Upper semi-continuous continuum-valued mappings onto circle-like continua

by

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In 1930 (therefore, without reference to inverse limits), D. van Dantzig proved, [3], that an $m$-adic solenoid is a continuous image of an $n$-adic solenoid if and only if $m$ is a factor of a power of $n$. In this paper, van Dantzig's theorem is generalized in several directions.

DEFINITIONS AND NOTATIONS. Throughout this paper, $R$ denotes the real line, $Z$ denotes the unit circle in the complex plane, and $\arg$ denotes the usual multi-valued argument function of $Z$ onto $R$. Denote by $\mathcal{P}$ the collection to which $\psi$ belongs if and only if $\psi$ is an upper semi-continuous mapping of $R$ onto a subset of $R$ which contains an interval of length $2\pi$ such that (1) for each real number $x$, $\psi(x)$ is either a point or a closed interval of length 1 or less; (2) $[\text{lub}_{\psi}(\pi) - \text{lub}_{\psi}(0)]/2\pi$ is an integer; (3) for each real number $x$,

$$
\text{lub}_{\psi}(x + 2\pi) - \text{lub}_{\psi}(x) = \text{glb}_{\psi}(x + 2\pi) - \text{glb}_{\psi}(x) = \text{lub}_{\psi}(2\pi) - \text{lub}_{\psi}(0).
$$

If $\psi$ is in $\mathcal{P}$, let

$$
W(\psi) = [\text{lub}_{\psi}(2\pi) - \text{lub}_{\psi}(0)]/2\pi
$$

and let

$$
M(\psi) = ([\text{lub}_{\psi}([0, 2\pi])] - [\text{glb}_{\psi}([0, 2\pi])])/2\pi.
$$

Denote by $\Phi$ the collection to which $\psi$ belongs if and only if $\psi$ is an upper semi-continuous mapping of $Z$ onto $Z$ such that, if $x$ is in $Z$, $\psi(x)$ is either a point or an arc of length 1 or less. If $\psi$ is in $\Phi$, let $E_\psi$ denote the set to which the point $(x, y)$ of the plane belongs if and only if there exist points $x$ and $y$ in $Z$ such that $x$ is in $\arg Z$, $y$ is in $\psi(x)$, and $y$ is in $\arg Z$. For each $\phi$ in $\Phi$, each component of $E_\phi$ is the graph of some element of $\mathcal{P}$; let $\phi^*$ denote one such element of $\mathcal{P}$. For each $\phi$ in $\Phi$, let $W(\phi) = W(\phi^*)$ and let $M(\phi) = M(\phi^*)$. If $\phi_1$, $\phi_2$, and $\phi_3 \in \Phi$, there is a positive integer $n$ such that, for each real number $x$, $[\text{p}(x)]^n(x) = \phi^*(\phi^*(x)) + 2\pi n$. If $\psi$ is in $\Phi$ and is single-valued, then $\phi^*$ is also single-valued, and if the integer $n$ is a factor of $W(\phi)$, $\phi$ has a continuous single-valued $n$th root.

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The following theorem can easily be established by induction:

\textbf{Theorem 1.} If \( \psi \) is in \( \Psi \), \( m \) is an integer, and \( x \) is a real number, then
\[
lub \psi(x + 2m\pi) - \lub \psi(x) = \glb \psi(x + 2m\pi) - \glb \psi(x) = 2m\pi W(\psi).
\]

In each of the following theorems the first statement is proved and the second is a simple corollary.

\textbf{Theorem 2.} If \( \psi_1 \) and \( \psi_2 \) are in \( \Psi \) and \( \psi_3 \) is single-valued, then \( \psi_1 \circ \psi_2 \) is in \( \Psi \) and \( W(\psi_1 \circ \psi_2) = W(\psi_1)W(\psi_2) \). If \( \psi_1 \) and \( \psi_2 \) are in \( \Phi \) and \( \psi_3 \) is single-valued, then \( \psi_1 \circ \psi_2 \) is in \( \Phi \) and \( W(\psi_1 \circ \psi_2) = W(\psi_1)W(\psi_2) \).

