Partially well ordered sets and partial ordinals

by

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1. Introduction. If P is a set and R is a relation on P which is reflexive, antisymmetric, and transitive, we shall call the pair (P, R) a *partially ordered set*. If x and y are distinct elements of P, we interpret xRy intuitively as "x precedes y". A partially ordered set (P, R) is *partially well ordered* (*pwo*) if and only if for every infinite sequence $\{x_n\}$ in P, there exist i and j with $i < j$ and $x_i R x_j$. Pwo-sets have been studied by Higman [6], Kruskal [8], [9], Michael [10], Nash-Williams [11], [12], R. Rado [15], and Tarkowski [18]. Some of these authors have considered quasi-orderings instead of partial orderings, and have of course adopted different terminologies. Also, several other definitions have been given which are equivalent to the one above.

The notion of partial well ordering is obviously a natural extension of the concept of well ordering, since a well-ordered set is a pwo-set which is linearly ordered by the relation R. The purpose of this paper is, in essence, to extend some of the well-known classical theory of well ordered sets and order types to pwo-sets. Certain basic theorems on order ideals and isomorphisms of pwo-sets are obtained. We define the order type of a pwo-set, which we call a "partial ordinal", and we show that the class $\mathcal{T}$ of all partial ordinals is partially ordered in a natural way. Furthermore, this ordering of $\mathcal{T}$ is an extension of the usual ordering of the class of all ordinals, which we are then able to characterize abstractly as a certain subclass of $\mathcal{T}$. The structure of the class $\mathcal{T}$ appears to be of some interest, and we obtain several theorems describing some of its properties.

2. Preliminary results and a representation theorem. We first give several more definitions. Let P be a set which is partially ordered by a relation $\leqslant$. If $x, y \in P$, we say that x and y are *incomparable* if and only if $x \nleqslant y$ and $y \nleqslant x$. A subset Q of P is *totally unordered* if and only if any two distinct elements of Q are incomparable. By $x < y$ we mean $x \leqslant y$ and $x \neq y$. An infinite sequence $\{x_n\}$ in P is *strictly increasing* (*strictly decreasing*) if and only if $x_n < x_{n+1}$ ($x_{n+1} < x_n$) for all n. If $Q \subseteq P$,

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we say that m is a *minimal element* of Q if and only if $m \in Q$ and there is no $x \in Q$ with $x < m$. A subset M of Q is a *minimal subset* of Q if and only if (i) each $m \in M$ is a minimal element of Q, and (ii) for all $x \in Q$, there exists $m \in M$ with $m \leqslant x$. The obvious dual definitions of *maximal element* and *maximal subset* will also be used. A mapping f of a partially ordered set $(P, \leqslant)$ into a partially ordered set $(P', \leqslant')$ is *order-preserving* if and only if $x, y \in P$ and $x \leqslant y$ imply $f(x) \leqslant' f(y)$. It is clear that the image of a pwo-set under an order-preserving mapping is pwo. A 1:1 mapping f of $(P, \leqslant)$ onto $(P', \leqslant')$ is an *isomorphism* if and only if both f and f^{-1} are order-preserving.

The cardinal number of a set P will be denoted by $|P|$. We denote set inclusion by $\subseteq$, reserving $\subset$ for proper inclusion. The empty set will be denoted by $\emptyset$.

The following theorem is well known ([9], [10]), and we therefore omit its proof.

THEOREM 1. *The following properties of a partially ordered set* $(P, \leqslant)$ *are equivalent.*

(a) $(P, \leqslant)$ *is* pwo.

(b) *Every infinite sequence of distinct elements of* P *contains a strictly increasing subsequence.*

(c) P *contains no infinite strictly decreasing sequence and no infinite totally unordered subset.*

(d) *Every subset of* P *contains a finite minimal subset.*

If $\{X_a\colon a \in I\}$ is any family of sets, we write $\prod\{X_a\colon a \in I\} = \{f\colon f$ is a function on I and $f(a) \in X_a$ for all $a \in I\}$. We call the elements of $\prod X_a$ *choice functions* for the family $\{X_a\}$. If R_a is a partial order on X_a for each $a \in I$, a partial order S on $\prod X_a$ may be defined by fSg if and only if $f(a) R_a g(a)$ for all $a \in I$. The partially ordered set $(\prod X_a, S)$ is called the *cardinal product* of the sets (X_a, R_a). The following theorem is also well known ([10], [12]).

