Continua which admit only the identity mapping onto non-degenerate subcontinua

by

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1. Introduction. J. de Groot has raised, [5], the question, "Does there exist a connected set which cannot be mapped continuously and non-degenerately on any proper subset?". R. D. Anderson has raised, [2], the questions, "Does there exist a non-degenerate continuum which admits only the identity or a constant mapping into itself? If so, does there exist one, all of whose non-degenerate subcontinua have this property?". And R. L. Moore has asked the author (in conversation) whether there exists an hereditarily indecomposable continuum no two of whose non-degenerate subcontinua are homeomorphic. These questions are all answered in the affirmative in Theorems 8, 9, and 10 of this paper.

The author has been encouraged to work on these results by Professors R. D. Anderson, R. L. Moore, and K. Borsuk. The constructions, Theorems 6 and 7 of this paper, are reminiscent of those of Anderson and Choquet, [1], of continua no two of whose non-degenerate subcontinua are homeomorphic. An earlier manuscript of a paper containing the example M_2 of section 3 was read by Mr. Bobby E. Wilder, a mathematics student at Auburn University, and by Doctor J. Mioduszewski, both of whom made useful suggestions toward eliminating errors; the notion of a preatomic mapping was suggested to the author by Doctor Mioduszewski as an instrument to circumvent one of those errors.

The author's paper, [4], contains theorems which were originally established for use in this chapter and are, indeed, essential technical theorems for this paper. Some of the terminology (e.g. solenoid, poly-adic solenoid, circle-like continuum, poly-adic circle-like continuum) and notations used in that paper are used, without being redefined, herein.

2. Preatomic, atomic and confluent mappings and inverse mapping systems. In this section are established some useful theorems on mappings. Theorems 6 and 7 establish the major constructions of this paper.

Definitions. Except where otherwise noted, the term *mapping* will mean continuous single-valued transformation. If f is a mapping of a continuum X onto a continuum Y and, for each subcontinuum K of X such that $f(K)$ is non-degenerate $K = f^{-1}(f(K))$, then f is said to be *preatomic*; if f is monotone and preatomic and maps X onto Y, then f is said to be *atomic*. If f is a mapping of a topological space X onto a topological space Y such that, for every subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped by f onto Q, then f is said to be *confluent*, [3].

Theorem 1. *Suppose that $\{X_n, \pi_n^m\}$ and $\{Y_n, \sigma_n^m\}$ are two inverse mapping systems such that, for each n, X_n and Y_n are compact (not necessarily metric) continua, π_n^m maps X_m into Y_n and σ_n^m maps Y_m into Y_n $(n < m)$, and let X_∞ and Y_∞ denote their respective inverse limit spaces. If, for each n, ζ_n is a preatomic mapping of X_n into Y_n and $\zeta_n(\pi_n^{n+1}) = \sigma_n^{n+1}(\zeta_n)$, then the mapping ζ of X_∞ into Y_∞ induced by the sequence $\zeta_1, \zeta_2, \ldots$ is also preatomic.*

Proof. Suppose that K is a subcontinuum of X_∞ such that $\zeta(K)$ is non-degenerate and $\zeta^{-1}(\zeta(K)) \neq K$. Now, K is a subset of $\zeta^{-1}(\zeta(K))$, thus there is a point x of $\zeta^{-1}(\zeta(K))$ not in K. There exists a positive integer m such that $\pi_m(x)$ is not in $\pi_m(K)$ and a positive integer n such that $\sigma_n(\zeta(K))$ is non-degenerate—let i denote the greater of m and n. Then $\zeta_i^{-1}(\zeta_i(\pi_i(K))) = \pi_i(K)$ and, hence, does not contain $\pi_i(x)$. Then $\zeta_i(\pi_i(x))$ is not in $\sigma_i(\zeta(K))$ and $\zeta(x)$ is not in $\zeta(K)$, a contradiction.

