[8] Ranga R. Rao, *Relations between weak and uniform convergence of measures with applications*, Ann. Math. Stat. 33 (1962), pp. 659-680.

[9] K. A. Ross, A. H. Stone, *Products of separable spaces*, Amer. Math. Monthly 71 (1964), pp. 398-403.

[10] N. A. Sanin, *On the product of topological spaces*, Trudy Math. Inst. Steklova 24 (1948) (Russian).

[11] D. Scott, *Measurable cardinals and constructible sets*, Bull. Acad. Polon. Sci, Sér. Sci. Math. Astron. 9 (1961), pp. 521-524.

[12] S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math. 16 (1930), pp. 140-150 (especially pp. 149 and 141).

[13] V. S. Varadarajan, *Measure on topological spaces*, Math. Sbornik 55 (1961), pp. 35-100 (Russian).

[14] I. Glicksberg, *The representation of functionals by integrals*, Duke Math. J. 19 (1951), pp. 253-261.

[15] P. Halmos, *Measure theory*, Van Nostrand 1959.

[16] A. P. and W. J. Roberston, *Topological vector spaces*, Cambridge 1964.

[17] A. Hulanicki, *Means and Følner condition on locally compact groups*, Studia Math. 27(1966), pp. 87-104.

[18] E. Hewitt, K. A. Ross, *Abstract harmonic analysis I*, Heidelberg 1963.

[19] J. L. Kelley, *General topology*, Toronto, New York, London 1955.

[20] R. H. Bing, *Extending a metric*, Duke Math. J. 14 (1947), pp. 511-519.

[21] P. Billingsley, *The invariance principle for dependent random variables*, Trans. A.M.S. 83 (1956), pp. 250-268.

[22] J. Namioka, *On a recent theorem of Reiter*, Proc. A. M. S. 17(1966), pp. 1101-1102.

CORNELL UNIVERSITY

Reçu par la Rédaction le 3. 7. 1965

Sequents in many valued logic I

by

G. Rousseau (Leicester)

The calculus of sequents for two-valued predicate logic is well known, either in the original formulation of Gentzen [1] or in any of the many variants in the literature. In this paper we show that there exists an analogous calculus for each finitely many valued predicate logic based on arbitrary connectives $F_1, \ldots, F_u$ and quantifiers $Q_1, \ldots, Q_w$.

1. Propositional calculus. Let $\boldsymbol{M} = \{0, 1, \ldots, M-1\}$ be the set of *truth-values* and for each k $(k = 1, \ldots, u)$ let f_k be a *truth-function* of $r = r_k \geqslant 1$ arguments, i.e., a mapping of $\boldsymbol{M} \times \ldots \times \boldsymbol{M} = \boldsymbol{M}^r$ into $\boldsymbol{M}$. Let $\mathfrak{A}$ and $\{F_1, \ldots, F_u\}$ be disjoint sets of symbols, the elements of which are called *atomic statements* and *connectives* respectively. The set $\mathfrak{S}$ of *statements* is the smallest set of expressions which contains all atomic statements and which, for each connective F_k, contains $F_k\,\alpha_1 \ldots \alpha_r$ whenever it contains $\alpha_1, \ldots, \alpha_r$. The *degree* of a statement is the number of occurrences of connectives in it. We will denote statements by the letters $\alpha, \beta, \gamma, \ldots$ and finite (possibly null) sequences of statements by $\Gamma, \Delta, \ldots$; in particular, the null sequence will be denoted by Λ.

A sequent is an expression of the form

$$\Gamma_0 \mid \Gamma_1 \mid \ldots \mid \Gamma_{M-2} \mid \Gamma_{M-1} \, . \tag{1}$$

We denote sequents by the letters $\Pi, \Sigma, \ldots$. If Π is the sequent (1) and Σ is the sequent

$$\Delta_0 \mid \Delta_1 \mid \ldots \mid \Delta_{M-2} \mid \Delta_{M-1} \, ,$$

then $\Pi\Sigma$ will denote the sequent

$$\Gamma_0\Delta_0 \mid \Gamma_1\Delta_1 \mid \ldots \mid \Gamma_{M-2}\Delta_{M-2} \mid \Gamma_{M-1}\Delta_{M-1} \, .$$

If R is a subset of $\boldsymbol{M}$ then the sequent (1) may be written $|\Gamma|_R$ (or $|\Gamma|_m$ if $R = \{m\}$) provided

$$\Gamma_i = \begin{cases} \Gamma & \text{if} \quad i \in R \, , \\ \Lambda & \text{if} \quad i \notin R \, . \end{cases}$$

A sequent is said to be *atomic* iff each statement occurring in it is atomic.

The elements of $M^{\mathfrak{A}}$ are called *valuations*. Each valuation v determines a unique mapping h_v of $\mathfrak{S}$ into M such that for each connective F_k

(2) $$h_v(F_k a_1 \ldots a_r) = f_k(h_v(a_1), \ldots, h_v(a_r))$$

whenever $a_1, \ldots, a_r \in \mathfrak{S}$.