THEOREM 2. *The cardinal product of finitely many* pwo-*sets is* pwo.

We shall also make repeated use of a fundamental theorem on choice functions due to R. Rado ([4], [14]).

THEOREM 3 (R. Rado). *Let* $\{X_i\colon i \in I\}$ *be a family of non-empty finite sets, and suppose that for each finite subset* A *of* I *we are given a choice function* f_A *for the family* $\{X_i\colon i \in A\}$. *Then there exists a choice function* f *for* $\{X_i\colon i \in I\}$ *such that, whenever* A *is a finite subset of* I, *there is a finite set* B *with* $A \subseteq B \subseteq I$ *and* $f(i) = f_B(i)$ *for all* $i \in A$.

If R_1 and R_2 are partial ordering relations on the same set P, and $R_1 \subseteq R_2$, we say that R_2 is an *extension* of R_1. Szpilrajn [17] has proved that any partial order on a set P has a linear extension. Following Dushnik

and Miller [3], we define the *dimension* of the partially ordered set (P, R) as the smallest cardinal k such that R is the intersection of k linear orders on P. This definition may also be given in the following equivalent form: the dimension of (P, R) is the smallest cardinal k such that (P, R) can be isomorphically embedded in the cardinal product of k linearly ordered sets. A proof of the equivalence of these definitions is given in [13]. It is clear that there exist pwo-sets of any finite dimension and also of countably infinite dimension. However, it is an open question whether there exist pwo-sets of uncountable dimension.

We now obtain another characterization of a pwo-set. For this purpose we first prove a lemma.

LEMMA 1. *If* (P, R) *is a partially ordered set containing an infinite totally unordered subset, then there is a linear extension of* R *which is not a well ordering.*

Proof. Let $\{q_1, q_2, \ldots, q_n, \ldots\}$ be an infinite totally unordered subset of P. We define a relation R_1 on P by xR_1y if and only if (i) xRy or (ii) xRq_2 and q_1Ry. Then R_1 is a partial ordering of P and is an extension of R. Also note that $q_2R_1q_1$, by (ii). We then define R_2 by xR_2y if and only if (i) xR_1y or (ii) xR_1q_3 and q_2R_1y. Continuing by induction, we may thus define, for each $n = 1, 2, \ldots$, a relation R_n on P by xR_ny if and only if (i) $xR_{n-1}y$ or (ii) $xR_{n-1}q_{n+1}$ and $q_nR_{n-1}y$. Each R_n is an extension of R, and $R_n \subset R_{n+1}$ for all n. Define $S = \bigcup\{R_n\colon n = 1, 2, \ldots\}$. Then S is a partial ordering of P, and the sequence $\{q_n\}$ is strictly decreasing in (P, S) (i.e., $q_{n+1}Sq_n$ for all n). If S^* is any linear extension of S, then (P, S^*) is not well ordered. But S^* is also a linear extension of R.

We now have

THEOREM 4. *A partially ordered set* (P, R) *is* pwo *if and only if every linear extension of* R *is a well ordering of* P.

Proof. Assume that (P, R) is pwo, and let L be any linear extension of R. Then the identity mapping is order-preserving on (P, R) onto (P, L), and hence L is a well ordering. To prove the converse, suppose that every linear extension of R is a well ordering. Then clearly P can contain no infinite sequence $\{x_n\}$ which is strictly decreasing in the order R, because such a sequence would be strictly decreasing in any extension of R. Furthermore, by Lemma 1, (P, R) contains no infinite totally unordered subset. Hence, by Theorem 1(c), (P, R) is pwo.

For pwo-sets of finite dimension, we now have the following useful "representation" theorem.

