A toroidal decomposition of E^3

by

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1. Introduction. An upper semi-continuous decomposition G of E^3 is defined to be a *toroidal* decomposition if and only if the following condition holds: There is a sequence $M_1, M_2, M_3, \ldots$ of compact 3-manifolds-with-boundary in E^3 such that (1) for each i, $M_{i+1} \subset \operatorname{Int} M_i$ and each component of M_i is a solid torus (cube with one handle) and (2) A is a non-degenerate element of G if and only if A is a non-degenerate component of $M_1 \cdot M_2 \cdot M_3 \ldots$.

Bing proved in [3] that the union of two solid Alexander horned spheres, sewed together along their boundaries, was homeomorphic to S^3. A major step in this proof consists of showing that for a certain toroidal decomposition H of E^3 into tame arcs and one-point sets, the decomposition space associated with H is homeomorphic to E^3. Keldyš raised the following question in [7]:

Does every toroidal decomposition of E^3 into tame arcs and one-point sets yield E^3 as its decomposition space?

In this paper, we give a negative answer to this question.

The example of this paper provides another monotone decomposition of E^3 into tame arcs and one-point sets such that the associated decomposition space is topologically distinct from E^3. The first such example given was Bing's dogbone decomposition of E^3 [4].

It was proved by Bing in [5] that there is a toroidal decomposition of E^3 into point-like continua such that the associated decomposition spaces is not homeomorphic to E^3. In this particular example, each non-degenerate element is an indecomposable continuum and hence is not locally connected.

Keldyš has announced in [7] that if G is a toroidal decomposition of E^3 into locally connected point-like continua such that the set of all non-degenerate elements of G is a continuous collection, then the de-

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composition space is homeomorphic to E^3. She also described an example to show that if "locally connected" is deleted in the preceding statement, the resulting proposition is false. Another such example can be obtained by a simple modification of the example of Bing's mentioned in the preceding paragraph.

It follows from a theorem proved by Andrews and Rubin in [1] that if X is the space of the decomposition described in this paper, then $X \times X^1$ is homeomorphic to E^4.

In Section 2 of this paper, we describe the decomposition to be studied and in Section 3, we prove that its decomposition space is not homeomorphic to E^3. In Section 4, we show that each non-degenerate element of the decomposition is a tame arc.

If M is a manifold with boundary, then Int M and Bd M denote the interior and boundary, respectively, of M.

2. The example. Let J_1 be a circle in E^3 of circumference 1 and T_1 be a tubular neighborhood of J_1 of small cross radius.

Let $J_{11}, I_{12}, \ldots, J_{1n_1}$ be a chain of simple closed curves as shown in Figure 1 such that each J_{1i} has length 1, adjacent J_{1i}'s are linked as shown with J_{1n_1} linked to J_{11} so that as a chain they go around T_1 more than $n_1/2$ times. Let T_{1i} be a tubular neighborhood of J_{1i} of very small cross radius so that no two of the T_{1i}'s intersect and each lies in T_1.

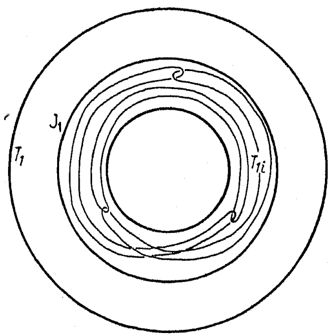

Fig. 1

For simplicity in drawing Figure 1, we showed n_1 as 3 but if the cross radius of T_1 were very small, n_1 would have to be more than 3 since each J_{1i} goes slightly more than half way around T_1 in order for the chain to go more than $n_1/2$ times around T_1. It is possible to find such an n_1 since there are cores of T_1 slightly shorter than J_1 and the J_{1i}'s can be placed close to such a shorter core.

In the interior of each T_{1i}, let $J_{1i1}, J_{1i2}, \ldots, J_{1in_{1i}}$ be a chain of simple closed curves which go around T_{1i} more than $n_{1i}/2$ times and such that each J_{1ij} is of length 1. Also, T_{1ij} is a tubular neighborhood of J_{1ij} of very very small cross radius so that no two of the T_{1ij}'s intersect and each lies in Int T_{1i}. Similarly we define T_{1ijk}'s in the interiors of the T_{1ij}'s more T's in these, and so on.