Theorem 2. *Suppose that $\{X_n, \pi_n^m\}$ is an inverse mapping system such that, for each n, X_n is a compact continuum and π_n^{n+1} is an atomic mapping of X_{n+1} onto X_n, and let X_∞ denote the inverse limit of that system. Then, for each n, the projection mapping π_n of X_∞ onto X_n is atomic.*

Referee's proof. "It suffices to replace the system $Y_1 \leftarrow Y_2 \leftarrow \ldots$ [in Theorem 1] by the system of identities $X_n \leftarrow X_n \leftarrow \ldots$ (with the limit X_n) and, instead of ζ_m: $X_m \to Y_m$ to set π_n^m: $X_m \to X_n$, where $m \geqslant n$." And, thus, from Theorem 1, we see that π_n is preatomic. Since π_n is also monotone and onto, it is atomic.

Theorem 3. *Suppose that f is an atomic mapping of the compact continuum X onto the compact, hereditarily indecomposable continuum Y, and, for each point y of Y, $f^{-1}(y)$ is either degenerate or hereditarily indecomposable. Then X is hereditarily indecompossable.*

Proof. Suppose that K_1 and K_2 are two intersecting non-degenerate subcontinua of X such that neither contains the other. Then $f(K_1+K_2) = f(K_1)+f(K_2)$ and is a non-degenerate indecomposable continuum (if it were degenerate, $f^{-1}(f(K_1+K_2))$ would be an hereditarily indecomposable continuum containing the decomposable continuum K_1+K_2). Now, one of $f(K_1)$ and $f(K_2)$ is a subcontinuum of the other, suppose that $f(K_1)$

is a subcontinuum of $f(K_2)$. Then $K_2 = f^{-1}(f(K_2)) = f^{-1}(f(K_1+K_2)) = K_1+K_2$, a contradiction.

Theorem 4. *If f is a mapping of the compact metric continuum X onto the hereditarily indecomposable continuum Y, then f is confluent.*

Proof. Suppose that Q is a subcontinuum of Y and C is a component of $f^{-1}(Q)$. Let $C_1, C_2, C_3, \ldots$ be a sequence of subcontinua of X such that (1) for each n, C_{n+1} is a subcontinuum of C_n; (2) for each n, C is a proper subcontinuum of C_n; and (3) C is the common part of continua of that sequence. Then, for each n, $f(C_n)$ contains a point not in Q and intersects Q; hence, $f(C_n)$ contains Q. The common part of $f(C_1), f(C_2), \ldots$ is $f(C)$ and contains Q, but Q contains $f(C)$. Then $f(C) = Q$.

The following theorem can be established with an argument similar to that for Lemma 1 of [1].

Definition. An *A^*-map* is an atomic mapping f of a compact continuum X onto a compact continuum Y such that there do not exist infinitely many points y of Y for which $f^{-1}(y)$ is non-degenerate, [2].

Theorem 5. *There exists an inverse mapping system $\{X_n, \pi_n^m\}$ such that X_1 is a simple closed curve in the plane and, for each n, (1) X_n is a bounded plane continuum; (2) π_n^{n+1} is an A^*-map such that, if x is a point of X_n, $\operatorname{inv}\pi_n^{n+1}(x)$ is non-degenerate, and T is an arc lying in X_n and containing x, then $\operatorname{inv}\pi_n^{n+1}(x)$ is a simple closed curve and $\operatorname{inv}\pi_n^{n+1}(T)$ spirals down on $\operatorname{inv}\pi_n^{n+1}(x)$; (3) if J is a simple closed curve lying in X_n then $\operatorname{inv}\pi_n^{n+1}(J)$ is not a simple closed curve; and (5) if T is an arc lying in X_n, there is an integer $m > n$ such that $\operatorname{inv}\pi_n^m(T)$ is not an arc.*

The inverse limit of an inverse mapping system such as in Theorem 5 might be a planar continuum each non-degenerate subcontinuum of which separates the plane, such as the example of G. T. Whyburn in [8].

Notations. If α is an ordered pair (i, j) of positive integers, denote i by $n_1(\alpha)$, denote j by $n_2(\alpha)$, denote the ordered pair $(i+1, j)$ by α^* and denote the ordered pair $(i, j+1)$ by α'—note that $\alpha^{*\prime} = \alpha'^{*} = (i+1, j+1)$. Let A denote the set of all ordered pairs of positive integers directed by the relation $\precsim$ such that $\alpha \precsim \beta$ if and only if α and β are two elements of A such that either $n_1(\alpha) < n_1(\beta)$ or $n_1(\alpha) = n_1(\beta)$ and $n_2(\alpha) < n_2(\beta)$.