THEOREM 5. *Let* (P, R) *be a partially ordered set of finite dimension. Then* (P, R) *is* pwo *if and only if* (P, R) *is isomorphic to a subset of the cardinal product of a finite number of well ordered sets.*

Proof. Suppose that (P, R) has finite dimension and is pwo. Then there exists a finite number of linear extensions $L_1, \ldots, L_n$ of R such that $R = \bigcap \{L_i\colon i = 1, \ldots, n\}$; and furthermore each L_i is a well ordering by Theorem 4. Let $(\Pi, \leqslant)$ denote the cardinal product of the well ordered sets $(P, L_1), \ldots, (P, L_n)$. For each $x \in P$, left f_x denote the function in Π defined by $f_x(i) = x$ for all $i = 1, 2, \ldots, n$. Note that $f_x \leqslant f_y$ if and only if xL_iy for all $i = 1, \ldots, n$; and this is also equivalent to xRy. Hence (P, R) is isomorphic to the subset $\{f_x\colon x \in P\} \subseteq \Pi$. The proof of the converse statement is trivial, since any subset of a pwo-set is pwo.

3. Order ideals. Throughout this section and the next we shall often refer to a partially ordered set P without explicit mention of the order relation, which will usually be denoted by $\leqslant$. If $W \subseteq P$, we say that W is an *order ideal* (*lower set*, in the terminology of Kruskal and Nash-Williams) in P if and only if whenever $y \in W$ and $x \leqslant y$, then also $x \in W$. The set of all order ideals of P (including $\emptyset$ and P itself) will be denoted by $L(P)$. With respect to the usual ordering of set inclusion, $L(P)$ forms a complete lattice.

If P is a pwo-set and the lattice $L(P)$ is also pwo, we shall say that P is *normal*. The following example, which is due to Kruskal [8], shows that there exist non-normal pwo-sets. Let ω denote the non-negative integers (with the usual order), and let $J_2 = \omega \times \omega$. We define a partial ordering $\leqslant$ on J_2 as follows. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be any elements of J_2. If $a_1 = b_1$, we define $a \leqslant b$ if and only if $a_2 \leqslant b_2$; if $a_1 < b_1$, we define $a < b$ if and only if $a_1 + a_2 \leqslant b_1$. Then $(J_2, \leqslant)$ is a pwo-set. For each $i = 0, 1, \ldots$, let $C_n = \{(x, y)\colon x = n\}$. Let A_n be the order ideal of J_2 generated by C_n, i.e., $A_n = \{a \in J_2\colon a \leqslant b \text{ for some } b \in C_n\}$. Then clearly A_i and A_j are incomparable sets if $i \neq j$. Thus $L(J_2)$ contains an infinite totally unordered subset and J_2 is non-normal.

Kruskal [8] has shown, furthermore, that *any non-normal pwo-set contains an isomorphic image of J_2.*

$L(P)$ does have the following property, however.

THEOREM 6. *If P is a* pwo-*set, then $L(P)$ contains no infinite strictly decreasing sequence.*

Proof. Suppose that there exists an infinite strictly decreasing sequence $\{W_n\}$ of order ideals of P. Then for each n, there exists $x_n \in W_n - W_{n+1}$. But P is pwo implies $x_i \leqslant x_j$ for some i, j with $i < j$. But this implies $x_i \in W_j$, contradicting the definition of the sequence $\{x_n\}$.

LEMMA 2. *If Q is a subset of a normal* pwo-*set P, then Q is normal.*

Proof. We are of course considering the ordering on Q to be that induced by the ordering on P. Let $\{W_n\}$ be any sequence of order ideals of Q. Let $A_n = \{x \in P\colon x \leqslant y \text{ for some } y \in W_n\}$. Then $\{A_n\}$ is a sequence of order ideals of P, and hence by hypothesis there exist i and j with $i < j$ and $A_i \subseteq A_j$. Hence $W_i = A_i \cap Q \subseteq A_j \cap Q = W_j$, completing the proof.

