Upper semi-continuous continuum-valued mappings onto circle-like continua

by

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In 1930 (therefore, without reference to inverse limits), D. van Dantzig proved, [3], that an m-adic solenoid is a continuous image of an n-adic solenoid if and only if m is a factor of a power of n. In this paper, van Dantzig's theorem is generalized in several directions.

DEFINITIONS AND NOTATIONS. Throughout this paper, R denotes the real line, Z denotes the unit circle in the complex plane, and arg denotes the usual multi-valued argument function of Z onto R. Denote by Ψ the collection to which ψ belongs if and only if ψ is an upper semi-continuous mapping of R onto a subset of R which contains an interval of length 2π such that (1) for each real number x, $\psi(x)$ is either a point or a closed interval of length 1 or less; (2) $[\operatorname{lub}\psi(\pi)-\operatorname{lub}\psi(0)]/2\pi$ is an integer; (3) for each real number x,

$$\operatorname{lub}\psi(x+2\pi)-\operatorname{lub}\psi(x) = \operatorname{glb}\psi(x+2\pi)-\operatorname{glb}\psi(x) = \operatorname{lub}\psi(2\pi)-\operatorname{lub}\psi(0)\,.$$

If ψ is in Ψ, let

$$W(\psi) = [\operatorname{lub}\psi(2\pi)-\operatorname{lub}\psi(0)]/2\pi$$

and let

$$M(\psi) = \{\operatorname{lub}\psi([0,2\pi])-\operatorname{glb}\psi([0,2\pi])\}/2\pi\,.$$

Denote by Φ the collection to which φ belongs if and only if φ is an upper semi-continuous mapping of Z onto Z such that, if z is in Z, $\varphi(z)$ is either a point or an arc of length 1 or less. If φ is in Φ, let E_φ denote the set to which the point (x, y) of the plane belongs if and only if there exist points z and w in Z such that x is in $\arg z$, w is in $\varphi(z)$, and y is in $\arg w$. For each φ in Φ, each component of E_φ is the graph of some element of Ψ; let φ^* denote one such element of Ψ. For each φ in Φ, let $W(\varphi) = W(\varphi^*)$ and let $M(\varphi) = M(\varphi^*)$. If φ_1, φ_2 and $\varphi_1\circ\varphi_2$ are in Φ, there is a positive integer n such that, for each real number x, $[\varphi_1\circ\varphi_2]^*(x) = \varphi^*(\varphi^*(x))+2\pi n$. If φ is in Φ and is single-valued, then φ^* is also single-valued, and if the integer n is a factor of $W(\varphi)$, φ has a continuous single-valued nth root.

The following theorem can easily be established by induction:

THEOREM 1. *If ψ is in Ψ, m is an integer, and x is a real number,*

$$\operatorname{lub}\psi(x+2\pi m)-\operatorname{lub}\psi(x)=\operatorname{glb}\psi(x+2\pi m)-\operatorname{glb}\psi(x)=2\pi mW(\psi)\,.$$

In each of the following four theorems the first statement is proved and the second is a simple corollary.

THEOREM 2. *If ψ_1 and ψ_2 are in Ψ and ψ_2 is single-valued, then $\psi_1\circ\psi_2$ is in Ψ and $W(\psi_1\circ\psi_2)=W(\psi_1)W(\psi_2)$. If φ_1 and φ_2 are in Φ and φ_2 is single-valued, then $\varphi_1\circ\varphi_2$ is in Φ and $W(\varphi_1\circ\varphi_2)=W(\varphi_1)W(\varphi_2)$.*

Proof. Evidently $\varphi_1\circ\varphi_2$ is upper semi-continuous and contains an interval of length 2π, and, for each real number x, $\psi_1(\psi_2(x))$ is either a point or a closed interval of length 1 or less. If x is a real number,

$$\begin{aligned}\operatorname{lub}\psi_1(\psi_2(x+2\pi))&=\operatorname{lub}\psi_1(\psi_2(x)+2\pi W(\psi_2))\\&=\operatorname{lub}\psi_1(\psi_2(x))+2\pi W(\psi_1)W(\psi_2)\end{aligned}$$

and

$$\begin{aligned}\operatorname{glb}\psi_1(\psi_2(x+2\pi))&=\operatorname{glb}\psi_1(\psi_2(x)+2\pi W(\psi_2))\\&=\operatorname{glb}\psi_1(\psi_2(x))+2\pi W(\psi_1)W(\psi_2)\,.\end{aligned}$$