The valuation v is said to *satisfy* the sequent (1) iff

(3) $$m \in h_v(\Gamma_m) \quad \text{for some } m \in M .$$

The set of valuations which satisfy Π is denoted by $\mathfrak{s}\Pi$; clearly we have

(4) $$\mathfrak{s}\Pi_1 \ldots \Pi_n = \mathfrak{s}\Pi_1 \cup \ldots \cup \mathfrak{s}\Pi_n .$$

We say that the sequent Π is *valid* (Val Π) iff $\mathfrak{s}\Pi = M^{\mathfrak{A}}$, and a set $\mathfrak{M}$ of sequents is *simultaneously satisfiable* iff $\bigcap_{\Pi \in \mathfrak{M}} \mathfrak{s}\Pi \neq \emptyset$.

Lemma 1. *Let f_k be a truth-function of $r = r_k$ arguments and let m be a truth-value. Then there exist subsets R_j^i of M $(i = 1, \ldots, n;\ j = 1, \ldots, r)$, where $n \leqslant M^{r-1}$, such that for all $x_1, \ldots, x_r \in M$*

(5) $$f_k(x_1, \ldots, x_r) = m \equiv \bigwedge_{i=1}^{n} (x_1 \in R_1^i \vee \ldots \vee x_r \in R_r^i) .$$

Proof. Each subset X of M^r can be represented as the union of not more than M^{r-1} cartesian products of subsets of M. In fact, if $r > 1$ then X is the union of all the products

$$\{m_1\} \times \ldots \times \{m_{r-1}\} \times \{m\colon (m_1, \ldots, m_{r-1}, m) \in X\}$$

such that $(m_1, \ldots, m_{r-1}) \in M^{r-1}$. The lemma now follows by an application of this remark to the set $M^r \setminus f_k^{-1}(m)$. We note that the bound M^{r-1} is best possible; for $\{(m_1, \ldots, m_r)\colon m_1 + \ldots + m_r \equiv 0 \pmod{M}\}$ has M^{r-1} elements but contains no product with more than one element.

If $a_1, \ldots, a_r$ are any statements, then for each i $(i = 1, \ldots, n)$ let $\Pi_i(a_1, \ldots, a_r)$ be the sequent $|a_1|_{R_1^i} \ldots |a_r|_{R_r^i}$. From (5) we obtain with the aid of (2) and (4) the formula

(6) $$\mathfrak{s}|F_k a_1 \ldots a_r|_m = \bigcap_{i=1}^{n} \mathfrak{s}\Pi_i(a_1, \ldots, a_r) .$$

If Π^0 and Π^1 are arbitrary sequents we have by (6) and (4)

(7) $$\mathfrak{s}\Pi^0|F_k a_1 \ldots a_r|_m \Pi^1 = \bigcap_{i=1}^{n} \mathfrak{s}\Pi^0\Pi_i(a_1, \ldots, a_r)\Pi^1 ,$$

whence also

(8) $$\mathrm{Val}\,\Pi^0|F_k a_1 \ldots a_r|_m \Pi^1 \equiv \bigwedge_{i=1}^{n} \mathrm{Val}\,\Pi^0\Pi_i(a_1, \ldots, a_r)\Pi^1 .$$

Formula (8) suggests characterizing the valid sequents by means of "introduction rules" (F_k, m) for each connective F_k and each truth-value m, of the form

$$\frac{\Pi^0\Pi_1(a_1, \ldots, a_r)\Pi^1; \ \ldots\ ; \Pi^0\Pi_n(a_1, \ldots, a_r)\Pi^1}{\Pi^0|F_k a_1 \ldots a_r|_m \Pi^1} .$$

We say that Π is an *immediate consequence* by rule (F_k, m) of $\Pi_1, \ldots, \Pi_n$ iff Π has the form $\Pi^0|F_k a_1 \ldots a_r|_m \Pi^1$ while each Π_i $(i = 1, \ldots, n)$ has the form $\Pi^0\Pi_i(a_1, \ldots, a_r)\Pi^1$. An atomic sequent of the form (1) is called an *initial* sequent iff the Γ_m $(m = 0, \ldots, M-1)$ have at least one statement in common.

The set of *provable* sequents is the least set which contains all initial sequents and which contains a sequent Π whenever it contains sequents $\Pi_1, \ldots, \Pi_n$ of which Π is an immediate consequence by some rule (F_k, m) $(k = 1, \ldots, u;\ m = 0, \ldots, M-1)$.

Theorem 1. *A sequent is valid iff it is provable.*

Proof. Each initial sequent is valid; by (8) an immediate consequence of valid sequents is valid; consequently every provable sequent is valid. We now prove the converse. For each sequent Π, let $\mathrm{m}\Pi$ be the maximum degree of statements occurring in Π and let $\mathrm{n}\Pi$ be the number of occurrences in Π of statements of degree $\mathrm{m}\Pi$. Let Π be a valid sequent. If $\mathrm{m}\Pi = 0$, then Π is atomic; but all valid atomic sequents are initial sequents; hence Π is provable. If $\mathrm{m}\Pi > 0$, we suppose that all valid sequents Σ are provable for which

(i) $\mathrm{m}\Sigma < \mathrm{m}\Pi$, or

(ii) $\mathrm{m}\Sigma = \mathrm{m}\Pi$ and $\mathrm{n}\Sigma < \mathrm{n}\Pi$.

Since $\mathrm{m}\Pi > 0$, Π may be expressed in the form $\Pi^0|F_k a_1 \ldots a_r|_m \Pi^1$ where $F_k a_1 \ldots a_r$ is of degree $\mathrm{m}\Pi$. By (8) the sequents $\Pi^0\Pi_i(a_1, \ldots, a_r)\Pi^1$ $(i = 1, \ldots, n)$ are valid; but for each of these sequents either (i) or (ii) holds, so by inductive hypothesis each is provable; hence Π is provable, being an immediate consequence by rule (F_k, m) of provable sequents. The proof is thus complete by induction.