Let G be the collection whose elements are the components of $T_1 \cdot (\bigcup T_{1i}) \cdot (\bigcup T_{1ij}) \cdot (\bigcup T_{1ijk}) \ldots$. We note that G has a Cantor set of elements. The decomposition of E^3 that we study is the decomposition whose only non-degenerate elements are the non-degenerate elements of G.

3. The decomposition space is topologically distinct from E^3**.** In this section we show that the space of the decomposition described in Section 2 is not homeomorphic to E^3. We do this by studying homeomorphisms from T_1 onto T_1 that are pointwise fixed on Bd T_1.

Suppose that D_1 and D_2 are disjoint disks in T_1 such that if $r = 1$ or 2, Bd D_r is a meridional simple closed curve on Bd T_1 and Int $D_r \subset$ Int T_1. We shall prove that if h is any homeomorphism from T_1 onto T_1, pointwise fixed on Bd T_1, there is an element A of G such that $h[A]$ intersects both D_1 and D_2. From this it follows that the decomposition space is not homeomorphic to E^3, because if it were, there would be, according to Theorem 3 of [2], a homeomorphism f from T_1 onto T_1, pointwise fixed on Bd T_1 and such that if A is any element of G, $f[A]$ intersects at most one of D_1 and D_2.

Suppose that M is a solid torus. By a *homotopy centerline* of M we mean any simple closed curve in M homotopic in M to a core of M.

LEMMA 1. *If M is any solid torus lying in T_1 and m is any positive integer, then the following two statements are equivalent:*

(1) *Each homotopy centerline of M intersects both D_1 and D_2.*

(2) *If J is a simple closed curve in M that goes around M m times, then there is a sequence of distinct points $p_1, q_1, p_2, q_2, \ldots, p_m, q_m$ of J in the order $p_1 q_1 p_2 q_2 \ldots p_m q_m p_1$ on J such that each p_i belongs to D_1 and each q_i belongs to D_2.*

Proof. It is clear that (2) implies (1). Suppose that (1) holds but (2) fails. We may assume that D_1, D_2 and Bd M are polyhedral and if $r = 1$ or 2, Bd M and D_r are in general position. Each component of $(D_1 \cup D_2) \cdot$ Bd M is a simple closed curve. Further, by Theorem 1 of [5], each component of $(D_1 \cup D_2) \cdot$ Bd M either bounds a disk on Bd M, or circles Bd M once longitudinally and no times meridionally, or circles Bd M once meridionally and no times longitudinally.

No component of $(D_1 \cup D_2) \cdot$ Bd M is a longitudinal simple closed curve on Bd M. If there were such a simple closed curve, then each component of $(D_1 \cup D_2) \cdot \mathrm{Bd}\, M$ would either be longitudinal or bounds a disk on $\mathrm{Bd}\, M$, and there would be a homotopy centerline of M on $\mathrm{Bd}\, M$ that would miss $D_1 \cup D_2$. This contradicts (1), so no component of $(D_1 \cup D_2) \cdot \mathrm{Bd}\, M$ is longitudinal. Further, some component of $D_1 \cdot \mathrm{Bd}\, M$ is meridional on $\mathrm{Bd}\, M$, or else some homotopy centerline of M misses D_1. Similarly, some component of $D_2 \cdot \mathrm{Bd}\, M$ is meridional on $\mathrm{Bd}\, M$.

Let L_r $(r = 1, 2)$ be a component of $D_r \cdot \mathrm{Bd}\, M$ that is meridional on $\mathrm{Bd}\, M$ and is an innermost simple closed curve on D_r with that property. The component E_r of $D_r \cdot M$ containing L_r is a disk or punctured disk such that each boundary curve of E_r distinct from L_r bounds a disk on $\mathrm{Bd}\, M$.

By supposition, there is a simple closed curve J in M that goes around M m times but for which there is no sequence of points satisfying (2). There exist disjoint disks E_1' and E_2' such that if $r = 1$ of 2, then $\mathrm{Bd}\, E_r' = L_r$, $\mathrm{Int}\, E_r' \subset \mathrm{Int}\, M$, and $E_r' \cdot J = E_r \cdot J$. We construct E_r' by capping, on $\mathrm{Bd}\, M$, boundary curves of E_r distinct from L_r and then deforming the caps slightly into $\mathrm{Int}\, M$.