\textbf{Proof.} Evidently \( \psi_1 \circ \psi_2 \) is upper semi-continuous and contains an interval of length \( 2\pi \), and, for each real number \( \alpha \), \( \psi_3(\psi_2(\alpha)) \) is either a point or a closed interval of length 1 or less. If \( \alpha \) is a real number,
\[
lub \psi_3(\psi_2(\alpha + 2\pi)) = \lub \psi_3(\psi_2(\alpha)) + 2\pi W(\psi_2)
\]
and
\[
\glb \psi_3(\psi_2(\alpha + 2\pi)) = \glb \psi_3(\psi_2(\alpha)) + 2\pi W(\psi_2)
\]
In particular,
\[
\lub \psi_3(\psi_2(2\pi)) = \lub \psi_3(\psi_2(0)) + 2\pi W(\psi_2) = \lub \psi_3 \Psi W(\psi_2).
\]

Note that Theorem 2 is a simple generalization of Theorem 1 of [1] and, in the present paper, is used in precisely the same context as was that theorem used in [1].

\textbf{Theorem 3.} If \( \psi_1 \), \( \psi_2 \) and \( \psi_3 \) are in \( \Psi \) and \( \psi_3 \) is single-valued, then \( W(\psi_1 \circ \psi_2) = W(\psi_1)W(\psi_2) \). If \( \psi_1 \), \( \psi_2 \), and \( \psi_3 \) are in \( \Phi \) and \( \psi_3 \) is single-valued, then \( W(\psi_1 \circ \psi_2) = W(\psi_1)W(\psi_2) \).

\textbf{Proof.} There exists a number \( \alpha \) in \( \psi_3(0) \) such that \( \psi_1(\alpha) = \lub \psi_3(\psi_2(0)) \) and a number \( \beta \) in \( \psi_3(2\pi) \) such that \( \psi_3(\beta) = \lub \psi_3(\psi_2(2\pi)) \). Now, \( \alpha + 2\pi W(\psi_2) \) is in \( \psi_3(2\pi) \) and
\[
\psi_3(\alpha + 2\pi W(\psi_2)) - \psi_3(\alpha) = 2\pi W(\psi_2)W(\psi_3);
\]
and \( \beta - 2\pi W(\psi_2) \) is in \( \psi_3(0) \) and
\[
\psi_3(\beta) - \psi_3(\beta - 2\pi W(\psi_2)) = 2\pi W(\psi_2)W(\psi_3).
\]

Clearly, if
\[
A = \psi_3(\beta) - \psi_3(\beta - 2\pi W(\psi_2)) \quad \text{and} \quad B = \psi_3(\alpha) - \psi_3(\alpha + 2\pi W(\psi_2)),
\]
\( A \) and \( B \) are both non-negative. But, from (1) and (2) it follows that \( A + B = 0 \). Then
\[
2\pi W(\psi_1 \circ \psi_2) = \psi_3(\beta) - \psi_3(\alpha) = \psi_3(\alpha + 2\pi W(\psi_2)) - \psi_3(\alpha) = 2\pi W(\psi_2)W(\psi_3).
\]
If \((X_n, \pi_n^0)\) and \((Y_n, \pi_n^0)\) are inverse mapping systems with inverse limit \(X_\infty\) and \(Y_\infty\), respectively; \(m_1, m_2, \ldots, n_1, n_2, \ldots\) are increasing sequences of positive integers; and, for each \(i\), \(\zeta_i\) is a mapping of \(X_{n_i}\) onto a subset of \(X_{m_i}\), then \(\pi_{n_i} = \pi_{m_i}^e\), \(\zeta_{i+1}\), then the mapping \(\zeta\) of \(X_\infty\) onto a subset \(K\) of \(Y_\infty\), such that, for each point \(x\) of \(X_\infty\), \(\pi_{n_i}(\zeta(x)) = \zeta_{i}(\pi_{m_i}^e(x))\) is called the mapping of \(X_\infty\) onto \(K\) induced by the sequence \(\zeta_i\), \(\zeta_{i+1}\), \ldots\) (if \(n\) is an integer, \(\pi_{n_i}\) denotes the projection of \(X_\infty\) onto a subset of \(X_n\) and \(\pi_{n_k}\) denotes the projection of \(Y_\infty\) onto a subset of \(Y_{n_k}\)).

If \(f\) is a function, \(\text{inv} f\) will sometimes be used to denote \(f^{-1}\).

If \(P\) is the sequence \(p_1, p_2, \ldots\) of non-zero integers and \(Q\) is the sequence \(q_1, q_2, \ldots\) of non-zero integers, then \(Q\) is said to be a factor of \(P\) provided there exists a positive integer \(v\) such that, if \(v' > v\), there is a positive integer \(\mu\) for which \(\prod_{i=1}^{v'} q_i = v\) is a factor of \(\prod_{i=1}^{v'} p_i\).

**THEOREM 6.** If \(P\) is a sequence of non-zero integers, \(Q\) is a sequence of non-zero integers which is a factor of \(P\), \(H\) is a \(P\)-adic circle-like continuum, and \(K\) is a \(Q\)-adic solenoid, then there is a continuous single-valued mapping of \(H\) onto \(K\).

