[14] R. Rado, *Axiomatic treatment of rank in infinite sets*, Canad. J. Math. 1 (1949), pp. 337-343.

[15] — *Partial well-ordering of sets of vectors*, Mathematika 1 (1954), pp. 89-95.

[16] W. Sierpiński, *Cardinal and ordinal numbers*, Monografie Matemat. 34, Warszawa 1958.

[17] E. Szpilrajn, *Sur l'extension de l'ordre partiel*, Fund. Math. 16 (1930), pp. 386-389.

[18] S. Tarkowski, *On the comparability of dendrites*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astron. Phys. 8 (1960), pp. 39-41.

UNIVERSITY OF CONNECTICUT
Storrs, Connecticut

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On a property of sets of positive measure

by

B. Sodnomov (Ulan-Ude)

1. For any subsets A, B of the real line we put $A+B=\{a+b\colon a\in A,\ b\in B\}$, $A-B=\{a-b\colon a\in A,\ b\in B\}$, $\mathfrak{D}(A,B)=\{|a-b|\colon a\in A,\ b\in B\}$, $D(A)=D(A,A)$. $\mathfrak{D}(A)$ is called the *set of distances* of A.

For every subset E of the real line, $E^{(\xi)}$ denotes the symmetrical reflection of E, ξ being the center of symmetry. It is evident that

$$A+B=A-B^{(0)}\,,\qquad A-B=A+B^{(0)}\,,$$

$$\mathfrak{D}(A,B)=[(A-B)\cup(B-A)]\cap[0,\infty)\,,\qquad \mathfrak{D}(A)=(A-A)\cap[0,\infty)\,.$$

If E_ξ denotes the intersection of E and $E^{(\xi)}$ then E_ξ is symmetric with respect to ξ. If ξ is a metric density point for E then ξ is also a metric density point for E_ξ.

mE denotes the Lebesgue measure of E.

We say that a system Σ_n of 3^n congruent and disjoint segments $\Delta_{t_1,t_2,\ldots,t_n}$, where $t_k\in\{0,1,2\}$, is *quite symmetric* if for any natural number k, $1\leqslant k\leqslant n$, it satisfies the following conditions:

(*) *$\Delta_{t_1,\ldots,t_{k-1},t_k,\ldots,t_n}$ and $\Delta_{t_1,\ldots,t'_{k-1},t'_k,\ldots,t'_n}$ are symmetric with respect to the center of $\Delta_{t_1,\ldots,t_{k-1},1,1,\ldots,1}$ provided the sequence $t'_k,t'_{k+1},\ldots,t'_n$ is obtained from the sequence $t_k,t_{k+1},\ldots,t_n$ by the substitution 0 for 2 and 2 for 0;*

(**) *$\Delta_{t_1,t_2,\ldots,t_{k-1},\nu_1,\nu_2,\ldots,\nu_{n-k+1}}$ will coincide with $\Delta_{\theta_1,\theta_2,\ldots,\theta_{k-1},\nu_1,\nu_2,\ldots,\nu_{n-k+1}}$ if the real line will be translated at a distance such that $\Delta_{t_1,\ldots,t_{k-1},1,1,\ldots,1}$ coincides with $\Delta_{\theta_1,\ldots,\theta_{k-1},1,1,\ldots,1}$.*

A perfect set F is called *quite symmetric* if there exists a sequence of quite symmetric systems $\Sigma_1,\Sigma_2,\ldots,\Sigma_n,\ldots$ such that the segment $\Delta_{t_1,t_2,\ldots,t_n}$ of Σ_n contains the segments $\Delta_{t_1,t_2,\ldots,t_n,t}$, $t\in\{0,1,2\}$, of Σ_{n+1} the centers of $\Delta_{t_1,t_2,\ldots,t_n}$ and $\Delta_{t_1,t_2,\ldots,t_n,1}$ coincide and $F=\bigcap_{n=1}^{\infty}S_n$, where $S_n=\bigcup_{\Delta\in\Sigma_n}\Delta$.