As a corollary we have the elimination theorem: If $\Pi|a|_R$ and $|a|_S\Sigma$ are provable sequents $(R \cap S = \emptyset)$, then $\Pi\Sigma$ is provable.

Theorem 2. *A set of sequents is simultaneously satisfiable iff every finite subset is simultaneously satisfiable.*

Proof. Let us assign to M the discrete topology and to $M^{\mathfrak{A}}$ the product topology. Since M is a compact Hausdorff space, the same is true of $M^{\mathfrak{A}}$ by Tychonoff's theorem. The proof will therefore be complete on showing that, for each Π, $\mathfrak{s}\Pi$ is a closed subset of $M^{\mathfrak{A}}$. Let $\mathfrak{A}_\Pi$ be the set of atomic statements occurring in Π. Since $\mathfrak{A}_\Pi$ is finite, the restrictions to $\mathfrak{A}_\Pi$ of the elements of $\mathfrak{s}\Pi$ form a finite set $\{w^1, \ldots, w^s\}$, say. Then $\mathfrak{s}\Pi$

may be expressed as the union of the sets $\bigcap_{a \in \mathfrak{A}\Pi} \overset{-1}{pr_a}(w^j(a))$ $(j = 1, \ldots, s)$, so that $s\Pi$ is in fact both open and closed.

2. Predicate calculus. A quantifier is a mapping q of $P(\boldsymbol{M})$ into $\boldsymbol{M}$. In this section we consider quantifiers $q_1, \ldots, q_w$ in addition to the truth-functions $f_1, \ldots, f_u$. We introduce a system $\mathcal{S}$ whose primitive symbols fall into the following (disjoint) categories: free variables $a_1, a_2, \ldots$; bound variables $x_1, x_2, \ldots$; constants c; functions φ (of $r = r_\varphi$ arguments); predicates π (of $r = r_\pi$ arguments); connectives $F_1, \ldots, F_u$; quantifiers $Q_1, \ldots, Q_w$.

The *terms* are the elements of the smallest set $\mathfrak{T}$ such that: (i) $\mathfrak{T}$ contains all constants and free variables; (ii) for each function φ, $\mathfrak{T}$ contains the expression $\varphi(t_1, \ldots, t_r)$ whenever it contains $t_1, \ldots, t_r$. The *elementary statements* are the expressions of the form $\pi(t_1, \ldots, t_r)$ where π is a predicate and $t_1, \ldots, t_r \in \mathfrak{T}$. The *statements* are the elements of the smallest set $\mathfrak{S}$ such that: (i) $\mathfrak{S}$ contains all elementary statements; (ii) $\mathfrak{S}$ contains $F_k\alpha_1\ldots\alpha_r$ whenever it contains $\alpha_1, \ldots, \alpha_r$; (iii) $\mathfrak{S}$ contains $Q_l x \alpha_a^x$ whenever it contains α, provided a is a free variable occurring in α and x is a bound variable not occurring in α (where α_a^x is the result of substituting x for a throughout α). We shall often write $\alpha = \alpha(a)$, $\alpha_a^t = \alpha(t)$, $\alpha_a^x = \alpha(x)$ when α is a statement containing the free variable a. The degree of a statement is the number of occurrences in it of connectives and quantifiers. The definition of sequent and associated notational conventions are taken over from § 1.

An *interpretation* in a (non-null) set D is a mapping $\sigma \to \sigma I$ of the constants, free variables, functions and predicates of the system $\mathcal{S}$ such that: (i) for each constant c and each free variable a, cI and $aI \in D$; (ii) for each function φ, $\varphi I \in D^{(D^r)}$; (iii) for each predicate π, $\pi I \in \boldsymbol{M}^{(D^r)}$. If a is a free variable and $d \in D$, then the interpretation I_a^d is defined by

$$\sigma I_a^d = \begin{cases} d & \text{if} \quad \sigma = a\,, \\ \sigma I & \text{if} \quad \sigma \neq a\,. \end{cases}$$

For each interpretation I in the set D, there is a unique mapping $t \to tJ$ of $\mathfrak{T}$ into D such that: (i) if t is a constant or free variable then $tJ = tI$; (ii) if φ is a function and $t_1, \ldots, t_r \in \mathfrak{T}$, then

$$\varphi(t_1, \ldots, t_r)J = \varphi I(t_1 J, \ldots, t_r J)\,.$$

The mapping J will always be denoted by I. The interpretation I determines, further, a unique mapping (again denoted by I) of $\mathfrak{S}$ into $\boldsymbol{M}$ such that: (i) if π is a predicate and $t_1, \ldots, t_r \in \mathfrak{T}$ then

$$\pi(t_1, \ldots, t_r)I = \pi I(t_1 I, \ldots, t_r I);$$

(ii) if $\alpha_1, \ldots, \alpha_r \in \mathfrak{S}$ then $(F_k\alpha_1\ldots\alpha_r)I = f_k(\alpha_1 I, \ldots, \alpha_r I)$; (iii) if $Q_l x \alpha_a^x$ is a statement then

$$(Q_l x \alpha_a^x)I = q_l\{\alpha I_a^d \colon d \in D\}\,.$$

It is a straight-forward matter to show that if a is a free variable and t is a term then for $\tau \in \mathfrak{T}$ and $\alpha \in \mathfrak{S}$

$$(9) \qquad \tau_a^t I = \tau I_a^{tI}; \qquad \alpha_a^t I = \alpha I_a^{tI}\,.$$

For each sequent Π, let $\mathfrak{T}_\Pi$ be the set of terms built up from the free variables $a_1, a_2, \ldots$ and the constants and functions occurring in Π; clearly $\mathfrak{T}_\Pi$ is denumerable. An interpretation in $\mathfrak{T}_\Pi$ is said to be *canonical* if $tI = t$ for every $t \in \mathfrak{T}_\Pi$.

