}^{\infty} Q_i$. For $n = 2, 3, \ldots$, let r_n be the mapping of T_1 into itself given by the formula

$$r_n(x) = \begin{cases} x & \text{for } x \in E \cup T^2 \cup \bigcup_{i=n+1}^{\infty} Q_i, \\ v_i & \text{for } x \in Q_i,\ i = 2, 3, \ldots, n. \end{cases}$$

It is easy to see that r_n is a retraction of T_1 onto $T_n = r_n(T_1)$. The sequence $\{T_n\}$ is r-decreasing. In fact: T_n is an AR-set for $n = 1, 2, \ldots$, and $T_1 \supset T_2 \supset T_3 \supset \ldots$, whence $T_1 \geqslant_r T_2 \geqslant_r \ldots$ On the other hand, let ζ be an embedding of T_n into T_m. Then $\zeta(D'^{\bullet}) = D'^{\bullet}$, or there exists an i such that $\zeta(D'^{\bullet}) \subset Q_i$. But if $\zeta(D'^{\bullet}) \subset Q_i$, then $\zeta(D') \subset Q_i$, $\zeta(v_1) \in Q_i$ and $\zeta(T^2) \subset Q_i$, which is impossible. Hence $\zeta(D'^{\bullet}) = D'^{\bullet}$ and further $\zeta(D') = D'$, $\zeta(J_k) = J_k$ (for $k = 1, 2, \ldots$) and if the dimension of T_n at the point v_i is equal to 3, then $\zeta(v_i) = v_i$ and the dimension of T_m at v_i is 3; therefore $m \leqslant n$. Thus we get $T_n \leqslant_r T_m$ if and only if $m \leqslant n$, whence $\{T_n\}$ is r-decreasing.

THEOREM 3. *The* r*-decreasing sequence* $\{T_n\}$ *constructed above attains the 2-dimensional* AR*-set* $T_0 = \bigcap_{n=1}^{\infty} T_n$.

Proof. T_0 is an AR-set and $\dim T_0 = 2$, because $T_0 = E \cup T^2$. Naturally, $T_0 \leqslant_r T_n$ for $n = 1, 2, \ldots$ Now let X be a space such that $T_0 \leqslant_r X \leqslant_r T_n$ for $n = 1, 2, \ldots$ We may assume that $T_0 \subset X$; by $X \leqslant_r T_1$, X is an AR-set also. Let φ_n be an embedding of X into T_n, for $n = 1, 2, \ldots$ $\varphi_n|T_0$ is also an embedding, thus we have succesively: $\varphi_n(D'^{\bullet}) = D'^{\bullet}$, $\varphi_n(D') = D'$, $\varphi_n(J_k) = J_k$ $(n, k = 1, 2, \ldots)$, and it follows that if X is separated by v_i for $i > 1$, then it is impossible to embed X into T_i; hence the points $v_2, v_3, \ldots$ do not separate X. But this means that the embedding φ_1 maps X into T_0, and we obtain $X \leqslant_r T_0$, which ends the proof.

Remark 3. Using the dendrite B from § 1 and an n-dimensional compact set K containing no arcs, we may obtain, in the same way as above, an r-decreasing sequence of n-dimensional spaces which attains a dendrite.

§ 4. The n-cube Q^n and the n-sphere S^n.

THEOREM 4. *For each* r*-increasing sequence* $\{X_k\}$ *such that* $X_k \leqslant_r Q^n$ *there exists a space* X *such that* $X_k \leqslant_r X <_r Q^n$.

Proof. For $n=1$ the theorem is trivially true. Let Q^n, for $n>1$, be the unique ball in the Euclidean n-space, and let Q_i be the ball in this space, with centre $(1/2^i+1/2^{i+1}, 0, 0, \ldots, 0)$ and radius $1/2^{i+1}$. The set $\hat{Q} = \overline{\bigcup_{i=1}^{\infty} Q_i}$ is an AR-set lying in Q^n, whence $\hat{Q} \leqslant_r Q^n$. Let q_i denote the point $(1/2^i, 0, 0, \ldots, 0)$ for $i = 0, 1, 2, \ldots$ For $i = 1, 2, \ldots$, q_i is the only common point of Q_i and Q_{i+1}. In each space X_i (for $i = 1, 2, \ldots$) there are two points s_i and t_i such that there is an embedding φ_i of X_i into Q_i, which sends s_i onto q_{i-1} and t_i onto q_i. Let $X = \overline{\bigcup_{i=1}^{\infty} \varphi_i(X_i)} \subset \hat{Q}$. There is a retraction of Q_i onto $\varphi_i(X_i)$, whence there is a retraction of $\hat{Q}$ onto X, and thus $X \leqslant_r \hat{Q} \leqslant_r Q^n$. Simultaneously $X_i \leqslant_r X$, because $\varphi_i(X_i) \subset X$.

For each $k, l, k \neq l$ there is a point q_i which separates X between $\varphi_k(X_k)$ and $\varphi_l(X_l)$, and no point separates Q^n. Thus, if there were an embedding of Q^n into X, it would be into some $\varphi_i(X_i)$, which is impossible by $X_i <_r Q^n$. Hence there is no embedding of Q^n into X and we obtain $X_k \leqslant_r X <_r Q^n$.

Remark 4. An analoguous theorem for the n-sphere S^n is also true. Indeed: if $X_i <_r S^n$, then $X_i \leqslant_r Q^n$.

References

[1] K. Borsuk, *Concerning the classification of topological spaces from the standpoint of the theory of retracts*, Fund. Math. 46 (1959), pp. 321-330.

[2] K. Sieklucki, *On a family of power* c *consisting of r-uncomparable dendrites*, Fund. Math. 46 (1959), pp. 331-335.

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