2. THEOREM 1. *If F is a quite symmetric perfect set, then there exists a perfect set P such that $P+P=F$.*

Proof. We denote the length of $\Delta_{t_1t_2,\ldots,t_n}$ by $2r_n$.

Let be $\Delta_t=[d_t,b_t]$, $t\in\{0,1,2\}$.

Now we define two segments

$$\delta_0 = \left[\frac{a_0}{2}, \frac{a_0}{2}+r_1\right], \qquad \delta_1 = \left[a_1-\frac{a_0}{2}, a_1-\frac{a_0}{2}+r_1\right].$$

These segments are congruent and disjoint. Put $P_1 = \delta_0 \cup \delta_1$. It is evident that $\delta_i + \delta_j = \Delta_{i+j}$ for $i, j \in \{0, 1\}$, i.e. $P_1 + P_1 = S_1$. Let us assume that 2^n congruent and disjoint segments $\delta_{i_1,i_2,\ldots,i_n}$, $i_k \in \{0, 1\}$, have been obtained and for these segments the following condition is fulfilled:

(1) $\delta_{i_1,i_2,\ldots,i_n} + \delta_{j_1,j_2,\ldots,j_n} = \Delta_{i_1+j_1,i_2+j_2,\ldots,i_n+j_n}$, where $i_k, j_k \in \{0, 1\}$.

Now we consider the segment $\Delta_\sigma = [a, b]$, where σ denotes the sequence $0, 0, \ldots, 0$.

Since $\delta_\sigma + \delta_\sigma = \Delta_\sigma$, we have $\delta_\sigma = [a/2, b/2]$. The centers of Δ_σ and $\Delta_{\sigma,1}$ coincide; therefore,

$$\Delta_{\sigma,1} = \left[\frac{a+b}{2} - r_{n+1}, \frac{a+b}{2} + r_{n+1}\right],$$

$$\Delta_{\sigma,0} = [a+\varrho, a+\varrho+2r_{n+1}],$$

$$\Delta_{\sigma,2} = [b-\varrho-2r_{n+1}, b-\varrho] \quad \text{where} \quad \varrho \geqslant 0.$$

Since $\Delta_{\sigma,0}, \Delta_{\sigma,1}, \Delta_{\sigma,2}$ are disjoint,

$$a+\varrho+2r_{n+1} < \frac{a+b}{2} - r_{n+1}, \qquad b-\varrho-2r_{n+1} > \frac{a+b}{2} + r_{n+1}.$$

In the segment δ_σ we define two segments

$$\delta_{\sigma,0} = \left[\frac{a+\varrho}{2}, \frac{a+\varrho}{2} + r_{n+1}\right], \qquad \delta_{\sigma,1} = \left[\frac{b-\varrho}{2} - r_{n+1}, \frac{b-\varrho}{2}\right].$$

Obviously $\delta_{\sigma,i} + \delta_{\sigma,j} = \Delta_{\sigma,i+j}$ for $i, j \in \{0, 1\}$. Now we put $\delta_{i_1,i_2,\ldots,i_n,i} = \delta_{\sigma,i} + \{\xi - a\}$ for $i \in \{0, 1\}$, where ξ is the center of $\delta_{i_1,i_2,\ldots,i_n}$ and a is the center of δ_σ.

Since $\delta_{i_1,i_2,\ldots,i_n} + \delta_{j_1,j_2,\ldots,j_n} = \Delta_{i_1+j_1,i_2+j_2,\ldots,i_n+j_n}$, $\delta_\sigma + \delta_\sigma = \Delta_\sigma$ and $\delta_{\sigma,i} + \delta_{\sigma,j} = \Delta_{\sigma,i+j}$, it is evident that $\delta_{i_1,\ldots,i_n,i} + \delta_{j_1,\ldots,j_n,j} = \Delta_{i_1+j_1,\ldots,i_n+j_n,i+j}$ for $i, j \in \{0, 1\}$.