**Proof.** Suppose that \(H\) is the inverse limit of the inverse mapping system \((Z_n, \pi_n^0)\) and \(K\) is the inverse limit of the system \((Z_n, \omega_n)\), where, for each \(n\), \(Z_n = Z_n, W(\omega_n) = 0\) and, for each number \(x\) in \(Z_n, \omega_n^e(x) = Z_n\). Let \(\nu\) be a positive integer such that, if \(\nu > v\), there is a positive integer \(\mu\) for which \(\prod_{i=1}^{\nu} q_i = \mu\) is a factor of \(\prod_{i=1}^{\nu} p_i\). Let \(m_1, m_2, \ldots\) be an increasing sequence of positive integers such that, for each integer \(j > 0\), \(\prod_{i=1}^{j} q_i\) is a factor of \(\prod_{i=1}^{j} p_i\) and, for each \(j > 0\), let \(\zeta_i\) denote the continuous \((\prod_{i=1}^{j} q_i)\)th root of \(\pi_{m_i}^e\) and consider it as a mapping of \(Z_{n_1}\) onto \(Z_{n_1}^{e_k}\). Then the diagram

\[
\begin{array}{cccc}
\pi_{m_1}^e & \pi_{m_2}^e & \pi_{m_3}^e & \pi_{m_4}^e \\
\downarrow \zeta_1 & \downarrow \zeta_2 & \downarrow \zeta_3 & \ldots \\
Z_{n_1}^{e_k} & Z_{n_2}^{e_k} & Z_{n_3}^{e_k} & Z_{n_4}^{e_k} \\
\end{array}
\]

is commutative and the mapping \(\zeta\) of \(H_\infty\) onto \(K_\infty\) induced by the sequence \(\zeta_1, \zeta_2, \ldots\) is a continuous single-valued mapping.

**THEOREM 7.** If \(H\) is a chainable continuum and \(K\) is a \(P\)-adic circle-like continuum, there does not exist an upper semi-continuous mapping \(f\) of \(H_\infty\) onto \(K_\infty\) such that, for each point \(x\) of \(H\), \(f(x)\) is a proper subcontinuum of \(K\).

**Proof.** Suppose that \(C\) is a chainable continuum, \(\Sigma\) is a \(P\)-adic semi-continuous mapping of \(C_\infty\) onto \(\Sigma\), such that, for each point \(x\) of \(C\), \(g(x)\) is a proper subcontinuum of \(\Sigma\). Then \(\Sigma\) is an indecomposable continuum. Let \(C'\) be a subcontinuum of \(C\) which is irreducible with respect to the property that \(g(C') = \Sigma\). Then, for each proper subcontinuum \(L\) of \(C\), \(g(L)\) is a proper subcontinuum of \(\Sigma\). Now, if \(\Sigma\) is the sum of two of its proper subcontinua \(L_1\) and \(L_2\), then \(g(C') = g(L_1) + g(L_2) = \Sigma\), contrary to the fact that \(\Sigma\) is indecomposable; thus \(C'\) is indecomposable. Therefore, according to a theorem of Burgess ([2], Theorem 7), \(C'\) is circle-like and, as can be seen from Burgess' proof of that theorem, there exists a sequence \(G_1, G_2, \ldots\) of collection of open sets properly covering \(C'\) such that (1) for each \(n\), each element of \(G_n\) has diameter less than \(1/2^n\); (2) for each \(n\), each element of \(G_{n+1}\) is a subcontinuum of \(G_n\); (3) for each odd \(n, G_4\) is a linear chain; and (4) for each even \(n, G_{n+1}\) is a linear chain. Then, for each even \(n, G_{n+1}\) is a linear chain. Then, for each even \(n, G_{n+1}\) circles in \(G_n\) times in the sense of Bing [1]; hence, \(C'\) is a zero-adic circle-like continuum.