The definitions of satisfaction etc. are analogous to those in § 1. Thus the interpretation I satisfies the sequent (1) iff

$$m \in (\Gamma_m)I \qquad \text{for some } m \in \boldsymbol{M}\,.$$

The set of interpretations which satisfy Π is denoted by $s\Pi$. A sequent Π is valid (ValΠ) iff $s\Pi$ contains every interpretation; a set $\mathfrak{M}$ of sequents is simultaneously satisfiable iff $\bigcap_{\Pi \in \mathfrak{M}} s\Pi \neq \emptyset$. It is clear that formulas (6)-(8) remain true in this new setting.

LEMMA 2. *Let q_l be a quantifier, m a truth-value. Then there exist p, q and $E_j^i \subseteq \boldsymbol{M}$ $(i = 1, \ldots, p;\ j = 0, \ldots, q)$ such that for every $E \subseteq \boldsymbol{M}$*

$$(10) \qquad q_l E = m \equiv \bigwedge_{l=1}^{p} (\exists x_i \in E)(\forall y_1, \ldots, y_q \in E)(x_i \in E_0^i \vee y_1 \in E_1^i \vee \ldots \vee y_q \in E_q^i)\,.$$

Proof. Let f be a truth-function of M arguments such that

$$f(x_0, \ldots, x_{M-1}) = q_l\{m \colon x_m = 1\}\,.$$

Then

$$q_l E = f\big(\chi_E(0), \ldots, \chi_E(M-1)\big) \qquad (E \subseteq \boldsymbol{M})\,,$$

(where χ_E is the characteristic function of E), and so (by Lemma 1) we have, for suitable $R_j^i \subseteq \boldsymbol{M}$,

$$q_l E = m \equiv \bigwedge_{i=1}^{p} \big(\chi_E(0) \in R_0^i \vee \ldots \vee \chi_E(M-1) \in R_{M-1}^i\big)\,.$$

There is no loss of generality in supposing that each R_j^i is a subset or indeed a proper subset of $\{0, 1\}$. Each term $\chi_E(j) \in R_j^i$ can be replaced by an equivalent term of the form $(Qz_j^i \in E)(z_j^i \in Z_j^i)$ where Q is $\forall$ or $\exists$ depending on R_j^i; the resulting expression can be readily brought into the form (10).

For each i $(i = 1, \ldots, p)$ we define the sequent

$$\Sigma_i\big(\alpha(t_i);\, a_1, \ldots, a_q\big) = |\alpha(t_i)|_{E_0^i} |\alpha(a_1)|_{E_1^i} \ldots |\alpha(a_q)|_{E_q^i}\,.$$

Simple computations using (9) and (10) yield:

LEMMA 3. (a) *If* $\mathrm{Val}\,\Pi^0\Sigma_i(\alpha(t_i); a_1, \ldots, a_q)\Pi^1$ $(i = 1, \ldots, p)$, *then* $\mathrm{Val}\,\Pi^0|Q_l x\alpha(x)|_m\Pi^1$, *provided* $a_1, \ldots, a_q$ *are distinct free variables not occurring in* $\Pi^0|Q_l x\alpha(x)|_m\Pi^1$ *or* $t_1, \ldots, t_p$.

(b) *Let* Σ *be any sequent and let* I *be a canonical interpretation in* $\mathfrak{T}_\Sigma$. *If* $I \in \mathfrak{s}|Q_l x\alpha(x)|_m$, *then there exist* $t_1, \ldots, t_p \in \mathfrak{T}_\Sigma$ *such that* $I \in \bigcap_{i=1}^{p} \mathfrak{s}\Sigma_i(\alpha(t_i); a_1, \ldots, a_q)$ *for any free variables* $a_1, \ldots, a_q$.

The introduction rule (F_k, m) is adapted from § 1 in the form

$$\frac{\Pi\Pi_1(a_1, \ldots, a_r)\,|F_k a_1 \ldots a_r|_m; \ \ldots\ ; \Pi\Pi_n(a_1, \ldots, a_r)\,|F_k a_1 \ldots a_r|_m}{|F_k a_1 \ldots a_r|_m \Pi}.$$

The introduction rule (Q_l, m) is the following:

$$\frac{\Pi\Sigma_1(\alpha(t_1); a_1, \ldots, a_q)|Q_l x\alpha(x)|_m; \ \ldots\ ; \Pi\Sigma_p(\alpha(t_p); a_1, \ldots, a_q)|Q_l x\alpha(x)|_m}{|Q_l x\alpha(x)|_m\Pi}$$

with the restriction that $a_1, \ldots, a_q$ are distinct free variables not occurring in $|Q_l x\alpha(x)|_m\Pi$ or $t_1, \ldots, t_p$.