Thus, a system of segments $\delta_{i_1,i_2,\ldots,i_n}$, $i_k \in \{0, 1\}$, $n = 1, 2, 3, \ldots$, satisfying (1), is constructed.

Put $P_n = \bigcup_{i_1,i_2,\ldots,i_n} \delta_{i_1,i_2,\ldots,i_n}$.

It follows from (1) that $P_n + P_n = S_n$ for $n = 1, 2, 3, \ldots$ The intersection of all sets P_n is some perfect set which is denoted by P.

Since $P + P = \bigcap_{n=1}^{\infty} (P_n + P_n)$ and $P_n + P_n = S_n$, we have finally $P + P = F$.

Remark. If F is symmetric with respect to 0, then P is also symmetric with respect to 0. In fact, it is easy to check by induction that a system of segments $\delta_{i_1,i_2,\ldots,i_n}$ is symmetric with respect to 0 if a system Σ_n is symmetric with respect to 0.

COROLLARY. *If F is a quite symmetric perfect set then there exist two perfect sets A and B such that $A - B = F$.*

It is sufficient to put $A = P$, $B = P^{(0)}$.

3. THEOREM 2. *Every linear set of positive Lebesgue measure contains a quite symmetric perfect subset.*

Proof. Let E be a certain perfect subset of positive measure of the given set. Now we take a certain metric density point ξ_1 of E and put $P = E \cap E^{(\xi_1)}$. Let ξ_0, $\xi_0 < \xi_1$, be an other metric density point of P. We take a number ε such that $0 < \varepsilon < \frac{1}{3}$.

Since ξ_0, ξ_1 are metric density points of P, there esists a number $\delta > 0$ such that the following inequalities hold:

$$\frac{m(P \cap [\xi_0 - r, \xi_0 + r])}{2r} > 1-\varepsilon, \qquad \frac{m(P \cap [\xi_1 - r, \xi_1 + r])}{2r} > 1-\varepsilon$$

provided $0 < r < \delta$.

Let r_0 be such a number that $0 < r_0 < \min(\delta, (\xi_1 - \xi_0)/2)$. Put $H = (P \cap [\xi_0 - r_0, \xi_0 + r_0])_{\xi_0}$. Let $P_1 = P \cap [\xi_1 - r_0, \xi_1 + r_0] \cap (H + \{\xi_1 - \xi_0\})$. Then, since $mH > 2r_0(1-2\varepsilon)$ and $m(P \cap [\xi_1 - r_0, \xi_1 + r_0]) > 2r_0(1-\varepsilon)$, we see that $mP_1 > 2r_0(1-3\varepsilon) > 0$.

Furthermore, ξ_1 is a metric density point of P_1.

Now we put $P_0 = P_1 - \{\xi_1 - \xi_0\}$, $P_2 = P_1 + \{\xi_1 - \xi_0\}$. Since P is symmetric with respect to ξ_1 and $P_0 \subset P$, we have that $P_2 \subset P$.

Let us assume that 3^n congruent perfect sets $P_{t_1,t_2,\ldots,t_n}$, $t_k \in \{0, 1, 2\}$, of positive measure have been obtained and for these sets the following conditions are fulfilled:

(i) *there is a quite symmetric system of segments $\Delta_{t_1,t_2,\ldots,t_n}$ which correspondingly contain the sets $P_{t_1,t_2,\ldots,t_n}$;*

(ii) *the center of $\Delta_{t_1,t_2,\ldots,t_n}$ coincides with the center of symmetry of $P_{t_1,t_2,\ldots,t_n}$;*

(iii) *the center of symmetry of $P_{t_1,t_2,\ldots,t_n}$ is simultaneously a metric density point of $P_{t_1,t_2,\ldots,t_n}$.*

If we make one of these sets $P_{t_1,t_2,\ldots,t_n}$, for instance $P_{0,0,\ldots,0}$, to play the role of the set P, then we shall obtain three perfect sets $P_{0,\ldots,0,i}$, $i \in \{0, 1, 2\}$. Now for every sequence $t_1, t_2, \ldots, t_n$ we put $P_{t_1,\ldots,t_n,i} = P_{0,0,\ldots,0,i} + \{\nu - a\}$, where ν is the center of $\Delta_{t_1,t_2,\ldots,t_n}$, and a is the center of $\Delta_{0,0,\ldots,0}$.