Suppose that \(C'\) is the inverse limit of the inverse mapping system \((Z_n, \pi_n^0)\) and \(\Sigma\) is the inverse limit of the system \((Z_n, \omega_n)\), where, for each \(n\), \(Z_n = Z_n, W(\omega_n) = 0\) and, for each number \(x\) in \(Z_1, \omega_1^e(x) = Z_n\), and \(|\omega_n| > 1\). Suppose that, for each \(n\), there exists a point \(x_n\) of \(C'\) such that \(\omega_n(g(\pi_{m_n}^e(x_n))) = Z_n\). There exists a subsequence \(x_n, x_{n+1}, \ldots\) of the sequence \(x_1, x_2, \ldots\) which converges to a point \(x\) of \(C'\) and \(\limsup \omega_n(x_n) = \Sigma\), which, since \(g\) is upper semi-continuous, is a subset of \(g(x)\). But this is contrary to the assumption that \(g(x)\) is a proper subcontinuum of \(\Sigma\). Thus, there exists a positive integer \(N_1\) such that, if \(x\) is a point of \(C'\), then \(\omega_{N_1}(g(x)) = \omega_{N_1}(g(x))\) is a proper subcontinuum of \(\Sigma_{N_1}\). Then there is a positive integer \(N > N_1\) such that, if \(n > N\) and \(x\) is a point of \(C'\), \(\omega_{N}(g(x))\) has length less than \(2n/\prod_{i>N} |\omega_i| < 1/3\).

Now, for each \(n > N\), there exists (1) a positive number \(\epsilon_n\) such that, if \(x\) and \(y\) are two points of \(C\) at a distance less than \(\epsilon_n\) from each other, then there is an arc of length less than \(1/3\) intersecting both \(g(x)\) and \(g(y)\) and (2) a positive integer \(M_n\) such that, if \(m > M_n\) and \(z\) is a number in \(Z_m\), then \(\pi_m(z)\) has diameter less than \(\epsilon_n\). For each \(i, j\), let \(n_i = N_i + i - 1\). Let \(m_1, m_2, \ldots\) be an increasing sequence of positive integers such that, for each \(i\) and each number \(z\) in \(Z_m, \pi_{m_i}(z)\) has diameter less than \(\epsilon_m\). Then, for each \(i\) and each number \(z\) in \(Z_m, \pi_{m_i}(g(\pi_{m_i}(z)))\) lies in an arc of \(Z_m\) having length less than \(1\).

For each positive integer \(i\) and each number \(z\) in \(Z_m\), let \(g(\pi_{m_i}(z))\) be the arc of \(Z_m\) with least arc length containing \(\rho_{m_i}(g(\pi_{m_i}(z)))\); for each \(i, q_i\) is in \(Q\). For each two integers \(i\) and \(j\) \((i < j)\) and each number \(z\) in \(Z_m\),
proof of Theorem 7, it can be shown that there exist increasing sequences \(m_1, m_2, \ldots\) and \(n_1, n_2, \ldots\) of positive integers and a sequence \(\varphi_1, \varphi_2, \ldots\) of elements of \(\Phi\) such that, for each two integers \(i\) and \(j\) \((i < j)\), \(\varphi_i \geq \varphi_j\) and \(\varphi_i \geq m_i\) are in \(\Phi\), \(W(\varphi_i + \varphi_j) = W(\varphi_i + m_i)\) and \(M(\varphi_i + \varphi_j) \leq M(\varphi_i + m_i)\). Now, suppose that \(W(\varphi_i) \neq 0\). Then, by Theorems 2 and 3, for each \(i\),

\[
W(\varphi_i) W(\varphi_i + m_i) \geq W(\varphi_i + m_i) = W(\varphi_i + \varphi_i) = W(\varphi_i) W(\varphi_i).
\]

Thus, for each \(i\), \(W(\varphi_i)\) is a factor of \(\prod_{k=i}^{m_i-1} \lambda_k\), a contradiction, so \(W(\varphi_i) = 0\). Then, by Theorems 4 and 5, for each \(i\),

\[
M(\varphi_i) = M(\varphi_i + m_i) \geq M(\varphi_i + \varphi_i) \geq |W(\varphi_i)| \geq \prod_{k=i}^{m_i-1} \lambda_k,
\]

again a contradiction and the theorem is proved.

**Theorem 9.** Suppose that \(H\) and \(K\) are two solenoids and there exist an upper semi-continuous mapping \(f\) of \(H\) onto \(K\) such that, for each point \(x\) of \(H\), \(f(x)\) is a proper subcontinuum of \(K\), and an upper semi-continuous mapping \(g\) of \(K\) onto \(H\) such that, for each point \(x\) of \(K\), \(g(x)\) is a proper subcontinuum of \(H\). Then \(H\) and \(K\) are homeomorphic.

**Proof.** Suppose that \(H\) is a \(P\)-adic solenoid and \(K\) is a \(Q\)-adic solenoid. Then each of \(P\) and \(Q\) is a factorant of the other. McCord has shown, [4], that if each of \(P\) and \(Q\) is a factorant of the other, then \(P\)-adic and \(Q\)-adic solenoids are homeomorphic.

**References**


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