We now have

THEOREM 7. *If P is a finite-dimensional* pwo-*set, then P is normal.*

Proof. By Lemma 11 of [12] we infer immediately that the cardinal product of finitely many well ordered sets is normal. By Theorem 5, P is isomorphic to a subset of such a product, and Theorem 7 thus follows from Lemma 2.

Theorem 7 shows, incidentally, that Kruskal's pwo-set J_2 must have infinite dimension.

4. M-decompositions. We shall now construct by transfinite induction a useful "canonical" decomposition of an arbitrary pwo-set $(P, \leqslant)$. In this section the small Greek letters α, β, γ, and λ will denote ordinal numbers. Let M_0 be the minimal subset of P. If M_α is defined for all ordinals $\alpha < \beta$, define M_β as the minimal subset of $P - \bigcup \{M_\alpha\colon \alpha < \beta\}$. If λ is the first ordinal for which $M_\lambda = \emptyset$, then the family of finite non-empty sets $\{M_\alpha\colon \alpha < \lambda\}$ will be referred to as the *M-decomposition* of P. The members of this family will be called *M-sets*. Clearly P is the union of all its M-sets; and if $y \in M_\beta$ for some $\beta < \lambda$, then for all $\alpha < \beta$ there exists $x \in M_\alpha$ with $x < y$.

We now state two simple lemmas whose proofs may be left to the reader.

LEMMA 3. *If P is a* pwo-*set with M-decomposition $\{M_\alpha\colon \alpha < \lambda\}$, and $W \in L(P)$, then the M-decomposition of W is $\{M_\alpha \cap W\colon \alpha < \gamma\}$ for some $\gamma \leqslant \lambda$.*

LEMMA 4. *Let P and P' be* pwo-*sets with M-decompositions $\{M_\alpha\colon \alpha < \lambda\}$ and $\{M'_\alpha\colon \alpha < \lambda'\}$, respectively.*

(i) *If f is an isomorphism of P onto P', then $f[M_\alpha] = M'_\alpha$ for all $\alpha < \lambda$ (and hence $\lambda = \lambda'$).*

(ii) *If f and g are both isomorphisms of P onto P', and x is any element of P, then $f(x)$ and $g(x)$ are in the same M-set of P'.*

If $\mathcal{F} = \{W_i\colon i \in I\}$ is a family of order ideals of P, let us say that $\mathcal{F}$ is *up-directed* if and only if for every $i, j \in I$, there exists $k \in I$ with $W_i \cup W_j \subseteq W_k$. We now have the following theorem, which we shall need in the next section.

THEOREM 8. *Let P and P' be* pwo-*sets, and I a set of indices. Let $\{W_i\colon i \in I\}$ and $\{W'_i\colon i \in I\}$ be up-directed families of order ideals of P and P', respectively. Suppose also that $P = \bigcup \{W_i\colon i \in I\}$, $P' = \bigcup \{W'_i\colon i \in I\}$, and W_i is isomorphic to W'_i for all $i \in I$. Then P is isomorphic to P'.*

Proof. Let $\{M_\alpha: \alpha < \lambda\}$ and $\{M'_\alpha: \alpha < \lambda'\}$ be the M-decompositions of P and P', respectively. For each $i \in I$, let f_i be an isomorphism of W_i onto W'_i. Let x be any element of P. Using Lemma 4(ii), it is easy to show that for all i with $x \in W_i$, all the elements $f_i(x)$ lie in the same M-set of P'. Thus for each $x \in P$, there is a unique M-set of P', which we denote by M_x, which contains all images $f_i(x)$ for all i with $x \in W_i$. (The sets M_x of course need not all be distinct.) Let A be any finite subset of P. Since each $x \in A$ is in some W_i, and $\{W_i: i \in I\}$ is up-directed, there is some W_k with $A \subseteq W_k$. We denote the restriction $f_k|A$ of the function f_k to the set A by f_A. Thus for each finite $A \subseteq P$, there is a choice function f_A for the family $\{M_x: x \in A\}$ such that f_A is an isomorphism of A into P'. Now by Rado's Theorem (Theorem 3), there is a choice function f for $\{M_x: x \in P\}$ such that, for all finite $A \subseteq P$, there is a finite set B with $A \subseteq B \subseteq P$ and $f(x) = f_B(x)$ for all $x \in A$. Clearly f is an isomorphism of P into P'.