In particular,

$$[\operatorname{lub}\psi_1(\psi_2(2\pi))-\operatorname{lub}\psi_1(\psi_2(0))]/2\pi=W(\psi_1)W(\psi_2)\,.$$

Note that Theorem 2 is a simple generalization of Theorem 1 of [1] and, in the present paper, is used in precisely the same context as was that theorem used in [1].

THEOREM 3. *If ψ_1,ψ_2 and $\psi_1\circ\psi_2$ are in Ψ and ψ_1 is single-valued, then $W(\psi_1\circ\psi_2)=W(\psi_1)W(\psi_2)$. If φ_1,φ_2, and $\varphi_1\circ\varphi_2$ are in Φ and φ_1 is single-valued, then $W(\varphi_1\circ\varphi_2)=W(\varphi_1)W(\varphi_2)$.*

Proof. There exists a number x in $\psi_2(0)$ such that $\psi_1(x)=\operatorname{lub}\psi(_1\psi_2(0))$ and a number y in $\psi_2(2\pi)$ such that $\psi_1(y)=\operatorname{lub}\psi_1(\psi_2(2\pi))$. Now, $x+2\pi W(\psi_2)$ is in $\psi_2(2\pi)$ and

(1) $$\psi_1(x+2\pi W(\psi_2))-\psi_1(x)=2\pi W(\psi_1)W(\psi_2)\,;$$

and $y-2\pi W(\psi_2)$ is in $\psi_2(0)$ and

(2) $$\psi_1(y)-\psi_1(y-2\pi W(\psi_2))=2\pi W(\psi_1)W(\psi_2)\,.$$

Clearly, if

$$A=\psi_1(y)-\psi_1(x+2\pi W(\psi_2))\quad\text{and}\quad B=\psi_1(x)-\psi_1(y-2\pi W(\psi_2))\,,$$

A and B are both non-negative. But, from (1) and (2) it follows that $A+B=0$. Then

$$2\pi W(\psi_1\circ\psi_2)=\psi_1(y)-\psi_1(x)=\psi_1(x+2\pi W(\psi_2))-\psi_1(x)=2\pi W(\psi_1)W(\psi_2)\,.$$

THEOREM 4. *If ψ_1 and ψ_2 are in Ψ, ψ_2 is single-valued, and $W(\psi_1)=0$, then $M(\psi_1)\geqslant M(\psi_1\circ\psi_2)$. If φ_1 and φ_2 are in Φ, φ_2 is single-valued, and $W(\varphi_1)=0$, then $M(\varphi_1)\geqslant M(\varphi_1\circ\varphi_2)$.*

Proof. There exist numbers x_0 and x_1 in the interval $[0,2\pi]$ such that $\operatorname{glb}\psi_1(\psi_2([0,2\pi]))=\operatorname{glb}\psi_1(\psi_2(x_0))$ and $\operatorname{lub}\psi_1(\psi_2([0,2\pi]))=\operatorname{lub}\psi_1(\psi_2(x))$; there also exist numbers t_0 and t_1 in $[0,2\pi]$ and integers n_0 and n_1 such that $\psi_2(x_0)=t_0+2\pi n_0$ and $\psi_2(x_1)=t_1+2\pi n_1$.

