Equationally compact algebras (III)

by

B. Węglorz (Wrocław)

This paper is a continuation of [3] and [5]. We discuss here some facts and questions related to the following theorem (Theorem 3.1): *If* $\mathfrak{A}$ *is a subalgebra of an equationally compact algebra, then there is an equationally compact algebra in* $\mathcal{HSP}(\mathfrak{A})$ (= the smallest equational class containing $\mathfrak{A}$), *which contains* $\mathfrak{A}$ *as a subalgebra.*

This theorem would be an obvious corollary if we knew that every equationally compact algebra is isomorphic to a subalgebra of a compact topological algebra. But this is an open problem. In [2] (Problem P484) J. Mycielski asked if moreover every equationally compact algebra is a retract of compact topological algebra? We know only that the answer is affirmative for Abelian groups, linear spaces, and Boolean algebras (see [2] and [5]).

Next we apply the above theorem to show that *every equationally compact semigroups with cancellation is a group* (Theorem 4.2) which refines the well-known fact (Numakura [4]) that this holds for the topologically compact case.

1. Terminology. The terminology and notation of [5] will be used troughout this paper. We suppose a theory of ordinal numbers such that for every ordinal α, we have $\alpha = \{\xi\colon\ \xi < \alpha\}$. We write often (as in [5]) $a \in \mathfrak{A}$, for $a \in A$ and $|\mathfrak{A}|$ for $|A|$ (= card A).

Let Σ be an arbitrary set of equations with constants in $\mathfrak{A}$; then, Σ is said to be *finitely satisfiable* in $\mathfrak{A}$ if each finite subset of Σ is satisfiable in $\mathfrak{A}$. The symbol $S(\Sigma)$ denotes the set of all indices of free variables (unknowns) of the set Σ; recall that in our considerations, $S(\Sigma)$ does not need to be denumerable. $\mathcal{HSP}(\mathfrak{A})$ denotes the smallest equational class containing $\mathfrak{A}$.

2. Some closures of algebras. Let $\mathfrak{m}$ be an infinite cardinal and let $\mathfrak{A}$ be an algebra. An algebra $\mathfrak{B}$ is said to be an $\mathfrak{m}$-*closure of* $\mathfrak{A}$ (in symbols $\mathfrak{B} \in \boldsymbol{c}_{\mathfrak{m}}\mathfrak{A}$) if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and each set Σ of equations with constants in $\mathfrak{A}$, with $|\Sigma| \leqslant \mathfrak{m}$, which is finitely satisfiable in $\mathfrak{A}$, is satisfiable in $\mathfrak{B}$.

An algebra $\mathfrak{B}$ is called a *closure* of $\mathfrak{A}$ if $\mathfrak{B} \in \boldsymbol{c}_{\mathfrak{m}}\mathfrak{A}$ for every cardinal $\mathfrak{m}$ (in symbols $\mathfrak{B} \in \boldsymbol{c}\mathfrak{A}$).

First note some properties of the operators $\boldsymbol{c}_{\mathfrak{m}}$ and $\boldsymbol{c}$:

(i) *If* $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$, *then for each* $\mathfrak{m}$: $\boldsymbol{c}_{\mathfrak{m}}\mathfrak{A}_1 \subseteq \boldsymbol{c}_{\mathfrak{m}}\mathfrak{A}_0$ *and* $\boldsymbol{c}\mathfrak{A}_1 \subseteq \boldsymbol{c}\mathfrak{A}_0$.

(ii) *If* $\mathfrak{B} \in \boldsymbol{c}_{\mathfrak{m}}\mathfrak{A}$ $(\mathfrak{B} \in \boldsymbol{c}\mathfrak{A})$ *and* $\mathfrak{B} \subseteq \mathfrak{C}$ *then* $\mathfrak{C} \in \boldsymbol{c}_{\mathfrak{m}}\mathfrak{A}$ $(\mathfrak{C} \in \boldsymbol{c}\mathfrak{A})$.

(iii) *If* $\mathfrak{m} \geqslant \mathfrak{n}$, *then* $\boldsymbol{c}_{\mathfrak{m}}\mathfrak{A} \subseteq \boldsymbol{c}_{\mathfrak{n}}\mathfrak{A}$.