Now we must prove that f is *onto* P'. To do this, we first show that $f[M_\alpha] \subseteq M'_\alpha$ for all $\alpha < \lambda$ (and hence $\lambda \leqslant \lambda'$). Given any $\alpha_0 < \lambda$, let $A = M_{\alpha_0}$. There exists a finite set B with $A \subseteq B \subseteq P$ and $f(x) = f_B(x)$ for all $x \in A$. Also there exists $k \in I$ such that $B \subseteq W_k$ and the function f_B is the restriction to B of the isomorphism f_k. By Lemma 3, the M-decompositions of W_k and W'_k are $\{W_k \cap M_\alpha: \alpha < \gamma\}$ and $\{W'_k \cap M'_\alpha: \alpha < \gamma\}$, respectively, for some $\gamma > \alpha_0$; and $f_k[W_k \cap M_\alpha] = W'_k \cap M'_\alpha$ for all $\alpha < \gamma$ (Lemma 4). But $M_{\alpha_0} \subseteq W_k$, and so $f_k[M_{\alpha_0}] \subseteq M'_{\alpha_0}$. But also for any $x \in M_{\alpha_0}$, we have $f(x) = f_B(x) = f_k(x)$. Hence $f[M_{\alpha_0}] \subset M'_{\alpha_0}$.

To conclude the proof of Theorem 8, we may reverse the roles of P and P' and prove, analogously, that there exists an isomorphism g of P' into P with $g[M'_\alpha] \subseteq M_\alpha$ for all $\alpha < \lambda'$ (and hence $\lambda' \leqslant \lambda$). From this we conclude that $\lambda = \lambda'$, and also that $|M_\alpha| = |M'_\alpha|$ for all $\alpha < \lambda$. But since M_α and M'_α are finite, we must have $f[M_\alpha] = M'_\alpha$ for all $\alpha < \lambda$, and hence f is *onto* P'.

The following simple example shows that Theorem 8 does not hold for arbitrary partially ordered sets P and P'. Let P be the set of all integers and P' the set of all negative integers, and let these sets have their usual ordering. For $i = 1, 2, \ldots$, let $W_i = \{k \in P: k \leqslant i\}$, and let $W'_\alpha = P'$ for all i. These families of order ideals are both up-directed. Since W_i is isomorphic to W'_i for all i, the hypotheses of Theorem 8 are all satisfied except for the condition of partial well ordering. But P is not isomorphic to P'.

We prove next (also for use in Section 5) a theorem which may be considered as an extension of König's "Infinity Lemma" ([7], p. 81).

THEOREM 9. *If P is a* pwo-*set with M-decomposition $\{M_\alpha: \alpha < \lambda\}$, then there is a chain in P containing precisely one element from each M-set of P.*

Proof. Let I denote the set of all ordinals less than λ, and let A be any finite subset of I. Using the properties of the M-decomposition and making a finite number of choices, we may construct a function $f_A \in \prod\{M_\alpha: \alpha \in A\}$ such that $f_A[A]$ is a chain. Thus we may apply Rado's Theorem to the family $\{M_\alpha: \alpha < \lambda\}$ to obtain a function $g \in \prod\{M_\alpha: \alpha < \lambda\}$ such that, for each finite $A \subseteq I$, there is a finite set B with $A \subseteq B \subseteq I$ and $g(\alpha) = f_B(\alpha)$ for all $\alpha \in A$. It is clear that the image $g[I]$ of I with respect to g is a chain in P containing precisely one element from each M-set.

The reader will observe that the König Infinity Lemma may be obtained from Theorem 9 by setting $\lambda = \omega$.

The following corollary of Theorem 9 is obvious.