Theorem 6. *There exists an inverse mapping system $\{X_n, \pi_n^m\}$ such that (1) X_1 is a solenoid and, for each n, X_n is a compact, metric continuum; (2) for each n and m $(n < m)$, π_n^m is an atomic mapping and, if x is a point of X_n, $\operatorname{inv}\pi_n^{n+1}(x)$ either is degenerate or is a solenoid; (3) if n is a positive integer and T is an arc lying in X_n, then there is an integer $m >$ such that $\operatorname{inv}\pi_n^m(T)$ contains a solenoid; (4) for each n, each solenoid lying in X_n is poly-adic; and (5) if Σ_1 is a solenoid lying in X_n and Σ_2 is a solenoid lying in X_m and there is an upper semi-continuous mapping f of Σ_1 onto Σ_2 such* *that, for each point x of Σ_1, $f(x)$ is a proper subcontinuum of Σ_2, then $n = m$ and $\Sigma_1 = \Sigma_2$.*

Proof. Let ϱ be a reversible function from the set of all ordered triples of positive integers onto a set of positive prime integers (if a is the ordered pair (i, j) in A and k is a positive integer, we shall write $\varrho(a, k)$ for $\varrho(i, j, k)$). There exists an inverse mapping system $\{C_\alpha, \sigma_\alpha^\beta\}$ over A and, for each α in A, a positive integer k_α and a finite sequence $J_1(\alpha), J_2(\alpha), \ldots, J_{k_\alpha}(\alpha)$ of simple closed curves such that (1) for each α in A, C_α is a bounded plane continuum; (2) if $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a finite sequence of elements of A such that, for each $i < n$, α_{i+1} is either α_i^* or α_i', then $\sigma_{\alpha_1}^{\alpha_n}$ is the composite mapping $\sigma_{\alpha_1}^{\alpha_2} \circ \sigma_{\alpha_2}^{\alpha_3} \circ \ldots \circ \sigma_{\alpha_{n-1}}^{\alpha_n}$; (3) if α is in A, k_α is the number of simple closed curves which are subcontinua of C_α and $J_1(\alpha)$, $J_2(\alpha), \ldots, J_{k_\alpha}(\alpha)$ are those simple closed curves; (4) if α is in A and $n_1(\alpha) = 1$, then C_α is a simple closed curve and $\sigma_\alpha^{\alpha'}$ is a $\varrho(\alpha, 1)$ to 1 local homeomorphism of $C_{\alpha'}$ onto C_α; (5) for each integer i, the inverse mapping system $\{C_{(n,i)}, \sigma_{(n,i)}^{(m,i)}\}$ satisfies all the conditions on the system of Theorem 5; and (6) if α is in A and x is a point of C_α, then (a) $\operatorname{inv} \sigma_\alpha^{\alpha^{*\prime}}(x)$ is the image under a homeomorphism, $h_{\alpha x}$, of the Cartesian product $[\operatorname{inv} \sigma_\alpha^{\alpha^*}(x)] \times [\operatorname{Inv} \sigma_\alpha^{\alpha'}(x)]$, (b) if $\operatorname{inv} \sigma_\alpha^{\alpha^*}(x)$ is a single point, a, and b is a point of $\operatorname{inv} \sigma_\alpha^{\alpha'}(x)$, then $\sigma_{\alpha^*}^{\alpha^{*\prime}}(h_{\alpha x}(a, b)) = a$, (c) if $\operatorname{inv} \sigma_\alpha^{\alpha^*}(x)$ is the simple closed curve $J_i(\alpha^*)$ $(i \leqslant k_{\alpha^*})$, and b is a point of $\operatorname{inv} \sigma_\alpha^{\alpha'}(x)$, then $\sigma_{\alpha^*}^{\alpha^{*\prime}} | h_{\alpha x}([J_i(\alpha^*)] \times \times \{b\})$ is a $\varrho(Y^*, i)$ to 1 local homeomorphism of the simple closed curve $h_{\alpha x}([J_i(\alpha^*)] \times \{b\})$ onto $J_i(\alpha^*)$, and (d) if a is a point of $\operatorname{inv} \sigma_\alpha^{\alpha^*}(x)$ and b is a point of $\operatorname{inv} \sigma_\alpha^{\alpha'}(x)$, then $\sigma_{\alpha'}^{\alpha^{*\prime}}(h_{\alpha x}(a, b)) = b$. If, for some such system $\{C_\alpha, \sigma_\alpha^\beta\}$, $\{X_n, \pi_n^m\}$ is the inverse mapping system over the set of positive integers such that (1) for each n, X_n is the inverse limit of the subsystem $\{C_\gamma, \sigma_\gamma^\delta\}$ of $\{C_\alpha, \sigma_\alpha^\beta\}$ for which $n_1(\gamma) = n_1(\delta) = n$, and (2) for each n and m $(n < m)$, π_n^m is the mapping from X_m onto X_n induced by the sequence $\sigma_{(1,n)}^{(1,m)}, \sigma_{(2,n)}^{(2,m)}, \ldots$, then conditions (1), (3), (4) of Theorem 6 hold for that system since each π_n^m is preatomic, monotone, and onto, condition (2) also holds, and that condition (5) holds is a direct consequence of Theorem 8 of [4].