(iv) The following conditions are equivalent: (a) $\mathfrak{A}$ *is equationally compact* ($\mathfrak{A}$ *is equationally* $\mathfrak{m}$-*compact*), (b) *for each* $\mathfrak{B} \subseteq \mathfrak{A}$, *we have* $\mathfrak{A} \in \boldsymbol{c}\mathfrak{B}$ $(\mathfrak{A} \in \boldsymbol{c}_{\mathfrak{m}}\mathfrak{B})$, (c) *for each* $\mathfrak{B} \supseteq \mathfrak{A}$, *we have* $\mathfrak{B} \in \boldsymbol{c}\mathfrak{A}$ $(\mathfrak{B} \in \boldsymbol{c}_{\mathfrak{m}}\mathfrak{A})$.

(v) *For each algebra* $\mathfrak{A}$ *and a cardinal* $\mathfrak{m}$, *there is an algebra* $\mathfrak{B} \in \boldsymbol{c}_{\mathfrak{m}}\mathfrak{A}$ *which is an elementary extension of* $\mathfrak{A}$ *(a refinament of this proposition is given in* (v), *in the next section).*

(vi) *There is an algebra* $\mathfrak{A}_0$ *for which* $\boldsymbol{c}\mathfrak{A}_0 = 0$.

(i)-(iii) are immediate consequences of the definitions of $\boldsymbol{c}_{\mathfrak{m}}$ and $\boldsymbol{c}$, and (iv) is a simple corollary of the definition of equational compactness ($\mathfrak{m}$-compactness). (v) and its refinement in the next section was proved in [3]. An algebra satisfying (vi) was defined in [2]. Namely, $\mathfrak{A}_0 = \langle \omega, 0, 1, \cdot \rangle$, where $x \cdot y = 0$ if $x = y$ and $x \cdot y = 1$ if $x \neq y$. Suppose that $\mathfrak{A}_0 \subseteq \mathfrak{B}$ and let η be an ordinal such that $|\eta| > |\mathfrak{B}|$. The following set of equations:

$$\{\text{“}x_\alpha \cdot x_\beta = 1\text{”}\colon \alpha \neq \beta,\ \alpha, \beta < \eta\} \cup \{\text{“}x_\alpha \cdot x_\alpha = 0\text{”}\colon \alpha < \eta\}$$

is finitely satisfiable in $\mathfrak{A}_0$ but cannot be satisfied in $\mathfrak{B}$. Thus (vi) follows.

Now we shall investigate sets of equations which are finitely satisfiable in a given algebra $\mathfrak{A}$. Note the following obvious lemma.

LEMMA 2.1. *Let* $\mathfrak{A}$ *be an arbitrary algebra and let for each* $i \in I$, Σ_i *be a set of equations with constants in* $\mathfrak{A}$, *which is finitely satisfiable in* $\mathfrak{A}$. *Then there exists a set* Σ *of equations which is finitely satisfiable in* $\mathfrak{A}$ *and such that* Σ *is satisfiable in an algebra* $\mathfrak{B} \supseteq \mathfrak{A}$ *if and only if for each* $i \in I$, *the set* Σ_i *is satisfiable in* $\mathfrak{B}$.

Lemma 2.1, and the definition of $\boldsymbol{c}_{\mathfrak{m}}$ easily imply the following proposition.

PROPOSITION 2.2. *For each algebra* $\mathfrak{A}$ *and each infinite cardinal* $\mathfrak{m}$ *there is a set* Σ *of equations with constants in* $\mathfrak{A}$, *which is finitely satisfiable in* $\mathfrak{A}$, *such that* $\mathfrak{B} \in \boldsymbol{c}_{\mathfrak{m}}\mathfrak{A}$ *if and only if* $\mathfrak{A} \subseteq \mathfrak{B}$ *and* Σ *is satisfiable in* $\mathfrak{B}$. *Moreover,* Σ *can be such that* $|\Sigma| \leqslant (|\mathfrak{A}| + t)^{\mathfrak{m}}$ *where* t *is the cardinality of the type of* $\mathfrak{A}$ (1).

(1) By the cardinality of the type of an algebra $\mathfrak{A} = \langle A, \{F_t\}_{t \in T}\rangle$ we mean the cardinality of T. In fact we could replace in this proposition t by $\min(t, 2^{|\mathfrak{A}|})$.

Finally we prove the following theorem.

THEOREM 2.3. *If* $\mathfrak{B} \in \boldsymbol{c}\mathfrak{A}$, *then there exists a* $\mathfrak{C} \subseteq \mathfrak{B}$ *such that* $\mathfrak{C} \in \boldsymbol{c}\mathfrak{A}$ *and* $\mathfrak{C} \in \mathcal{HSP}(\mathfrak{A})$.