Since J goes around M m times, it can be shown that there exist m mutually disjoint open arcs $A_1, A_2, \ldots, A_m$ on J such that each $\bar{A}_i$ abuts on different sides of E_1' and each A_i is disjoint from E_1'. Since each such A_i intersects E_2 and the endpoints of $\bar{A}_i$ belong to E_1, there exist points $p_1, q_1, p_2, q_2, \ldots, p_m$ and q_m of J satisfying (2). This is a contradiction, and Lemma 1 is proved.

Lemma 2. *Suppose that h is a homeomorphism from T_1 onto T_1, pointwise fixed on $\mathrm{Bd}\, T_1$. Suppose that each homotopy centerline of $h[T_{1ij\ldots k}]$ intersects both D_1 and D_2. Then for some positive integer m, $m \leqslant n_{1ij\ldots k}$, each homotopy centerline of $h[T_{1ij\ldots km}]$ intersects both D_1 and D_2.*

Proof. Suppose that Lemma 2 is false. If $t = 1, 2, \ldots,$ or $n_{1ij\ldots k}$, let L_t be a homotopy centerline of $h[T_{1ij\ldots kt}]$ that is disjoint from one of D_1, D_2. Observe that $L_1, L_2, \ldots, L_{n_{1ij\ldots k}}$, as a chain, goes around $h[T_{1ij\ldots k}]$ more than $n_{1ij\ldots k}/2$ times.

Let T^* be the universal covering space of $h[T_{1ij\ldots k}]$ and let p be the projection map. Let L_1' be an image under p^{-1} of L_1, L_2' be the image under p^{-1} of L_2 that is linked with $L_1', \ldots, L_{n_{1ij\ldots k}}'$ the image under p^{-1} of $L_{n_{1ij\ldots k}}$ that is linked with $L_{(n_{1ij\ldots k})-1}'$, and L_1'' the image under p^{-1} of L_1 that is linked with $L_{n_{1ij\ldots k}}$. By Theorem 3 of [4], if $t = 1, 2, \ldots,$ or $(n_{1ij\ldots k})-1$, there is an arc K_t in T^* from a point of L_t' to a point of L_{t+1}' missing $p^{-1}(D_1 \cup D_2)$, and there is such an arc $K_{n_{1ij\ldots k}}$ from a point of $L_{n_{1ij\ldots k}}'$ to a point of L_1''. Near $L_1 \cup \ldots \cup L_{n_{1ij\ldots k}} \cup p[K_1] \cup p[K_2] \cup \ldots \cup$ $\cup\, p[K_{n_{1ij\ldots k}}]$ there lies a simple closed curve J such that (1) J goes around M more than $n_{1ij\ldots k}/2$ times and (2) J is the union of arcs $a_1, a_2, \ldots, a_{n_{1ij\ldots k}}$ such that these arcs chain J circularly and no one intersects both D_1

and D_2. Since J goes around M more than $n_{1ij\ldots k}/2$ times, this contradicts Lemma 1.

We now show that the decomposition space is not homeomorphic to E^3. Suppose that h is any homeomorphism from T_1 onto T_1, pointwise fixed on $\mathrm{Bd}\, T_1$. It is clear that each homotopy centerline of $h[T_1]$ intersects both D_1 and D_2. By induction and Lemma 2, there is a sequence $1, i, j, k, \ldots$ such that for each t, each homotopy centerline of $h[T_{1ij\ldots t}]$ intersects both D_1 and D_2. Let A be $T_1 \cdot T_{1i} \cdot T_{1ij} \cdot T_{1ijk} \ldots$. Then $h[A]$ intersects both D_1 and D_2.

As we noted in the second paragraph of this section, this is sufficient to show that the decomposition space is not homeomorpic to E^3.

4. Arcs in G are tame. Let A be a non-degenerate element of G. We show in this section that A is a tame arc.

Let $T_1, T_2, T_3, \ldots$ be the decreasing sequence of solid tori such that T_i is the torus at the ith stage in the definition of G which contains A. Let J_i be the core of T_i used in defining T_i in Section 2. We suppose that T_i is fibered into a simple closed curve of mutually exclusive disks such that each of these disks is of diameter less than $1/4^i$, each intersects J_i in precisely one point, and each which intersects T_{i+1} intersects it in precisely two fibers of T_{i+1} except possibly those fibers of T_i that can be joined in T_i to one of the two bends of J_{i+1} by an arc of length less than $1/4^{i+1}$. These exceptional fibers of T_i near the bends of J_{i+1} intersect T_{i+1} in either a point, a disk, the union of two tangent disks, or the union of two mutually exclusive disks which need not be fibers of T_{i+1}.