COROLLARY 1. *Any infinite* pwo-*set P contains a chain C with $|C| = |P|$.*

As another corollary to Theorem 9, of incidental interest, we may easily prove a well-known theorem of Lindenbaum on well orderings [5]. If (P, R) is a partially ordered set and $Y \subseteq P$, let us denote the ordering induced by R on Y by R/Y: i.e., $R/Y = R \cap (Y \times Y)$. We then have

COROLLARY 2. *Let X be an infinite set, and suppose that $\{R_1, \ldots, R_k\}$ is any finite set of well orderings of X. Then there exists $Y \subseteq X$ with $|Y| = |X|$ and $R_i/Y = R_j/Y$ for all i and j $(i, j = 1, \ldots, k)$.*

Proof. Consider the well ordered sets (X, R_i), for $i = 1, \ldots, k$, and let $(P, \leqslant)$ denote their cardinal product. Each $x \in X$ occurs as an element, say x_i, of (X, R_i). The correspondence $x \to (x_1, \ldots, x_k)$ is then a $1:1$ mapping of X onto a subset of P, which we identify with X itself. By Theorem 2, P is pwo, and hence so is its subset X. By Corollary 1 to Theorem 9, X contains a chain Y with $|Y| = |X|$. If $x, y \in Y$, then by definition of the cardinal product order we have xR_iy for all i, or yR_ix for all i. Thus all the orderings R_i coincide on the set Y: i.e., $R_i/Y = R_j/Y$ for all i and j.

5. Partial ordinals. Let P be any partially ordered set. We define the *order type* of P, denoted by τP, as the class of all partially ordered sets which are isomorphic to P. The order type of a pwo-set will be called a *partial ordinal*. We shall denote partial ordinals by small Greek letters, such as μ, σ, and ξ. Sets of partial ordinals will be denoted by capital Greek letters. The class of all partial ordinals will be denoted by $\mathfrak{I}$, and the subclass of $\mathfrak{I}$ consisting of the ordinals will be denoted by $\mathfrak{O}$.

We now define a relation $\leqslant$ on the class $\mathfrak{I}$ as follows. If $\tau P, \tau Q \in \mathfrak{I}$, then $\tau P \leqslant \tau Q$ if and only if P is isomorphic to an order ideal of Q. The relation $\leqslant$ is obviously reflexive and transitive. The following theorem and corollary show that it is also antisymmetric.

THEOREM 10. *If* $(P, \leqslant)$ *is a* pwo-*set, and* f *is an isomorphism of* P *onto an order ideal* $f[P]$ *of* P*, then* $f[P] = P$.

Proof. The proof of the theorem will be by contradiction. Let us define f^0 as the identity function, and for n a positive integer, let $f^n = f \circ f^{n-1}$. Assume that $f[P] \neq P$. We shall then prove by induction that for all n, (i) $f^n[P]$ is a proper subset of $f^{n-1}[P]$, and (ii) $f^n[P]$ is an order ideal of P.

Note that (i) is true for the case $n = 1$ by assumption. Suppose, then, that $f^n[P] \subset f^{n-1}[P]$ for some n. Let $x_0 \in f^{n-1}[P] - f^n[P]$. Then $f(x_0) \in f^n[P]$. If $f^{n+1}[P] = f^n[P]$, then $f(x_0) \in f^{n+1}[P]$ and hence there exists $y \in f^n[P]$ with $f(y) = f(x_0)$. Since $y \neq x_0$, this contradicts the hypothesis that f is 1:1. Hence $f^{n+1}[P] \subset f^n[P]$.

Next we prove (ii) by induction. The case $n = 1$ is trivially true by hypothesis. Suppose, then, that $f^n[P]$ is an order ideal of P for some n. Let $u \in f^{n+1}[P]$ and let $v \leqslant u$. We shall prove that $v \in f^{n+1}[P]$. Since $u \in f^n[P]$, and $f^n[P]$ is an order ideal, we know that $v \in f^n[P]$. Hence there exists $x \in f^{n-1}[P]$ with $f(x) = v$. Also, there exists $y \in f^{n-1}[P]$ with $f(f(y)) = u$. Thus we have $f(x) \leqslant f(f(y))$; and since f is an isomorphism, it follows that $x \leqslant f(y)$. But $f(y) \in f^n[P]$ and $f^n[P]$ is an order ideal: hence $x \in f^n[P]$. Hence $v = f(x) \in f^{n+1}[P]$. This completes the proof of (ii).