Proof. Let $\Sigma_{\mathfrak{m}}$ be a set satisfying Proposition 2.2, for $\mathfrak{A}$ and $\mathfrak{m}$, let $S_{\mathfrak{m}} = S(\Sigma_{\mathfrak{m}})$, and let Λ be a set of equations which are axioms for the class $\mathcal{HSP}(\mathfrak{A})$. Let $W_{\mathfrak{m}}$ be the set of all terms with constants in $\mathfrak{A}$ and free variables x_α $(\alpha \in S_{\mathfrak{m}})$. Finally, let $\Xi_{\mathfrak{m}}$ be the smallest set of equations such that if “$\tau(x_1, \ldots, x_n) = \vartheta(x_1, \ldots, x_n)$” $\in \Lambda$ then all equations of the form $\tau(\sigma_1, \ldots, \sigma_n) = \vartheta(\sigma_1, \ldots, \sigma_n)$, where $\sigma_1, \ldots, \sigma_n \in W_{\mathfrak{m}}$, belong to $\Xi_{\mathfrak{m}}$, i.e. $\Xi_{\mathfrak{m}}$ contains all results of substitutions of elements of $W_{\mathfrak{m}}$ for free variables in Λ. Clearly, the set $\Xi_{\mathfrak{m}}$ is satisfied by any sequence $\{a_\alpha\}_{\alpha \in S_{\mathfrak{m}}}$ of elements of $\mathfrak{A}$. Moreover, if $\Xi_{\mathfrak{m}}$ is satisfied in an algebra $\mathfrak{A}' \supseteq \mathfrak{A}$ by a sequence $\{a_\alpha\}_{\alpha \in S_{\mathfrak{m}}}$, then the subalgebra of $\mathfrak{A}'$ generated by the set $A \cup \{a_\alpha\colon \alpha \in S_{\mathfrak{m}}\}$ belongs to $\mathcal{HSP}(\mathfrak{A})$.

It is easy to see that $\Xi_{\mathfrak{m}} \cup \Sigma_{\mathfrak{m}}$ is finitely satisfiable in $\mathfrak{A}$; hence it is satisfiable in $\mathfrak{B}$ by a sequence $\{c_\alpha\}_{\alpha \in S_{\mathfrak{m}}}$. Let $\mathfrak{C}_{\mathfrak{m}}$ be the subalgebra of $\mathfrak{B}$ which is generated by the set $A \cup \{c_\alpha\colon \alpha \in S_{\mathfrak{m}}\}$. It is easy to see that $\mathfrak{C}_{\mathfrak{m}} \in \boldsymbol{c}_{\mathfrak{m}}\mathfrak{A}$ since the set $\Sigma_{\mathfrak{m}}$ is satisfied in $\mathfrak{C}_{\mathfrak{m}}$. Also we have $\mathfrak{C}_{\mathfrak{m}} \in \mathcal{HSP}(\mathfrak{A})$ since $\Xi_{\mathfrak{m}}$ is satisfied in $\mathfrak{C}_{\mathfrak{m}}$ by $\{c_\alpha\}_{\alpha \in S_{\mathfrak{m}}}$ and $\mathfrak{C}_{\mathfrak{m}}$ is generated by $A \cup \{c_\alpha\colon \alpha \in S_{\mathfrak{m}}\}$.

Let $\boldsymbol{M}$ be the set of all maximal subalgebras of $\mathfrak{B}$ which belong to $\mathcal{HSP}(\mathfrak{A})$. Since for each infinite cardinal $\mathfrak{m}$, $\mathfrak{C}_{\mathfrak{m}} \in \mathcal{HSP}(\mathfrak{A})$ is a subalgebra of $\mathfrak{B}$, there is an $\mathfrak{A}_{\mathfrak{m}} \in \boldsymbol{M}$ such that $\mathfrak{C}_{\mathfrak{m}} \subseteq \mathfrak{A}_{\mathfrak{m}}$ and $\mathfrak{A}_{\mathfrak{m}} \in \boldsymbol{c}_{\mathfrak{m}}\mathfrak{A}$ by (ii). Obviously such an $\mathfrak{A}_{\mathfrak{m}} \in \boldsymbol{c}_{\mathfrak{n}}\mathfrak{A}$ for all $\mathfrak{n} \leqslant \mathfrak{m}$. Thus there is a $\mathfrak{C} \in \boldsymbol{M}$ such that $\mathfrak{A}_{\mathfrak{m}} = \mathfrak{C}$ for arbitrary large $\mathfrak{m}$ and hence $\mathfrak{C} \in \boldsymbol{c}\mathfrak{A}$. Q.E.D.