Then

$$\begin{aligned}2\pi M(\psi_1\circ\psi_2)&=\operatorname{lub}\psi_1(\psi_2(x))-\operatorname{glb}\psi_1(\psi_2(x_0))\\&=\operatorname{lub}\psi_1(t_1+2\pi n_1)-\operatorname{glb}\psi_1(t_0+2\pi n_0)\\&=\operatorname{lub}\psi_1(t_1)-\operatorname{glb}\psi_1(t_0)\leqslant 2\pi M(\psi_1)\,.\end{aligned}$$

THEOREM 5. *If ψ_1,ψ_2 and $\psi_1\circ\psi_2$ are in Ψ and ψ_1 is single-valued, then $M(\psi_1\circ\psi_2)\geqslant|W(\psi_1)|$. If φ_1,φ_2 and $\varphi_1\circ\varphi_2$ are in Φ and φ_1 is single-valued, then $M(\varphi_1\circ\varphi_2)\geqslant|W(\varphi_1)|$.*

Proof. There exist numbers x_0 and x_1 in $[0,2\pi]$ and numbers y_0 in $\psi_2(x_0)$ and y_1 in $\psi_2(x_1)$ such that $y_1-y_0=2\pi$. Then

$$\begin{aligned}2\pi M(\psi_1(\psi_2))&\geqslant|\psi_1(y_1)-\psi_1(y_0)|\\&=|\psi_1(y_0+y_1-y_0)-\psi_1(y_0)|\\&=|\psi_1(y_0)+W(\psi_1)-\psi_1(y_0)|=2\pi|W(\psi_1)|\,.\end{aligned}$$

DEFINITIONS. If P is a sequence $p_1,p_2,p_3,\ldots$ of integers and, for each n, $Z_n=Z$, π_n^{n+1} is a single-valued element of Φ, and $W(\pi_n^{n+1})=p_n$, then any continuum topologically equivalent to the inverse limit, Z_∞, of the inverse mapping system $\{Z_n,\pi_n^m\}$ (except where otherwise noted, an inverse mapping system will be assumed to be taken over the set of all positive integers directed by $<$) will be called a *P-adic circle-like continuum*. If, for each n, π_n^{n+1} is a local homeomorphism (or, equivalently, $(\pi_n^{n+1})^*$ is strictly increasing or strictly decreasing) then any continuum topologically equivalent to Z_∞ will be called a *P-adic solenoid*. It is known that, if Σ is a P-adic solenoid, then Σ is topologically equivalent to the inverse limit of the inverse mapping system $\{Z_n,\sigma_n^m\}$ where, for each n, $Z_n=Z$ and, for each number z in Z, $\sigma_n^{n+1}(z)=z^{p_n}$. If n is a positive integer and N is the sequence $n,n,n,\ldots$, then an N-adic solenoid is an n-adic solenoid in the sense of van Dantzig, [3]. If P is a sequence of integers infinitely many terms of which are 0, then P-adic circle-like continua will be called *zero-adic circle-like continua*. If all but a finite number of terms of P have absolute value 1, then P-adic circle-like continua will be called *monadic circle-like continua*. Circle-like continua which are neither zero-adic nor monadic will be said to be *poly-adic*.

If $\{X_n, \pi_n^m\}$ and $\{Y_n, \sigma_n^m\}$ are inverse mapping systems with inverse limit X_∞ and Y_∞ respectively; $m_1, m_2, \ldots$ and $n_1, n_2, \ldots$ are increasing sequences of positive integers; and, for each i, ζ_i is a mapping of X_{m_i} onto a subset of X_{n_i} and $\zeta_i \circ \pi_{m_i}^{m_{i+1}} = \sigma_{n_i}^{n_{i+1}} \circ \zeta_{i+1}$, then the mapping ζ of X_∞ onto a subset K of Y_∞ such that, for each point x of X_∞, $\sigma_{n_i}(\zeta(x)) = \zeta_i(\pi_{m_i}(x))$ is called the *mapping of X_∞ onto K induced by the sequence* $\zeta_1, \zeta_2, \ldots$ (if n is an integer, π_n denotes the projection of X_∞ onto a subset of X_n and σ_n denotes the projection of Y_∞ onto a subset of Y_n).

If f is a function, inv$f$ will sometimes be used to denote f^{-1}.

If P is the sequence $p_1, p_2, \ldots$ of non-zero integers and Q is the sequence $q_1, q_2, \ldots$ of non-zero integers, then Q is said to be a *factorant of P* provided there exists a positive integer ν such that, if $\nu' \geqslant \nu$, there is a positive integer μ for which $\prod_{i=\nu}^{\nu'} q_i$ is a factor of $\prod_{i=1}^{\mu} p_i$.