The sequent (1) is said to be *fundamental* iff the Γ_m $(m = 0, \ldots, M-1)$ have at least one statement in common. Then the *provable* sequents are the elements of the smallest set containing the fundamental sequents and closed under the application of the rules (F_k, m), (Q_l, m) $(k = 1, \ldots, u;$ $l = 1, \ldots, w;$ $m = 0, \ldots, M-1)$.

The proof of the next theorem requires certain preliminary definitions. The letter $\boldsymbol{s}$ will always denote a finite sequence of positive integers. If $\boldsymbol{s}$ is the sequence $(s_0, \ldots, s_{n-1})$, then $\boldsymbol{s}, i$ will denote the sequence $(s_0, \ldots, s_{n-1}, i)$; the null sequence will be denoted by $\mathbf{0}$. If $\boldsymbol{s}$ is an initial segment of the (finite or infinite) sequence $\boldsymbol{t}$, we write $\boldsymbol{s} \leqslant \boldsymbol{t}$. A non-empty set T of sequences $\boldsymbol{s}$ is called a *tree* iff

(i) if $\boldsymbol{s} \leqslant \boldsymbol{s}' \in T$, then $\boldsymbol{s} \in T$,

(ii) $\{i: \boldsymbol{s}, i \in T\}$ is finite for each $\boldsymbol{s} \in T$.

If a tree T is infinite, then there exists an infinite sequence $\boldsymbol{t}$ such that $\boldsymbol{s} \in T$ for each $\boldsymbol{s} \leqslant \boldsymbol{t}$. (For each n, let $\boldsymbol{t}(n)$ be the least i such that $(\boldsymbol{t}(0), \ldots, \boldsymbol{t}(n-1), i)$ is an initial segment of infinitely many elements of T.)

For each sequent Σ we define inductively a mapping $\boldsymbol{s} \to \Sigma_{\boldsymbol{s}}$ as follows:

(a) $\Sigma_{\mathbf{0}} = \Sigma$;

(b) $\Sigma_{\boldsymbol{s},i}$ is defined only if $\Sigma_{\boldsymbol{s}}$ is not fundamental, and then only as specified below. Let $\boldsymbol{s}$ be of length $qM + m$ $(m \in \boldsymbol{M})$:

(i) if $\Sigma_{\boldsymbol{s}}$ is of the form $|F_k a_1 \ldots a_r|_m\Pi$, then $\Sigma_{\boldsymbol{s},i}$ is the sequent $\Pi\Pi_i(a_1, \ldots, a_r)\,|F_k a_1 \ldots a_r|_m$ $(i = 1, \ldots, n)$;

(ii) if $\Sigma_{\boldsymbol{s}}$ is of the form $|Q_l x\alpha(x)|_m\,\Pi$ then $\Sigma_{\boldsymbol{s},i}$ is the sequent $\Pi\Sigma_i(\alpha(t_i); a_1, \ldots, a_q)|Q_l x\alpha(x)|_m$ $(i = 1, \ldots, p)$, where $(t_1, \ldots, t_p)$ is the kth element of $\mathfrak{T}_\Sigma \times \ldots \times \mathfrak{T}_\Sigma$ in some fixed enumeration, k being the number of proper initial segments $\boldsymbol{s}' \leqslant \boldsymbol{s}$ of length congruent to $m \pmod{M}$ such that $\Sigma_{\boldsymbol{s}'}$ is of the form $|Q_l x\alpha(x)|_m\Pi'$, and where $a_1, \ldots, a_q$ are the first distinct free variables not occurring in $\Sigma_{\boldsymbol{s}}$ or $t_1, \ldots, t_p$;

(iii) if $\Sigma_{\boldsymbol{s}}$ is not of one of the forms mentioned then $\Sigma_{\boldsymbol{s},1} = \Sigma_{\boldsymbol{s}}$.

Evidently the domain of the mapping $\boldsymbol{s} \to \Sigma_{\boldsymbol{s}}$ is a tree.

THEOREM 3. *A sequent is valid iff it is provable.*

Proof. Each fundamental sequent is valid; by (8) the application of rule (F_k, m) preserves validity, while by lemma 3(a) rule (Q_l, m) preserves validity; hence every provable sequent is valid. Suppose now that Σ is unprovable; it follows that $\Sigma_{\boldsymbol{s}}$ is defined for infinitely many $\boldsymbol{s}$ and hence that there exists an infinite sequence $\boldsymbol{t}$ such that $\Sigma_{\boldsymbol{s}}$ is defined for each $\boldsymbol{s} \leqslant \boldsymbol{t}$.

For each $m \in \boldsymbol{M}$ let S_m be the set of all statements which occur in the mth place of some $\Sigma_{\boldsymbol{s}}$ $(\boldsymbol{s} \leqslant \boldsymbol{t})$. Let I be any canonical interpretation in $\mathfrak{T}_\Sigma$; we show that for each non-elementary statement α

(11) *if* $\alpha \in S_{\alpha I}$ *then* $\alpha' \in S_{\alpha' I}$ *for some* α' *of lower degree.*

Suppose $\alpha I = m$, $\alpha \in S_m$.

