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Upper semi-continuous decompositions of irreducible continua

by

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Introduction. In 1935, B. Knaster showed [6] that there is a continuous collection of nondegenerate continua which is an arc with respect to its elements and such that the union of its elements is a compact irreducible continuum. In 1949, E. E. Moise [7] showed that there is no such collection with the additional property that each of its members is an arc. Moise's theorem was improved in 1952 by M. E. Hamstrom [4], and further improved in 1953 when E. Dyer showed [3] that there is no continuous collection of decomposable continua such that the union of the members is a compact irreducible continuum. It is known [6] that there is an upper semi-continuous collection of arcs which is an arc with respect to its elements and such that the union of its elements is a compact irreducible continuum. In this paper we consider an upper semi-continuous collection G having the following property.

(A) *If $g \in G$, each point of g is a limit point of the union of the members of each component of $G-g$.*

As a corollary to Theorem 4 of this paper we have that there is no upper semi-continuous collection G of arcs such that G has property (A), G is an arc with respect to its elements, and G^* is a compact irreducible metric continuum.

Continua with degenerate E-continua. If M is a compact, hereditarily decomposable, irreducible, metric continuum, M contains two continua such that each is the complement in M of a composant of M. These are called the *E-continua* of M by H. C. Miller [9]. We see from the following theorem that if M is chainable, M contains a continuum which has a degenerate E-continuum. The constructive proof of this theorem is essentially identical with that of G. W. Henderson for Theorem 13 of [5]. The theorem may also be compared with Theorem 6 of [1].

THEOREM 1. *If M is a hereditarily decomposable, compact, chainable metric continuum and P' and Q' are points of M and $\varepsilon > 0$, there is a sub-*

continuum K of M such that K is irreducible from a point P within a distance ε of P' to a point Q within a distance ε of Q' and irreducible from P to no point except Q.

Before beginning our proof of Theorem 1 (¹) we make some definitions and state several lemmas. For each of the lemmas of this section we assume that our space is compact and metric. If $\varepsilon > 0$, by an *ε-chain* is meant a finite ordered collection $d_1, d_2, \ldots, d_n$ of open sets each of diameter less than ε such that d_i intersects d_j if and only if $|i-j| \leqslant 1$. A *simple chain* is a chain such that no link is a subset of any other and such that each two non-intersecting links have non-intersecting closures. A chain D is said to be *embedded* in a chain E if and only if the closure of each link of D is a subset of some link of E. By a *subchain* D of a chain E is meant a chain each link of which is a link of E. It may of course be ordered in two ways. A chain *minimally covers* a set M if and only if it covers M but no proper subchain does so. By a *chain sequence* for a set M is meant a sequence $C_1, C_2, \ldots$ such that (1) for each $i > 0$, C_i is a simple $(1/i)$-chain minimally covering M, (2) for each $i > 0$, C_{i+1} is embedded in C_i, and (3) $M = \bigcap_{i=1}^{\infty} C_i^*$.

LEMMA 1. *If M is a chainable continuum there is a chain sequence for M.*

For a method of proof of Lemma 1, see the proof of Theorem 4 of [2].

The statement that the chain D loops in the chain E means that E has at least three links, D is embedded in E, the closure of the union of the end links of D is a subset of an end link of E and the other end link of E contains the closure of a link of D.

LEMMA 2. *If $C_1, C_2, \ldots$ is a sequence such that for each $i > 0$, C_i is a simple $(1/i)$-chain and C_{i+1} loops in C_i, then $\bigcap_{i=1}^{\infty} C_i^*$ is a nondegenerate compact indecomposable continuum.*

If C denotes a sequence of chains, by a *C-chain* is meant a subchain of a chain of the sequence C. If D is a chain, $F(D)$ and $L(D)$ denote the first and last links of D, respectively. By a *terminal subchain* of D is meant a subchain of D having $L(D)$ as its last link. A chain D is said to be *embedded in a chain E from $F(E)$* if and only if D is embedded in E and $\overline{F(D)} \subset F(E)$.