THEOREM 7. *There exists an inverse mapping system $\{Y_n, \xi_n^m\}$ such that (1) Y_1 is an hereditarily indecomposable circle-like continuum; (2) for each n, Y_n is an hereditarily indecomposable compact metric continuum; (3) for each n and m $(n < m)$, ξ_n^m is an atomic mapping of Y_m onto Y_n and, if y is a point of Y_n, $\operatorname{inv} \xi_n^{n+1}(y)$ either is degenerate or is an hereditarily indecomposable circle-like continuum; (4) if n is a positive integer and P is a pseudo-arc lying in Y_n, then there is an integer $m > n$ such that $\operatorname{inv} \xi_n^m(P)$ contains an hereditarily indecomposable circle-like continuum; (5) for each n_1 each circle-like continuum lying in Y_n is paly-adic; and (6) if Σ_1 is a circle-like continuum in Y_n and Σ_2 is a circle-like continuum in Y_m and, if there is an upper semi-continuous mapping f of Σ_1 onto Σ_2 such that, for each point y of Σ_1, $f(y)$ is a proper subcontinuum of Σ_2, then $n = m$ and $\Sigma_1 = \Sigma_2$.*

Outline of proof. We again wish to construct an inverse mapping system $[C_\alpha, \sigma_\alpha^\beta\}$ over A as in the proof of Theorem 6, altering the mappings $\sigma_\alpha^{\alpha^*}$, for each α, so that, for each n, the circle-like continua which appear in Y_n, the inverse limit of the subsystem $\{C_\gamma, \sigma_\gamma^\delta\}$ of $\{C_\alpha, \sigma_\alpha^\beta\}$ for which $n_1(\gamma) = n_1(\delta) = n$, are hereditarily indecomposable instead of being solenoids. For each α and simple closed curve J in α^*, this can be done by requiring sufficient crookedness on the mapping $\sigma_\alpha^{\alpha^*} | J$. However, in trying to place $C_{\alpha^{*\prime}}$, $\sigma_{\alpha^*}^{\alpha^{*\prime}}$ and $\sigma_{\alpha'}^{\alpha^{*\prime}}$ into the diagram

$$\begin{array}{ccc} C_\alpha & \leftarrow & C_{\alpha^*} \\ \uparrow & & \\ C_{\alpha'} & & \end{array}$$

we cannot follow the precise procedure of the proof of Theorem 6. Instead, we fit a continuum D and mappings $f_{\alpha 1}$ and $f_{\alpha 2}$ into a diagram

$$\begin{array}{ccc} C_\alpha & \leftarrow - & C_{\alpha^*} \\ \uparrow & f_{\alpha 1} & \uparrow f_{\alpha 2} \\ C_{\alpha'} & \longleftarrow & D \end{array}$$

in precisely the same manner that $C_{\alpha^{*\prime}}$, $\sigma_{\alpha'}^{\alpha^{*\prime}}$ and $\sigma_{\alpha^*}^{\alpha^{*\prime}}$ were fit in the proof of Theorem 6, with, for each simple closed curve J of D, $f_{\alpha 1} | J$ being a local homeomorphism. We then place $C_{\alpha^{*\prime}}$ and $f_{\alpha 3}$ into the diagram as follows

