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On the equivalence of an exhaustion principle and the axiom of choice

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INTRODUCTION. An interesting and very general abstract formulation of the exhaustion principle used in measure theory was given in the paper [1]. The aim of this note is to give, in a direct way, an abstract formulation of the following simple form of the exhaustion principle and to show that it is equivalent to the axiom of choice.

THEOREM I (MEASURE EXHAUSTION THEOREM). *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a finite measure. Then there exists a set $P \in \mathfrak{M}$ such that $E \subset \mathfrak{M}$, $E \subset X-P$, implies $\mu(E) = 0$.*

Notation. We shall use the notation according to [2]. Let us explain some further symbols which we shall use in the paper.

(a) S will denote a fixed set and $\mathfrak{m}$ a cardinal number such that $\mathfrak{m} \leqslant \overline{\overline{S}}$ ($\overline{\overline{A}}$ denotes the cardinal number of the set A).

(b) $R \subset S \times S$ will be a relation (see [4], p. 54), xRy means $\langle x, y\rangle \in R$, x non Ry means $\langle x, y\rangle \notin R$.

(c) $Y \subset S$ will be a non-void set. $\hat{S}_Y$ stands for a system of subsets E of Y for the elements of which the following is true:

$$x \in E,\ y \in E,\ x \neq y \quad \Rightarrow \quad xRy \text{ or } yRx,$$
$$x \in E \ \Rightarrow\ x \text{ non } Rx .$$

(d) $\varphi^{(\mathfrak{m})}$ stands for an S-valued function with the domain consisting of all $E \in \hat{S}_Y$ for which $\overline{\overline{E}} \leqslant \mathfrak{m}$.

(e) The function $\varphi^{(\mathfrak{m})}$ and the relation R fulfil the following condition:

$$y \in S,\ \varphi^{(\mathfrak{m})}(E)Ry \ \Rightarrow\ xRy \text{ for each } x \in E .$$

Now we shall formulate an abstract form of Theorem I.

PRINCIPLE OF EXHAUSTION. *Let S, Y, $\hat{S}_Y$, R and $\varphi^{(\mathfrak{m})}$ fulfil assumptions* (a)-(e). *Let $\overline{\overline{E}} \leqslant \mathfrak{m}$ and $\varphi^{(\mathfrak{m})}(E) \in Y$ for each $E \in \hat{S}_Y$ and let there exist at least one $x \in Y$ for which x non Rx. Then there exists $z \in Y$ such that, for each $y \in Y$, zRy implies yRy.*

THEOREM 1. *In the set theory including the axiom of choice the principle of exhaustion holds.*

At first let us present some consequences of the principle of exhaustion.

Theorem I follows from the exhaustion principle if we put $S = \mathfrak{M}$. For the set $Y \subset \mathfrak{M}$ we may choose the system of all measurable sets with positive measure. Further we put $\mathfrak{m} = \aleph_0$. The relation R we define in the following way: ERF means $E \cap F = \emptyset$.

$\varphi^{(\mathfrak{m})}(\boldsymbol{E})$ denotes the union of sets of the system $\boldsymbol{E}$. The assumptions (a)-(e) are evidently fulfilled and so Theorem I is a consequence of the exhaustion principle.

In a quite analogical way we obtain from the exhaustion principle the following theorem:

THEOREM II (THEOREM OF EXHAUSTION OF THE VECTOR MEASURE). *Let $(X, \mathfrak{M}, \mu)$ be a vector measure space (i.e. let $(X, \mathfrak{M})$ be a measurable space and μ be a σ-additive set function with the domain $\mathfrak{M}$ and with the range in a linear topological space). Let each system of pairwise disjoint sets with non-zero measure be at most countable. Then there exists a set $P \in \mathfrak{M}$ such that $\mu(E) = 0$ for each $E \in \mathfrak{M}$, $E \subset X-P$.*

Proof of Theorem 1. Let α be the least of the ordinal numbers of the power $\aleph(\mathfrak{m})$ (cf. [4], p. 220). For each $\xi < \alpha$ the sequence $\{x_n^\xi\}$ will be constructed in the following way: For $\xi = 1$, x_1^1 is any element of Y such that $x_1^1 \operatorname{non} R x_1^1$. For $n \geqslant 2$, x_n^1 is any element of Y such that $x_i^1 R x_n^1$, $i = 1, 2, \ldots, n-1$, and simultaneously $x_n^1 \operatorname{non} R x_n^1$. If for some $n \geqslant 2$ such an element does not exist, then the proof is completed, since for the element z appearing in the principle of exhaustion we can take $\varphi^{(\mathfrak{m})}(E)$, where $E = \{x_1^1, \ldots, x_{n-1}^1\}$. Hence we may assume that x_n^1 exists for $n = 1, 2, \ldots$

Let $\xi < \alpha$ be any ordinal number and let $\{x_n^\eta\}$ be constructed for each $\eta < \xi$ such that the set $E = \{x_n^\eta\colon \eta < \xi,\ n = 1, 2, \ldots\}$ belongs to the domain of $\varphi^{(\mathfrak{m})}$. The element x_1^ξ will be chosen such that $x_1^\xi \in Y$, $\varphi^{(\mathfrak{m})}(E) R x_1^\xi$, $x_1^\xi \operatorname{non} R x_1^\xi$. If there is no element with the above-mentioned property the proof is finished. If such an element exists, then $\{x_n^\xi\}_{n=1}^\infty$ is constructed in the same way as in the case of $\xi = 1$. Evidently there must exist an ordinal number $\xi_0 < \alpha$ and a natural number N such that $x_N^{\xi_0}$ with the above-mentioned property does not exist. In the opposite case we might construct an element $E \in \hat{S}_Y$ the power of which would be $\aleph(\mathfrak{m})$. But this is impossible. Hence the element $z = \varphi(E_{\xi_0} \cup \{x_1^{\xi_0}\} \cup \ldots \cup \{x_{N-1}^{\xi_0}\})$ is the desired element and the proof is finished.

THEOREM 2. *The axiom of choice follows from the principle of exhaustion.*

Proof. Let $T \neq 0$ be a set and F a function defined on T with values in a given family of disjoint sets. Let S denote the set of all functions g each of which is defined on a subset of the set T and for which $g(t) \in F(t)$. Let $\mathfrak{m}$ be the cardinal number of the set S. The functions $g(t)$ represent some relations when considered as subsets $T \times \bigcup_{t \in T} F(t)$, so the set inclusion defines a partial ordering in S. Let us define a relation R on S in the following way. If $g, h \in S$, then $gRh \iff g \subset h,\ g \neq h$.

Let $Y = S$. Evidently $Y \neq 0$. Let us define the function $\varphi^{(\mathfrak{m})}$ as the set-theoretical union. The assumptions (a)-(e) are fulfilled. According to the principle of exhaustion there exists a function $f(t)$ such that $g(t) \in S$, $g(t) \supset f(t)$ implies $g(t) = f(t)$. The domain of such a function f must be the whole set T. The proof is finished.

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