(i) If $\alpha = F_k a_1 \ldots a_r$ then by (6) $I \in \mathfrak{s}\Pi_i(a_1, \ldots, a_r)$ $(i = 1, \ldots, n)$; for some $\boldsymbol{s} \leqslant \boldsymbol{t}$, $\Sigma_{\boldsymbol{s}}$ is of the form $|F_k a_1 \ldots a_r|_m\Pi$ and

$$\Sigma_{\boldsymbol{s},i} = \Pi\Pi_i(a_1, \ldots, a_r)\,|F_k a_1 \ldots a_r|_m \quad (i = 1, \ldots, n);$$

choose that i such that $\boldsymbol{s}, i \leqslant \boldsymbol{t}$; there exist α' and m' such that $\alpha' I = m'$ and α' occurs in the m'th place of $\Pi_i(a_1, \ldots, a_r)$; hence $\alpha' \in S_{\alpha' I}$ and α' is of degree lower than α.

(ii) If $\alpha = Q_l x\alpha(x)$ then by lemma 3(b) there exist $t_1, \ldots, t_p \in \mathfrak{T}_\Sigma$ such that for any $a_1, \ldots, a_q$ $I \in \mathfrak{s}\Sigma_i(\alpha(t_i); a_1, \ldots, a_q)$ $(i = 1, \ldots, p)$; for some $\boldsymbol{s} \leqslant \boldsymbol{t}$ $\Sigma_{\boldsymbol{s}}$ is of the form $|Q_l x\alpha(x)|_m\Pi$ and

$$\Sigma_{\boldsymbol{s},i} = \Pi\Sigma_i(\alpha(t_i); a_1, \ldots, a_q)|Q_l x\alpha(x)|_m \quad (i = 1, \ldots, p)$$

for suitable $a_1, \ldots, a_q$; choose that i such that $\boldsymbol{s}, i \leqslant \boldsymbol{t}$; there exist α' and m' such that $\alpha' I = m'$ and α' occurs in the m'th place of $\Sigma_i(\alpha(t_i); a_1, \ldots, a_q)$; hence $\alpha' \in S_{\alpha' I}$ and α' is of degree lower than α.

Let I be a canonical interpretation in $\mathfrak{T}_\Sigma$ such that for each predicate π, if $t_1, \ldots, t_r \in \mathfrak{T}_\Sigma$ then $\pi I(t_1, \ldots, t_r)$ is the least m such that $\pi(t_1, \ldots, t_r) \notin S_m$; (the existence of such an I is clear). If Σ is valid we have $\alpha \in S_{\alpha I}$ for some α occurring in Σ; thus by repeated application of (11) there exists an elementary statement α' such that $\alpha' \in S_{\alpha' I}$; since this contradicts the assumption on I, it follows, as required, that Σ cannot be valid.

THEOREM 4. *A set of sequents is simultaneously satisfiable iff every finite subset is simultaneously satisfiable.*

Proof. Let $\mathfrak{M}$ be a set of sequents (of $\mathcal{S}$) such that each finite subset of $\mathfrak{M}$ is simultaneously satisfiable. We may suppose without loss of generality that the truth-functions $f_1, \ldots, f_u$ include $g_1, \ldots, g_w$ where $g_l(x_0, \ldots, x_{M-1}) = q_l\{x_0, \ldots, x_{M-1}\}$ $(l = 1, \ldots, w)$. Consider the infinite sequence of systems $\mathcal{S}_0, \mathcal{S}_1, \ldots$ such that: (i) $\mathcal{S}_0 = \mathcal{S}$; (ii) for each n the primitive symbols of $\mathcal{S}_{n+1}$ are those of $\mathcal{S}_n$ together with new constants $\varepsilon_{m,a,\alpha}$ in one-one correspondence with the triples (m, a, α), where $m \in \boldsymbol{M}$, a is a free variable occurring in α and α is a statement of $\mathcal{S}_n$ but not a statement of $\mathcal{S}_{n-1}$. Let $\mathcal{S}^*$ be the union of the systems $\mathcal{S}_n$; we denote by $\mathfrak{T}^*$ resp. $\mathfrak{S}^*$ the set of terms resp. statements of $\mathcal{S}^*$. We write $\alpha(\varepsilon_m)$ for $\alpha(\varepsilon_{m,a,\alpha(a)})$.

It can be shown inductively that if $\mathfrak{N}$ is a finite set of statements of $\mathcal{S}_{n-1}$ then each interpretation I of (the primitive symbols of) $\mathcal{S}$ in D can be extended to an interpretation I_n of $\mathcal{S}_n$ in D such that for each $\alpha(a) \in \mathfrak{N}$ (with free variable a)

$$\{\alpha(a) I_{na}^{\;d}\colon d \in D\} = \{\alpha(\varepsilon_0) I_n, \ldots, \alpha(\varepsilon_{M-1}) I_n\} .$$

Hence if $\mathfrak{N}$ is a finite subset of $\mathfrak{S}^*$, then each interpretation I of $\mathcal{S}$ in D can be extended to an interpretation I^* of $\mathcal{S}^*$ in D such that for $\alpha(a) \in \mathfrak{N}$

$$\{\alpha(a) I^{*d}_{\;a}\colon d \in D\} = \{\alpha(\varepsilon_0) I^*, \ldots, \alpha(\varepsilon_{M-1}) I^*\} .$$