Thus, we shall obtain 3^{n+1} congruent perfect sets $P_{t_1,\ldots,t_n,t_{n+1}}$ for which conditions (i)-(iii) are fulfilled.

Having supposed $F_n = \bigcup_{t_1,\ldots,t_n} P_{t_1,t_2,\ldots,t_n}$, we obtain $E \supseteq F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$

Let $F = \bigcap_{n=1}^{\infty} F_n$. From conditions (i)-(iii) it is easy to see that F is a quite symmetric perfect set.

COROLLARY 1. *For every linear set E of positive Lebesgue measure there exists a perfect set P such that $P+P \subseteq E$.*

COROLLARY 2. *If a closed set E has 0 for a metric density point, then there exists a perfect set P such that $\mathfrak{D}(P) \subseteq E$.*

We take a quite symmetric subset F which is symmetric with respect to 0. There exists a perfect set P such that $P+P = F$.

Since P is also symmetric with respect to 0, we have $P-P = P+P^{(0)} = P+P$ and $\mathfrak{D}(P) = (P-P) \cap [0, \infty] = F \cap [0, \infty)$. Thus, $\mathfrak{D}(P) \subset F \subseteq E$.

Corollary 1 solves a problem stated by J. Mycielski in § 4.2 of [1]. J. Mycielski informs us, while this paper was still in preparation, that he found a more direct proof of Corollaries 1 and 2 which is to appear in [2]. I am greatly thankful to him.

References

[1] Jan Mycielski, *Independent sets in topological algebras*, Fund. Math. 55 (1964), pp. 139-147.

[2] — *Complete subgraphs and measure*, Actes des Journées sur la Théorie des Graphes — ICC, Dunod 1967.

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On distributive quasi-lattices

by

Jerzy Płonka (Wrocław)

Introduction. In the present paper we are concerned with certain algebras which we call *distributive quasi-lattices* (DQL). These algebras generalize distributive lattices and the main result proved here is a representation theorem which essentially reduce their structure to that of a certain direct spectrum of distributive lattices. To get this we need first analogous results on certain semigroups called here *idempotent quasi-Abelian semigroups*.

Finally, we characterize the independence in idempotent quasi-Abelian semigroups and in distributive quasi-lattices.

§ 1. Idempotent quasi-Abelian semigroups. Let us consider an algebra $\mathfrak{P} = \langle X; \circ \rangle$ in which the fundamental operation $\circ$ satisfies the following axioms:

$$(x \circ y) \circ z = x \circ (y \circ z), \tag{1}$$

$$x \circ x = x, \tag{2}$$

$$x \circ y \circ z = x \circ z \circ y. \tag{3}$$

In view of (1) and (2) $\mathfrak{P}$ is an idempotent semigroup. Such semigroups will be called *idempotent right quasi-Abelian semigroups* (QAS).

If $\circ$ fulfils (1), (2), and

$$x \circ y \circ z = y \circ x \circ z, \tag{3'}$$

then $\mathfrak{P}$ is called an *idempotent left quasi-Abelian semigroup*. In the sequel we deal only with idempotent right quasi-Abelian semigroups. The left case is dual.

We shall give now some examples of such semigroups.

EXAMPLES. 1. Let a be a positive real number. Let the set X of the algebra $\mathfrak{P}$ be $\{x\colon 0 < |x| \leqslant a\}$.

Operation $\circ$ will be defined as follows: If $\operatorname{sgn} x = \operatorname{sgn} y$ then $x \circ y = x$; in the remaining cases, $x \circ y = |x|$. It is easy to see that here $\circ$ depends on both variables.