THEOREM 6. *If P is a sequence of non-zero integers, Q is a sequence of non-zero integers which is a factorant of P, H is a P-adic circle-like continuum, and K is a Q-adic solenoid, then there is a continuous single-valued mapping of H onto K.*

Proof. Suppose that H is the inverse limit of the inverse mapping system $\{Z_n, \pi_n^m\}$ and K is the inverse limit of the system $\{Z_n, \sigma_n^m\}$ where, for each n, $Z_n = Z$, $W(\pi_n^{n+1}) = p_n$ and, for each number z in Z, $\sigma_n^{n+1}(z) = Z^{q_n}$. Let ν be a positive integer such that, if $\nu' \geqslant \nu$, there is a positive integer μ for which $\prod_{i=\nu}^{\nu'} q_i$ is a factor of $\prod_{i=1}^{\mu} p_i$. Let $m_0, m_1, m_2, \ldots$ be an increasing sequence of positive integers such that, for each integer $j \geqslant 0$, $\prod_{i=\nu}^{\nu+j} q_i$ is a factor of $\prod_{i=1}^{m_j} p_k$ and, for each $j > 0$, let ζ_i denote the continuous $[\prod_{i=\nu}^{\nu+j} q_i]$th root of $\pi_{m_0}^{m_j}$ and consider it as a mapping of Z_{m_j} onto $Z_{\nu+i}$. Then the diagram

$$\begin{array}{ccccccc}
Z_{m_1} & \xleftarrow{\pi_{m_1}^{m_2}} & Z_{m_2} & \xleftarrow{\pi_{m_2}^{m_3}} & Z_{m_3} & \leftarrow & \\
\downarrow \zeta_1 & & \downarrow \zeta_2 & & \downarrow \zeta_3 & & \ldots \\
Z_{\nu+1} & \xleftarrow{\sigma_{\nu+1}^{\nu+2}} & Z_{\nu+2} & \xleftarrow{\sigma_{\nu+2}^{\nu+3}} & Z_{\nu+3} & \leftarrow &
\end{array}$$

is commutative and the mapping ζ of H onto K induced by the sequence $\zeta_1, \zeta_2, \ldots$ is a continuous single-valued mapping.

THEOREM 7. *If H is a chainable continuum and K is a poly-adic circle-like continuum, there does not exist an upper semi-continuous mapping f of H onto K such that, for each point x of H, $f(x)$ is a proper subcontinuum of K.*

Proof. Suppose that C is a chainable continuum, Σ is a poly-adic solenoid, and g is an upper semi-continuous mapping of C onto Σ such that, for each point x of C, $g(x)$ is a proper subcontinuum of Σ. Then Σ is an indecomposable continuum. Let C' be a subcontinuum of C which is irreducible with respect to the property that $g(C') = \Sigma$. Then, for each proper subcontinuum L of C', $g(L)$ is a proper subcontinuum of Σ. Now, if C' is the sum of two of its proper subcontinua L_1 and L_2, then $g(C') = g(L_1) + g(L_2) = \Sigma$, contrary to the fact that Σ is indecomposable; thus C' is indecomposable. Therefore, according to a theorem of Burgess ([2], Theorem 7), C' is circle-like and, as can be seen from Burgess' proof of that theorem, there exists a sequence $G_1, G_2, \ldots$ of collections of open sets properly covering C' such that (1) for each n, each element of G_n has diameter less than $1/n$; (2) for each n, each element of G_{n+1} is a subset of some element of G_n; (3) for each odd n, G_n is a linear chain; and (4) for each even n, G_n is a circular chain. Then, for each even n, G_{n+2} circles in G_n zero times in the sense of Bing [1]; hence, C' is a zero-adic circle-like continuum.