Let $\mathfrak{M}^* \subseteq \mathfrak{S}^*$ be the set of sequents obtained by adding to $\mathfrak{M}$ all sequents of the forms

$$|\alpha(t)|_E|\alpha(\varepsilon_0), \ldots, \alpha(\varepsilon_{M-1})|_{E'} , \tag{12}$$

$$|Q_l x \alpha(x)|_E|G_l \alpha(\varepsilon_0) \ldots \alpha(\varepsilon_{M-1})|_{E'} , \tag{13}$$

where $l = 1, \ldots, w$; $E \subseteq \boldsymbol{M}$; $E' = \boldsymbol{M} \setminus E$; $t \in \mathfrak{T}^*$; $\alpha(a) \in \mathfrak{S}^*$ and a is a free variable occurring in $\alpha(a)$. Then it follows that every finite subset of $\mathfrak{M}^*$ is simultaneously satisfiable, i.e., for each finite subset $\mathfrak{N}$ of $\mathfrak{M}^*$ there is an interpretation I^* of $\mathcal{S}^*$ such that I^* satisfies each element of $\mathfrak{N}$.

Let $\mathfrak{A}$ be the set of statements $\alpha \in \mathfrak{S}^*$ not of one of the forms $F_k \alpha_1 \ldots \alpha_r$ $(k = 1, \ldots, u)$; $\mathfrak{S}^*$ may be considered as the set of statements built up from atomic statements $\mathfrak{A}$ using the connectives $F_1, \ldots, F_u$; by Theorem 2 there is a valuation $v \in \boldsymbol{M}^{\mathfrak{A}}$ such that for each $\Pi \in \mathfrak{M}^*$, $h_v \in \mathfrak{s}\Pi$.

Let I be the canonical interpretation of $\mathfrak{S}^*$ in $\mathfrak{T}^*$ such that for each predicate π

$$\pi I(t_1, \ldots, t_r) = h_v \pi(t_1, \ldots, t_r) \qquad (t_1, \ldots, t_r \in \mathfrak{T}^*) .$$

One shows by induction that

$$\alpha I = h_v \alpha \quad \text{for all } \alpha \in \mathfrak{S}^* . \tag{14}$$

Indeed it suffices to show that

$$h_v Q_l x \alpha(x) = q_l\{h_v \alpha(t)\colon t \in \mathfrak{T}^*\}; \tag{15}$$

but

$$\{h_v \alpha(t)\colon t \in \mathfrak{T}^*\} \supseteq \{h_v \alpha(\varepsilon_0), \ldots, h_v \alpha(\varepsilon_{M-1})\} ,$$

while conversely, since each sequent of the form (12) is an element of $\mathfrak{M}^*$,

$$\{h_v \alpha(t)\colon t \in \mathfrak{T}^*\} \subseteq \{h_v \alpha(\varepsilon_0), \ldots, h_v \alpha(\varepsilon_{M-1})\};$$

thus

$$\begin{aligned} q_l\{h_v \alpha(t)\colon t \in \mathfrak{T}^*\} &= q_l\{h_v \alpha(\varepsilon_0), \ldots, h_v \alpha(\varepsilon_{M-1})\} \\ &= g_l(h_v \alpha(\varepsilon_0), \ldots, h_v \alpha(\varepsilon_{M-1})) = h_v(G_l \alpha(\varepsilon_0) \ldots \alpha(\varepsilon_{M-1})); \end{aligned}$$

Formula (15) follows since each sequent of the form (13) is an element of $\mathfrak{M}^*$.

From (14) it follows that I satisfies every element of $\mathfrak{M}^*$; hence the restriction of I to $\mathcal{S}$ satisfies every element of $\mathfrak{M}$. This completes the proof.

For any given truth-function or quantifier, introduction rules can be determined explicitly, often in a form more elegant than that given by the proofs of lemmas 1 and 2. For example, consider the three-valued Łukasiewicz connectives with the following truth-tables, and the quantifier $\bigvee E = \sup E$:

C	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

A	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

	N
0	2
1	1
2	0

The corresponding introduction rules, in simplified form, are as follows:

$$\frac{|\;|\alpha \quad \beta|\;|}{C\alpha\beta|\;|} \qquad \frac{|\alpha,\beta| \quad \beta|\;|\alpha}{|C\alpha\beta|} \qquad \frac{\alpha|\alpha|\beta \quad \alpha|\beta|\beta}{|\;|\alpha\beta C}$$

$$\frac{\alpha|\;| \quad \beta|\;|}{A\alpha\beta|\;|} \qquad \frac{\alpha|\alpha| \quad |\alpha,\beta| \quad \beta|\beta|}{|A\alpha\beta|} \qquad \frac{|\;|\alpha,\beta}{|\;|A\alpha\beta}$$

$$\frac{|\;|\alpha}{N\alpha|\;|} \qquad \frac{|\alpha|}{|N\alpha|} \qquad \frac{\alpha|\;|}{|\;|N\alpha}$$

$$\frac{\alpha(a)|\;|}{\bigvee x \alpha(x)|\;|} \qquad \frac{\alpha(a)|\alpha(a)| \quad |\alpha(t)|}{|\bigvee x \alpha(x)|} \qquad \frac{|\;|\alpha(t)}{|\;|\bigvee x \alpha(x)}$$