Finally, as in [5], we can characterize closures and $\mathfrak{m}$-closures of a given algebra in terms of ultrapowers and homomorphisms.

Let $\mathfrak{B}$ and $\mathfrak{C}$ be two algebras which contain a given algebra $\mathfrak{A}$. A homomorphism h of $\mathfrak{B}$ into $\mathfrak{C}$ is called an $\mathfrak{A}$-*homomorphism* if h restricted to $\mathfrak{A}$ is the identity mapping.

THEOREM 2.4. *The following conditions are equivalent:*

(i) $\mathfrak{B}$ *is a closure of* $\mathfrak{A}$ (i.e. $\mathfrak{B} \in \boldsymbol{c}\mathfrak{A}$);

(ii) $\mathfrak{B}$ *contains an* $\mathfrak{A}$-*homomorphic image of every algebra in which* $\mathfrak{A}$ *is pure;*

(iii) $\mathfrak{B}$ *contains an* $\mathfrak{A}$-*homomorphic image of every elementary extension of* $\mathfrak{A}$;

(iv) $\mathfrak{B}$ *contains an* $\mathfrak{A}$-*homomorphic image of every ultrapower of* $\mathfrak{A}$.

The proof is the same as the proof of Theorem 2.3, in [5]. A similar characterization of $\mathfrak{m}$-closures can be obtained but then some restrictions on $\mathfrak{A}$ or $\mathfrak{m}$ are needed.

3. Equational compactifications. An algebra $\mathfrak{B}$ is said to be an *$\mathfrak{m}$-compactification of* $\mathfrak{A}$ (in symbols $\mathfrak{B} \in \boldsymbol{C}_{\mathfrak{m}}\mathfrak{A}$) if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B}$ is equationally $\mathfrak{m}$-compact.

An algebra $\mathfrak{B}$ is called a *compactification of* $\mathfrak{A}$ if $\mathfrak{B} \in \boldsymbol{C}_{\mathfrak{m}}\mathfrak{A}$ for each $\mathfrak{m}$ (in symbols $\mathfrak{B} \in \boldsymbol{C}\mathfrak{A}$), i.e. $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{B}$ is equationally compact.

First let us note a few properties of $\boldsymbol{C}_{\mathfrak{m}}$ and $\boldsymbol{C}$, similar to (i)-(vi) of Section 2.

(i) *If* $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$, *then* $\boldsymbol{C}_{\mathfrak{m}}\mathfrak{A}_1 \subseteq \boldsymbol{C}_{\mathfrak{m}}\mathfrak{A}_0$ *for each* $\mathfrak{m}$ *and* $\boldsymbol{C}\mathfrak{A}_1 \subseteq \boldsymbol{C}\mathfrak{A}_0$.

(ii) *If* $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$ *and* $\mathfrak{B} \in \boldsymbol{C}_{\mathfrak{m}}\mathfrak{A}$ $(\mathfrak{B} \in \boldsymbol{C}\mathfrak{A})$, *then* $\mathfrak{B} \in \boldsymbol{C}_{\mathfrak{m}}\mathfrak{C}$ $(\mathfrak{B} \in \boldsymbol{C}\mathfrak{C})$.

(iii) *If* $\mathfrak{m} \geqslant \mathfrak{n}$, *then* $\boldsymbol{C}_{\mathfrak{m}}\mathfrak{A} \subseteq \boldsymbol{C}_{\mathfrak{n}}\mathfrak{A}$.

(iv) *An algebra* $\mathfrak{A}$ *is equationally* $\mathfrak{m}$*-compact (equationally compact) if and only if for each* $\mathfrak{B} \subseteq \mathfrak{A}$, *we have* $\mathfrak{A} \in \boldsymbol{C}_{\mathfrak{m}}\mathfrak{B}$ $(\mathfrak{A} \in \boldsymbol{C}\mathfrak{B})$.

(v) *For each algebra* $\mathfrak{A}$ *and cardinal* $\mathfrak{m}$, *there is an algebra* $\mathfrak{B} \in \boldsymbol{C}_{\mathfrak{m}}\mathfrak{A}$, *which is an elementary extension of* $\mathfrak{A}$ *and* $|\mathfrak{B}| \leqslant |\mathfrak{A}|^{\mathfrak{m}}$ (see [3], Theorem 1).