LEMMA 3. *If M is a chainable and hereditarily decomposable continuum and $C = C_1, C_2, \ldots$ is a chain sequence for M and B is a C-chain, then there is a C-chain D and a terminal subchain T of D such that* (1) *D is embedded in B from $F(B)$,* (2) *T has at least three links and is embedded in $L(B)$, and* (3) *if E is a C-chain embedded in D from $F(D)$, then no terminal subchain of E loops in T.*

(¹) A proof of Theorem 1 which is like that of Henderson is included for the sake of completeness at the suggestion of the referee.

Proof. Suppose that M is a chainable and hereditarily decomposable continuum, $C = C_1, C_2, \ldots$ is a chain sequence for M, and B is a C-chain. There is a C-chain B_1 and a terminal subchain T_1 of B_1 such that (1) B_1 is embedded in B from $F(B)$ and (2) T_1 has at least three links and is embedded in $L(B)$. Suppose that the lemma is false. It follows that there is a C-chain B_2 embedded in B_1 from $F(B_1)$ such that some terminal subchain of B_2 loops in T_1. Let T_2 denote one such terminal subchain. B_2 is a C-chain embedded in B from $F(B)$ and T_2 is a terminal subchain of B_2 which has at least three links and is embedded in $L(B)$. Thus there is a C-chain B_3 embedded in B_2 from $F(B_2)$ such that some terminal subchain T_3 of B_3 loops in T_2. This process may be continued to establish the existence of a sequence $T_1, T_2, \ldots$ of C-chains such that for each $i > 0$, T_{i+1} loops in T_i. It follows with the aid of Lemma 2 that $\bigcap_{i=1}^{\infty} T_i^*$ is a nondegenerate compact indecomposable continuum. But M contains no such continuum and the lemma follows from this contradiction.

We now begin our proof of Theorem 1. Let M denote a hereditarily decomposable, compact, chainable metric continuum. Suppose that P' and Q' are two points of M and that $\varepsilon > 0$. Let $C = C_1, C_2, \ldots$ denote a chain sequence for M. There is a C-chain B_1 which has at least three links such that $P' \in F(B_1)$, $Q' \in L(B_1)$, and each link of B_1 is of diameter less than ε. By Lemma 3, there is a C-chain B_2 and a terminal subchain T_2 of B_2 such that (1) B_2 is embedded in B_1 from $F(B_1)$, (2) T_2 has at least three links and is embedded in $L(B_1)$, and (3) if E is a C-chain embedded in B_2 from $F(B_2)$, then no terminal subchain of E loops in T_2. By repeated application of Lemma 3, we see that there is a sequence $B_1, B_2, \ldots$ and a sequence $T_2, T_3, \ldots$ such that for each $i > 0$, (1) B_i is a C-chain, (2) T_{i+1} is a terminal subchain of B_{i+1}, (3) B_{i+1} is embedded in B_i from $F(B_i)$, (4) T_{i+1} has at least three links and is embedded in $L(B_i)$, and (5) if E is a C-chain embedded in B_{i+1} from $F(B_{i+1})$, then no terminal subchain of E loops in T_{i+1}. Let $K = \bigcap_{i=1}^{\infty} B_i^*$, $P = \bigcap_{i=1}^{\infty} \overline{F(B_i)}$ and $Q = \bigcap_{i=1}^{\infty} \overline{L(B_i)}$. K is a compact continuum irreducible from P to Q, and, since $(P+P') \subset F(B_1)$, the distance from P to P' is less than ε. Similarly, the distance from Q to Q' is less than ε. Let E denote the set of all points x such that K is irreducible from P to x, and suppose that E is nondegenerate. Let $A \in E-Q$. Let n denote a positive integer such that no link of B_n contains both A and Q. By Theorems 37 and 116 of Chapter I of [8], Q is a limit point of $K-E$, and $L(B_{n+2})$ contains Q, so $L(B_{n+2})$ contains a point B in $K-E$. By Theorem 116 of Chapter I of [8], E is a continuum, so there is a positive integer m such that no link of B_m contains both B and a point of E. B_m is embedded in B_{n+1} and there exist three distinct links U_A, U_B, and $U_Q = L(B_m)$ of B_m containing A, B, and Q, respectively. Since $\overline{F(B_m)} \subset F(B_{n+1})$ and $\overline{U}_B \subset L(B_{n+1})$, there is a link U of B_m which precedes U_B such that $\overline{U} \subset F(T_{n+1})$ and the closure of each link of B_m between U and U_B is a subset of a link of T_{n+1}. U_B does not intersect the connected set E, so U_A is between the links U_B and U_Q in B_m. Further since A is not in $L(B_n)$, $\overline{U}_A$ is a subset of a link in B_{n+1} which is not in T_{n+1}. Thus there is a link V of B_m which follows U_B such that $\overline{V} \subset F(T_{n+1})$ and each link of B_m between V and U_B has a closure which is a subset of a link of T_{n+1}. But we now have that the subchain of B_m having $F(B_m)$ as its first link and V as its last link is a C-chain embedded in B_{n+1} from $F(B_{n+1})$ which has a terminal subchain which loops in T_{n+1}. This is a contradiction from which it follows that E is degenerate.