Certain semantic and syntactical notions can be defined for statements, relative to a subset $\boldsymbol{D}$ of designated truth-values. We write $\Gamma \| \varDelta$ for the sequent $|\Gamma|_{\boldsymbol{M} \setminus \boldsymbol{D}}|\varDelta|_{\boldsymbol{D}}$. The interpretation I is said to *satisfy* α $(I \in \mathfrak{s}^*(\alpha))$ iff $\alpha I \in \boldsymbol{D}$; clearly we have

$$\mathfrak{s}^*(\gamma) = \mathfrak{s}\,\|\gamma; \qquad \mathbf{C}\mathfrak{s}^*(\delta) = \mathfrak{s}\delta\| .$$

We make the following definitions: (i) α is *valid* iff every interpretation is an element of $\mathfrak{s}^*(\alpha)$; α is *provable* iff the sequent $\|\alpha$ is provable; (ii) $\mathfrak{C}$ is *simultaneously satisfiable* iff $\bigcap_{\alpha \in \mathfrak{C}} \mathfrak{s}^*(\alpha) \neq \emptyset$; $\mathfrak{C}$ is *consistent* iff for no $\Gamma \subseteq \mathfrak{C}$ is $\Gamma\|$ provable. Then we have:

THEOREM 5. *Validity and provability are equivalent properties; $\mathfrak{C}$ is simultaneously satisfiable iff $\mathfrak{C}$ is consistent.*

Proof. By Theorems 3 and 4 we can prove, more generally, that the following are equivalent:

(a) $$\bigcap_{\gamma \in \mathfrak{C}} \mathfrak{s}^*(\gamma) \subseteq \bigcup_{\delta \in \mathfrak{D}} \mathfrak{s}^*(\delta) .$$

(b) $\Gamma\|\Delta$ *is provable for some* $\Gamma \subseteq \mathfrak{C}$, $\Delta \subseteq \mathfrak{D}$.

If a suitable quantifier (x) and connectives $\supset$ and J_m $(m \in \boldsymbol{M})$ are definable in terms of $F_1, \ldots, F_u$ and $Q_1, \ldots, Q_w$, theorem 5 can be used to obtain a set of axiom schemes which, with *modus ponens* and the rule of generalization, yield as theorems precisely the valid statements. For each $m \in \boldsymbol{M}$ and each $\gamma \in \mathfrak{S}$ we define the statement $\Gamma^{[m]}\gamma$ inductively as follows:

$$\begin{cases} \Lambda^{[m]}\gamma = \gamma , \\ \alpha, \Gamma^{[m]}\gamma = (J_m\alpha \supset \Gamma^{[m]}\gamma) . \end{cases}$$

If Π is the sequent (1), then $\Pi^*\gamma$ is defined as the statement

$$\Gamma_0^{[0]}\Gamma_1^{[1]} \ldots \Gamma_{M-2}^{[M-2]}\Gamma_{M-1}^{[M-1]}\gamma .$$

Consider the following axiom schemes:

(A1) $\alpha \supset (\beta \supset \alpha)$,

(A2) $\alpha \supset (\beta \supset \gamma) .\supset. (\alpha \supset \beta) \supset (\alpha \supset \gamma)$,

(A3) $(x)\big(\alpha \supset \beta(x)\big) .\supset. \alpha \supset (x)\beta(x)$,

(A4) $|\alpha|_{\boldsymbol{M}}{}^*\gamma$,

(A5) $\Pi_1(\alpha_1, \ldots, \alpha_r)^*\gamma \supset. \ldots \supset. \Pi_n(\alpha_1, \ldots, \alpha_r)^*\gamma \supset |F_k\alpha_1 \ldots \alpha_r|_m{}^*\gamma$ $\quad (k = 1, \ldots, u;\ m \in \boldsymbol{M})$,

(A6) $(x_1) \ldots (x_q)\big(\Sigma_1(\alpha(t_1); x_1, \ldots, x_q)^*\gamma\big) \supset. \ldots \supset.$
$\supset. (x_1) \ldots (x_q)\big(\Sigma_p(\alpha(t_p); x_1, \ldots, x_q)^*\gamma\big) \supset |Q_l x\alpha(x)|_m{}^*\gamma \quad (l = 1, \ldots, w;\ m \in \boldsymbol{M})$,

(A7) $J_m\alpha \supset \alpha \quad (m \in \boldsymbol{D})$.

In (A3) it is assumed that x does not occur in α, while in (A6) $x_1, \ldots, x_q$ do not occur in γ.

One shows quite readily that $\Pi^*\gamma$ is a consequence of (A1)-(A6) whenever Π is a provable sequent. But if α is a valid statement then, by Theorem 5, $\|\alpha$ is a provable sequent so that $\|\alpha^*\alpha$ and hence α are consequences of (A1)-(A7). We therefore have

THEOREM 6. *If the axioms* (A1)-(A7) *are valid and if validity is preserved by modus ponens and the rule of generalization, then a statement is valid iff it is a consequence of* (A1)-(A7).

References

[1] G. Gentzen, *Untersuchungen über das logische Schliessen*, Mathematische Zeitschrift 39 (1934-5), pp. 176-210, 405-431.

[2] H. Rasiowa and R. Sikorski, *On the Gentzen theorem*, Fund. Math. 48 (1960), pp. 57-69.

[3] A. Mostowski, *Axiomatizability of some many valued predicate calculi*, Fund. Math. 50 (1961), pp. 165-190.

[4] J. B. Rosser and A. Turquette, *Many valued logics*, Studies in Logic and the Foundations of Mathematics, Amsterdam 1952.

Reçu par la Rédaction le 21. 7. 1965