$$\begin{array}{ccccc} C_\alpha & \longleftarrow & C_{\alpha^*} & & \\ \uparrow & & \uparrow & & \\ C_{\alpha'} & \longleftarrow & D & \overset{f_{\alpha 3}}{\longleftarrow} & C_{\alpha^{*\prime}} \end{array}$$

where $C_{\alpha^{*\prime}}$ is homeomorphic to D but, for each simple closed curve J of $C_{\alpha^{*\prime}}$, $f_{\alpha 3} | J$ has degree one but is a crooked map. We thus obtain a system $\{Y_n, \xi_n^m\}$ such that (1) for each n, Y_n is the inverse limit of the subsystem $\{C_\gamma, \sigma_\gamma^\delta\}$ of $\{C_\alpha, \sigma_\alpha^\beta\}$ for which $n_1(\gamma) = n_1(\delta) = n$, and (2) for each n and m $(n < m)$, ξ_n^m is the mapping of X_m onto X_n induced by the sequence $\sigma_{(1,n)}^{(1,m)}, \sigma_{(2,n)}^{(2,m)}, \ldots$ Then conditions (1), (4) and (5) of Theorem 7 hold, and, since each ξ_n^m is preatomic, monotone, and onto, condition (3) holds. Condition (3) together with Theorem 3 implies that condition (2) holds. That condition (6) holds is a consequence of Theorem 8 of [4].

3. The continua M_1 and M_2.

Let M_1 denote the inverse limit of an inverse mapping system $\{Y_n, \xi_n^m\}$ as in Theorem 7 and let M_2 denote the inverse limit of an inverse mapping system $\{X_n, \pi_n^m\}$ as in Theorem 6.

THEOREM 8. *If H is a subcontinuum of M_1 and f is a mapping of H onto the non-degenerate subcontinuum K of M_1, then $H = K$ and f is the identity mapping of H onto itself.*

Proof. Suppose that H is a non-degenerate subcontinuum of M_1 and f is a mapping of H onto the non-degenerate subcontinuum K of M_1.

Suppose that H does not intersect K. Let n denote the least positive integer i such that $\xi_i(K)$ contains a circle-like continuum and let Σ denote a circle-like continuum lying in $\xi_n(K)$. Let C denote a component of $[\xi_n \circ f]^{-1}(\Sigma)$. Then, by Theorem 4, $\xi_n f(C) = \Sigma$. Let C' denote a subcontinuum of C such that $\xi_n(f(C')) = \Sigma$ but, if C'' is any proper subcontinuum of C', $\xi_n(f(C''))$ is a proper subcontinuum of Σ. Let m be the least positive integer j such that $\xi_j(C')$ is non-degenerate. For each point x of $\xi_m(C')$, let $\tau(x) = \xi_n(f(\operatorname{inv}\xi_m(x)))$; then τ is an upper semi-continuous mapping of $\xi_m(C')$ onto Σ such that, for each point x of $\xi_m(C')$, $\tau(x)$ is a proper subcontinuum of Σ. Now, Σ is a poly-adic circle-like continuum; hence, by Theorem 7 of [4], $\xi_m(C')$ is not chainable. Thus $\xi_m(C')$ is circle-like. By Theorem 7 of this paper, $m = n$ and $\xi_m(C') = \Sigma$. Since H and K are mutually exclusive and ξ_m is monotone, $\xi_n(H)$ and $\xi_n(K)$ are mutually exclusive, a contradiction since both contain Σ.

Suppose that x is a point of H and $f(x)$ is not. Then there is a domain D with respect to K containing $f(x)$ whose closure does not intersect H. Then, if H' is the component containing x of the closure of $f^{-1}(D)$, $f(H')$ contains $f(x)$ and a point of the boundary with respect to K of D, and H' and $f(H')$ are mutually exclusive non-degenerate subcontinua of M_1 and $f|H'$ maps H' onto $f(H')$, a contradiction. Thus K is a subcontinuum of H.