Suppose that C' is the inverse limit of the inverse mapping system $\{Z_n, \pi_n^m\}$ and Σ is the inverse limit of the system $\{Z_n, \sigma_n^m\}$ where, for each n, $Z_n = Z$, $W(\pi_n^{n+1}) = 0$ and, for each number z in Z, $\sigma_n^{n+1}(z) = Z^{q_n}$ and $|q_n| > 1$. Suppose that, for each n, there exists a point x_n of C' such that $\sigma_n(g(x_n)) = Z_n$. There exists a subsequence $x_{n_1}, x_{n_2}, \ldots$ of the sequence $x_1, x_2, \ldots$ which converges to a point x of C' and $\limsup g(x_{n_i}) = \Sigma$, which, since g is upper semi-continuous, is a subset of $g(x)$. But this is contrary to the assumption that $g(x)$ is a proper subcontinuum of Σ. Thus, there exists a positive integer N_1 such that, if x is a point of C', then $\sigma_{N_1}(g(x))$ is a proper subcontinuum of Z_{N_1}. Then there is a positive integer $N > N_1$ such that, if $n \geqslant N$ and x is a point of C', $\sigma_n(g(x))$ has arc length less than $2\pi / \prod_{i=N_1}^{N-1} |q_i| < 1/3$.

Now, for each $n \geqslant N$, there exist (1) a positive number ε_n such that, if x and y are two points of C' at a distance less than ε_n from each other, then there is an arc of length less than $1/3$ intersecting both $g(x)$ and $g(y)$ and (2) a positive integer M_n such that, if $m \geqslant M_n$ and z is a number in Z_m, then $\operatorname{inv}\pi_m(z)$ has diameter less than ε_n. For each i, let $n_i = N + i - 1$. Let $m_1, m_2, \ldots$ be an increasing sequence of positive integers such that, for each i and each number z in Z_{m_i}, $\operatorname{inv}\pi_{m_i}(z)$ has diameter less than ε_{n_i}. Then, for each i and each number z in Z_{m_i}, $\sigma_{n_i}(g(\operatorname{inv}\pi_{m_i}(z)))$ lies in an arc of Z_{n_i} having length less than 1.

For each positive integer i and number z in Z_{m_i}, let $\varphi_i(z)$ be the arc of Z_{n_i} with least arc length containing $\sigma_{n_i}(g(\operatorname{inv}\pi_{m_i}(z)))$; for each i, φ_i is in Φ. For each two integers i and j $(i < j)$ and each number z in Z_{m_j},

$\sigma_{ni}^{nj}(\varphi_j(z))$ is a subset of $\varphi_i(\pi_{mi}^{mj}(z))$; thus, $\sigma_{nj}^{ni}\circ\varphi_j$ and $\varphi_i\circ\pi_{mj}^{mi}$ are both in Φ,

$$W(\sigma_{ni}^{nj}\circ\varphi_j)=W(\varphi_i\circ\pi_{mi}^{mj}),\quad\text{and}\quad M(\sigma_{ni}^{nj}\circ\varphi_j)\leqslant M(\varphi_i\circ\pi_{mi}^{mj}).$$

Now, suppose that, for some i, $W(\varphi_i)=0$. Then, by Theorems 4 and 5, for each integer $j>i$,

$$M(\varphi_i)\geqslant M(\varphi_i\circ\pi_{mi}^{mj})\geqslant M(\sigma_{ni}^{nj}\circ\varphi_j)\geqslant|W(\sigma_{ni}^{nj})|\geqslant\prod_{k=n_i}^{n_j}|q_k|,$$

a contradiction. But then, by Theorems 2 and 3,

$$0=W(\varphi_1)W(\pi_{m_1}^{m_2})=W(\varphi_1\circ\pi_{m_1}^{m_2})=W(\sigma_{n_1}^{n_2}\circ\varphi_2)=W(\sigma_{n_1}^{n_2})W(\varphi_2)$$

and, thus, $W(\varphi_2)=0$. Thus we have reached a contradiction and there can be no such upper semi-continuous mapping g of a chainable continuum onto a poly-adic solenoid.

Suppose that f is an upper semi-continuous mapping of a chainable continuum H onto a poly-adic circle-like continuum K such that, for each point x of H_1, $f(x)$ is a proper subcontinuum of K. By Theorem 6. there is a continuous single-valued mapping h of K onto a poly-adic solenoid Σ'. Then $h\circ f$ is an upper semi-continuous mapping of H onto Σ; and, since each proper subcontinuum of K is chainable, for each point x of H, $h(f(x))$ is a proper subcontinuum of Σ', a contradiction. Thus, the theorem is proved.