Upper semi-continuous collections with property A. In the following let G denote an upper semi-continuous collection of mutually disjoint continua such that G has property A, G is an arc with respect to its elements, and G^* is a compact metric continuum.

THEOREM 2. *If G^* is irreducible between two of its points and M is a subcontinuum of G^* which is not a subset of a member of G, there is a subarc H of G such that $H^* = M$. Moreover, if A and B are points of different end elements of H, then M is irreducible from A to B.*

Proof. Suppose that G^* is irreducible and M is a subcontinuum of G^* which is not a subset of a member of G. There is a subarc H of G with end elements h and k each intersecting M and such that no member of $G-H$ intersects M. We shall show that each subcontinuum of M which intersects both h and k must contain H^* from which it follows that $M = H^*$ and that M is irreducible from each point of h to each point of k. Let M' denote a subcontinuum of M which intersects both h and k, and let K denote the union of M' with those members of G not between h and k. K is a subcontinuum of G^* which contains the end elements of G and thus is not a proper subset of G^*. So each member of G between h and k is a subset of K, and thus of M'. But G has property A, so both h and k must be contained in M' and we have that $H^* \subset M'$.

We next note that with certain conditions on the elements of G, G^* is chainable.

THEOREM 3. *If G^* is irreducible and each element of G is hereditarily decomposable and hereditarily irreducible, then G^* is chainable and hereditarily decomposable.*

Proof. If M is a nondegenerate subcontinuum of G^* contained in a member of G, then M is decomposable and irreducible by our hypothesis. If M intersects two members of G, then M is decomposable and irreducible by Theorem 2. Thus G^* is hereditarily decomposable and hereditarily irreducible. It then follows from a theorem of H. C. Miller [9] that G^* is atriodic and hereditarily unicoherent and from a theorem of R. H. Bing [2] that G^* is chainable.

THEOREM 4. *If each member of G is nondegenerate, hereditarily decomposable and hereditarily irreducible, then G^* is not an irreducible continuum.*

Proof. If G^* is irreducible, then by Theorem 3, G^* is chainable and hereditarily decomposable. By Theorem 1, G^* must contain a continuum M intersecting two members of G such that M is irreducible between two points P and Q but not irreducible from P to any point except Q. But by Theorem 2, there must be two elements h and k of G such that M is irreducible from each point of h to each point of k. It follows that P is in one of h or k and that the other of these sets is degenerate, contrary to our hypothesis.

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