Suppose that x is a point of K such that $f(x) \neq x$. There exists a domain D with respect to K containing $f(x)$ such that the closure of D and $f^{-1}(D)$ do not intersect. Let H' be the component of the closure of $f^{-1}(D)$ containing x; then $f(H')$ contains both $f(x)$ and a point of the boundary with respect to K of D, again a contradiction. Thus f is a retraction.

Let k denote the least positive integer j such that $\xi_j(H)$ is non-degenerate. Since ξ_k is an atomic mapping of H onto $\xi_k(H)$, there is no proper subcontinuum H' of H such that $\xi_k(H') = \xi_k(H)$. Thus H is indecomposable ([6], p. 146). Suppose that K is a proper subcontinuum of H and let x and y be two distinct points of K. Let L be a composant of H not containing K and let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ be sequences of points of L converging to x and y respectively. For each n, let K_n be a proper subcontinuum of H containing both x_n and y_n. Then, for some n, $f(K_n)$ is non-degenerate, but $f(K_n)$ does not intersect K_n, a contradiction. Thus H is K and f is the identity mapping of H onto itself.

THEOREM 9. *There exists an hereditarily indecomposable continuum no two of whose non-degenerate subcontinua are homeomorphic.*

Clearly, M_1 is such a continuum.

Although Theorems 8 and 9 completely resolve the main problems under attack in this paper, the continuum M_2, which the author devised some considerable time before he devised M_1, has some special properties which seem worthy of study.

THEOREM 10. *If H is a subcontinuum of M_2 and f is a mapping of H onto a non-degenerate subcontinuum K of M_2, then K is a subcontinuum of H and f is a retraction.*

Proof. Suppose that H is a non-degenerate subcontinuum of M, and f is a mapping of H onto the non-degenerate subcontinuum K of M_2.

Suppose that H does not intersect K. Let n denote the least positive integer i such that $\pi_i(K)$ contains a solenoid and let Σ denote a solenoid lying in $\pi_n(K)$. Let H' denote a subcontinuum of H such that $\pi_n(f(H'))$ contains Σ but, if H'' is any proper subcontinuum of H', $\pi_n(f(H''))$ does not contain Σ. Let m denote the least positive integer j such that $\pi_j(H')$ is non-degenerate. For each point x of $\pi_m(H')$, let $\tau(x) = \pi_n(f(\operatorname{inv}\pi_m(x)))$. If L is a proper subcontinuum of $\pi_m(H')$, $\operatorname{inv}\pi_m(L)$ is a proper subcontinuum of H' and, thus, $\tau(L)$ does not contain Σ. But then $\tau(L)$ is either a proper subcontinuum of Σ or does not intersect Σ. Now, if $\pi_m(H')$ is the sum of two proper subcontinua L_1 and L_2, one of the two sets $\tau(L_1)$ and $\tau(L_2)$ is a proper subcontinuum of Σ and the other intersects that one, in which case both $\tau(L_1)$ and $\tau(L_2)$ are proper subcontinua of Σ. Then $\tau(L_1)+\tau(L_2) = \Sigma$, but, since Σ is an indecomposable continuum, this is a contradiction. Thus, $\pi_m(H')$ is an indecomposable continuum, and, since it is either an arc or a solenoid, it is a solenoid; denote $\pi_m(H')$ by Σ'. There is a point, x, of Σ' such that $\tau(x)$ is a subcontinuum of Σ and, if y is a point of the composant, C, of Σ' containing x and L is a proper subcontinuum of Σ' lying in C and containing both x and y, then $\tau(L)$ is a proper subcontinuum of Σ. Then $\tau(C)$ is a subset of Σ, and, since the closure of C is Σ', $\tau(\Sigma')$ is a subset of Σ. Thus $\tau(\Sigma') = \Sigma$. Then, by Theorem 6, $n = m$ and $\Sigma' = \Sigma$. Since H and K are mutually exclusive and π_n is monotone, $\pi_n(H)$ and $\pi_n(K)$ are mutually exclusive, a contradiction since both contain Σ.

Now, exactly as in the proof of Theorem 8, it follows that K is a subcontinuum of H and f is a retraction.