THEOREM 8. *Suppose that P and Q are sequences of non-zero integers; H is a P-adic circle-like continuum; K is a Q-adic circle-like continuum; and f is an upper semi-continuous mapping of H onto K such that, for each point x of H, $f(x)$ is a proper subcontinuum of K. Then Q is a factorant of P.*

Proof. Suppose that Q is not a factorant of P. Then there exist an increasing sequence $\nu_1,\nu_2,\ldots$ of positive integers and a sequence $\lambda_1,\lambda_2,\ldots$ of positives primes such that, for each two positive integers i and μ, λ_i is a factor of greater multiplicity of $\prod_{j=\nu_1}^{\nu_{i+1}-1}q_j$ than for $\prod_{j=1}^{\mu}p_j$. For each i, let $q_i'=\prod_{j=\nu_i}^{\nu_{i+1}-1}q_i$ and denote the sequence $q_1',q_2',\ldots$ by Q'. Now, Q' is a factorant of Q, so there is a continuous single-valued mapping h of K onto Σ, the inverse limit of the inverse mapping system $\{Z_n,\sigma_n^m\}$ where, for each n, $Z_n=Z$ and, for each number z in Z, $\sigma_n^{n+1}(z)=z^{q_n'}$. Now, $h\circ f$ is an upper semi-continuous mapping of H onto Σ; for each i, $|q_i'|>1$; and, for each x in H, $f(x)$ is chainable. Thus, for each point x in H, $h(f(x))$ is a proper subcontinuum of Σ. Let $g=h\circ f$.

Suppose that H is the inverse limit of the inverse mapping system $\{Z_n,\pi_n^m\}$ where, for each n, $Z_n=Z$ and $W(\pi_n^{n+1})=p_n$. Now, as in the proof of Theorem 7, it can be shown that there exist increasing sequences $m_1,m_2,\ldots$ and $n_1,n_2,\ldots$ of positive integers and a sequence $\varphi_1,\varphi_2,\ldots$ of elements of Φ such that, for each two integers i and j $(i<j)$, $\sigma_{ni}^{nj}\circ\varphi_j$ and $\varphi_i\circ\pi_{mi}^{mj}$ are in Φ, $W(\sigma_{ni}^{nj}\circ\varphi_j)=W(\varphi_i\circ\pi_{mi}^{mj})$ and $M(\sigma_{ni}^{nj}\circ\varphi_j)\leqslant M(\varphi_i\circ\pi_{mi}^{mj})$. Now, suppose that $W(\varphi_1)\neq 0$. Then, by Theorems 2 and 3, for each i,

$$W(\varphi_1)W(\pi_{m_1}^{mi})=W(\varphi_1\circ\pi_{m_1}^{mi})=W(\sigma_{n_1}^{ni}\circ\varphi_i)=W(\sigma_{n_1}^{ni})W(\varphi_i).$$

Thus, for each i, $W(\varphi_1)$ is a factor of $\prod_{k=n_1}^{n_i-1}\lambda_j$, a contradiction, so $W(\varphi_1)=0$. Then, by Theorems 4 and 5, for each i,

$$M(\varphi_1)\geqslant M(\varphi_1\circ\pi_{m_1}^{mi})\geqslant M(\sigma_{n_1}^{ni}\circ\varphi_i)\geqslant|W(\sigma_{n_1}^{ni})|\geqslant\prod_{j=1}^{n_1-1}\lambda_i,$$

again a contradiction and the theorem is proved.

THEOREM 9. *Suppose that H and K are two solenoids and there exist an upper semi-continuous mapping f of H onto K such that, for each point x of H, $f(x)$ is a proper subcontinuum of K, and an upper semi-continuous mapping g of K onto H such that, for each point x of K, $g(x)$ is a proper subcontinuum of H. Then H and K are homeomorphism.*

Proof. Suppose that H is a P-adic solenoid and K is a Q-adic solenoid. Then each of P and Q is a factorant of the other. McCord has shown, [4], that if each of P and Q is a factorant of the other, then P-adic and Q-adic solenoids are homeomorphic.

References

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