Let C_i be the union of the disks in the fibering of T_i that intersect A. In case where diameter $A \geqslant 1/4^i$, C_i is a cylinder.

Since A can be covered by the C's, it is snakelike. Since the length of the J's is 1, this snakelike continuum does not oscillate much and is an arc. We show that it is tame by showing that it is locally tame except possibly at its endpoints and has penetration index 1 there.

The fibers of C_{i+1} may not match up with the fibers of C_i near the bends of J_{i+1} so there is no assurance that $C_{i+1} \subset C_i$. Let C_i^+ be the union of all fibers of T_i that can be joined to C_i by an arc in T_i of length less than or equal to $1/4^i$. We find that $C_{i+1}^+ \subset \mathrm{Int}\, C_i^+$. See Figure 2.

It i is large, then except possibly near the ends of C_i, each of the disk in the fibering of C_i intersects C_{i+1}^+ in precisely a fiber of C_{i+1}. We explain why in the next three paragraphs.

In a certain sense, each C_{i+1} is longer than C_i but not much longer. Let $M(C_i) = L(C_i) - 1/4^i$ where $L(C_i)$ is the shortest length of an arc in C_i joining the two bases of C_i. Any arc in C_{i+1} joining the ends of C_{i+1} can be joined in C_i to either end of C_i by an arc in a fiber of C_{i+1}. Hence $L(C_i) < L(C_{i+1}) + 2/4^{i+1}$, and $M(C_1), M(C_2), \ldots$ is a monotonic increasing

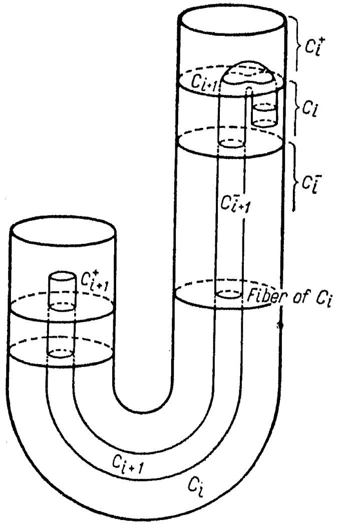

Fig. 2

sequence. Since $\{M(C_i)\}$ is bounded above by 1, it has a limit $M(A)$. Let $\varepsilon_i = M(A) - M(C_i)$. Then $M(C_i) < M(C_{i+j}) < M(C_i) + \varepsilon_i$ and $L(C_{i+j}) < L(C_i) + \varepsilon_i + 1/4^i$.

Each end of C_{i+1} can be joined in C_{i+1} by an arc of length less than $\varepsilon + 6/4^{i+1}$ to a fiber of C_{i+1} that intersects an end of C_i or else an arc in C_{i+1} of length less than $L(C_i)$ intersects the two ends of C_i. Any bend of J_{i+1} in C_i would be within $1/4^{i+1}$ of an end of C_i so each fiber of C_i that fails to intersect C_{i+1} in precisely a fiber of C_{i+1} can be joined to an end of C_i by an arc in C_{i+1} of length less than $\varepsilon_i + 7/4^{i+1}$.

Let C_i^- denote the union of all disks in the fibering of C_i such that each arc in C_i from such a disk to an end of C_i is of length greater than or equal to $\varepsilon_i + 2/4^i$. Then each fiber in C_i^- intersects C_{i+1}^+ only in a fiber of C_{i+1} which in turn is a fiber of C_{i+1}^-.

Some of the C_i^-'s may be disks or null but for convenience we suppose that each is a cylinder. We note that $A \cdot C_i^- = C_i^- \cdot C_{i+1}^- \cdot C_{i+2}^- \ldots$ and hence A intersects each fiber of C_i^- in precisely one point. Hence A is locally tame at each point of $A \cdot \operatorname{Int} C_i$. The only points of A that do not belong to some $\operatorname{Int} C_i$'s are the endpoints of A and the boundaries of the closures of the components of $C_i^+ - C_i^-$ show that A has penetration index 1 at each endpoint. It follows from results of [6] that A is a tame arc.

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