(vi) *If* $\mathfrak{B} \in \boldsymbol{C}_{\mathfrak{m}}\mathfrak{A}$ *and* $|\mathfrak{B}| \leqslant \mathfrak{m}$, *then* $\mathfrak{B} \in \boldsymbol{C}\mathfrak{A}$ (by [3], Theorem 3).

(vii) $\boldsymbol{C}_{\mathfrak{m}}\mathfrak{A} \subseteq \boldsymbol{c}_{\mathfrak{m}}\mathfrak{A}$ *and* $\boldsymbol{C}A \subseteq \boldsymbol{c}\mathfrak{A}$.

THEOREM 3.1. *If* $\mathfrak{B} \in \boldsymbol{C}\mathfrak{A}$, *then there exists a* $\mathfrak{C} \subseteq \mathfrak{B}$ *such that* $\mathfrak{C} \in \boldsymbol{C}\mathfrak{A}$ *and* $\mathfrak{C} \in \mathcal{HSP}(\mathfrak{A})$ *and such is any maximal subalgebra of* $\mathfrak{B}$ *containing* $\mathfrak{A}$ *and belonging to* $\mathcal{HSP}(\mathfrak{A})$.

The proof is easy (a simplification of the proof of Theorem 2.3).

For completeness, let us state the following obvious proposition mentioned in the introduction.

PROPOSITION 3.2. *If* $\mathfrak{A}$ *is a subalgebra of a compact topological algebra* $\mathfrak{B}$, *then there is a compact topological algebra in* $\mathcal{HSP}(\mathfrak{A})$ *which contains* $\mathfrak{A}$, *and such is the topological closure of* $\mathfrak{A}$ *in* $\mathfrak{B}$.

The following problems are open. Let $\mathfrak{A}$ be a subalgebra of a weakly equationally compact algebra. Does there exist a weakly equationally compact algebra in $\mathcal{HSP}(\mathfrak{A})$ which contains $\mathfrak{A}$? All examples of algebras, which I know, satisfying $\boldsymbol{C}\mathfrak{A} = 0$, are such that $\boldsymbol{c}\mathfrak{A} = 0$. Does $\boldsymbol{C}\mathfrak{A} = 0$ imply $\boldsymbol{c}\mathfrak{A} = 0$ for all algebras?

Theorem 3.1 can be generalized to algebraic systems in the following way (the proof does not change essentially):

THEOREM 3.1′. *Let* $\mathfrak{A}$ *be a subsystem of an atomic compact algebraic system* $\mathfrak{B}$. *Then there is an atomic compact system* $\mathfrak{C} \subseteq \mathfrak{B}$ *such that* $\mathfrak{A} \subseteq \mathfrak{C}$ *and* $\mathfrak{C}$ *satisfies each universal positive sentence with constants in* $\mathfrak{A}$, *which is valid in* $\mathfrak{A}$.

I do not know if such a $\mathfrak{C}$ can be chosen as an elementary extension of $\mathfrak{A}$?

4. Equationally compact semigroups. Now, we shall apply Theorem 3.1 to prove the result on semigroups mentioned in the introduction. First we prove a special case of this theorem.

LEMMA 4.1. *An Abelian equationally compact semigroup with cancellation is a group.*

Proof. Let us consider the following set of equations:

$$\Sigma = \{\text{“}x = ay_a\text{”}:\ a \in \mathfrak{S}\}\,.$$

It is easy to see that each finite subset of Σ can be solved in $\mathfrak{S}$. Thus, by equational compactness of $\mathfrak{S}$, Σ can be solved in $\mathfrak{S}$. Let $x = d$ and $y_a = c_a$ be such a solution. Thus we have:

$$d = d \cdot c_d\,.$$

Using cancellation, we see that c_d is a unity of $\mathfrak{S}$.

Next, for each $b \in \mathfrak{S}$ we have

$$b \cdot d \cdot c_{bd} = d = d \cdot c_d\,,$$

and using cancellation again, we see that c_{bd} is an inverse of b. Thus $\mathfrak{S}$ is a group.

THEOREM 4.2. *An equationally compact semigroup with left and right cancellation is a group.*

Proof. It follows from 4.1, using 3.1, that such a semigroup is a union of Abelian groups. It is easy to check that their unit elements coincide, whence this is a group.

This proof, shorter than the earlier proof of the author, was found by A. Hulanicki.

In an analogous way, we can eliminate the topological assumptions from several theorems proved in [1] and [4] concerning semigroups, semirings, semimodules etc.

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