THEOREM 11. *The identity is the only mapping of M_2 onto a non-degenerate subcontinuum of M_2.*

Proof. Since π_1 is atomic and $\pi_1(M_2) = X_1$ is indecomposable, then, as in the proof of Theorem 8, M_2 is indecomposable. Now, if f is a mapping of M_2 onto a non-degenerate subcontinuum of M_2, f is a retraction, and, since M_2 is indecomposable, as in the proof of Theorem 8, we can show that $f(M_2) = M_2$ and, thus, f is the identity mapping of M_2 onto itself.

THEOREM 12. *If H is a non-degenerate subcontinuum of M_2, H contains a continuum K which can be retracted onto a non-degenerate proper subcontinuum L.*

Proof. Let H be a non-degenerate subcontinuum of M_2 and n a positive integer such that $\pi_n(H)$ is non-degenerate. There exists in $\pi_n(H)$ an arc axb such that $\mathrm{inv}\,\pi_n(x)$ is degenerate, thus $\mathrm{inv}\,\pi_n(x)$ is a separating point of $\mathrm{inv}\,\pi_n(axb) = K$. Then K can be retracted to $\mathrm{inv}\,\pi_n(ax) = L$, where ax denotes the appropriate subarc of axb.

NOTE. Since each non-degenerate subcontinuum of either M_1 or M_2 contains a continuum which can be mapped onto a poly-adic solenoid and no plane continuum can be mapped onto a polyadic solenoid, [7], no non-degenerate subcontinuum of either M_1 or M_2 can be embedded in the plane. Since all of the continua involved in the constructions of M_1 and M_2 as inverse limit spaces are one-dimensional, then M_1 and M_2 are one-dimensional and, hence, can be embedded in E^3.

4. Some other continua. In a conversation in which the author mentioned the continuum M_2, G. S. Young asked him whether there exist continua with more than one, but only a finite number, of mappings onto non-degenerate subcontinua and whether there exists a continuum N such that the space of all mappings of N onto non-degenerate subcontinua of N is topologically equivalent to the Cantor set. These questions are answered in this section.

THEOREM 3. *If n is a positive integer, there exists a compact metric continuum H_n, with an atomic mapping onto a simple closed curve, such that there exist n, and only n, mappings of H_n onto H_n, each of them is a homeomorphism, and there exists no mapping of H_n onto a non-degenerate proper subcontinuum.*

Proof. Let n be a positive integer and ab be an arc in $\pi_1(M_2)$ such that $\mathrm{inv}\,\pi_1(a)$ and $\mathrm{inv}\,\pi_1(b)$ are degenerate. For each $i \leqslant n$, let h_i be a homeomorphism of $\mathrm{Inv}\,\pi_i(ab)$ onto a continuum K_i such that (1) K_i intersects K_j $(i < j < n)$ if and only if $j-i$ is either 1 or $n-1$, in which case $K_i \cdot K_j$ is degenerate (or, if $n = 2$, $K_1 \cdot K_2$ has two points), and (2) $h_n(b) = h_1(a)$ and, if $i < n$, $h_i(b) = h_{i+1}(a)$. Then the continuum $H_n = K_1 + K_2 + \ldots + K_n$ is a continuum as described in Theorem 6.

THEOREM 14. *There exists a compact metric continuum N such that* (1) *each mapping of N onto a non-degenerate subcontinuum of N is a homeomorphism of N onto N, and* (2) *the space of all homeomorphism of N onto N is topologically equivalent to the Cantor set.*

Proof. For each n, let Q_n be the continuum H_{2^n} in the proof of Theorem 6 (using the same arc ab, for each n) and let σ_n^{n+1} be a 2 to 1 local homeomorphism of Q_{n+1} onto Q_n. Let N be the inverse limit of the inverse mapping system $\{Q_n, \sigma_n^m\}$. It can easily be shown that every mapping of N onto a non-degenerate subcontinuum of N is a homeomorphism of N onto N. Now, N is the sum of c copies of $\mathrm{inv}\,\pi_1(ab)$ and, if K is one of those copies and K' is one of those copies, it can be shown that there is only one homeomorphism of N onto N which throws K onto K', and that the space of all homeomorphisms of N onto N is homeomorphic to the space of all subcontinua of N that are copies of $\mathrm{inv}\,\pi_1(ab)$ with the Hausdorff topology. This latter space is a totally disconnected, perfect, and compact metric space.

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