(i) and (ii) now imply that $\{f^n[P]\}$ is an infinite strictly decreasing sequence in $L(P)$, which contradicts Theorem 6. This completes the proof of Theorem 10.

The following corollary, which proves that the relation $\leqslant$ is antisymmetric on $\mathfrak{I}$, is now obvious.

COROLLARY 1. *If* P *and* Q *are* pwo-*sets, and each is isomorphic to an order ideal of the other, then* P *and* Q *are isomorphic.*

It should be noted at this point that the partial ordering relation $\leqslant$, which we defined above for partial ordinals, does not provide us with a partial ordering of the class of order types of *all* partially ordered sets, because of the failure of the antisymmetric property. A simple example to show that Corollary 1 fails for arbitrary partially ordered sets is obtained by taking P as the set of all real numbers (with the usual order), and Q as the set of all real numbers with a greatest element adjoined.

The reader will also note that the ordering induced by $\leqslant$ on the subclass $\mathfrak{O}$ of $\mathfrak{I}$ agrees with the usual ordering of the ordinal numbers.

We have two more simple corollaries of Theorem 10.

COROLLARY 2. *If* A *and* B *are distinct order ideals of a* pwo-*set* P *and* A *is isomorphic to* B*, then* A *and* B *are incomparable sets.*

COROLLARY 3. *If* P *is a normal* pwo-*set, then every family of mutually isomorphic order ideals of* P *is finite.*

Now let P be any pwo-set, and write $\Gamma(P) = \{\sigma \in \mathfrak{I}: \sigma \leqslant \tau P\}$. The correspondence $Q \to \tau Q$ then defines a mapping of the complete lattice $L(P)$ onto $\Gamma(P)$. This mapping is order-preserving but it need not be 1:1, since mutually isomorphic order ideals map into the same image in $\Gamma(P)$. $\Gamma(P)$ may fail to be pwo if P is non-normal, and the reader may verify that this is indeed the case for $\Gamma(J_2)$. Also $\Gamma(P)$ need not be complete, nor even a lattice. As an example of this, consider the pwo-sets with the following Hasse diagrams.

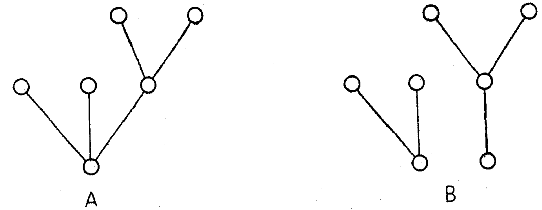

Fig. 1

Let P denote the cardinal sum $A + B$, in the sense of Birkhoff ([1], p. 7). Also, let the two "components" of B be B_1 and B_2. Then, in $\Gamma(P)$, the pair of partial ordinals τA and τB has no greatest lower bound; for the set of common lower bounds of τA and τB contains the two maximal elements τB_1 and τB_2. Hence $\Gamma(P)$ is not a lattice.

Although $\Gamma(P)$ may fail to be a complete lattice, it does have a certain completeness property in the normal case, which is a consequence of the following theorem.

THEOREM 11. *If* P *is a normal* pwo-*set and* D *is a chain in* $L(P)$*, then*

$$\tau(\bigcup \{W: W \in D\}) = \text{l.u.b. } \{\tau W: W \in D\}.$$

Proof. Let Δ be the chain of partial ordinals $\{\tau W: W \in D\}$ in $\Gamma(P)$, and let Δ^* be the set of all upper bounds of Δ in $\Gamma(P)$. Let $A = \bigcup \{W: W \in D\}$, and suppose that B is any member of $L(P)$ with $\tau B \in \Delta^*$. We shall show that $\tau A \leqslant \tau B$. For each $\sigma \in \Delta$, there exists a set E_σ of order ideals in B such that $W_\sigma \in E_\sigma$ implies $\tau W_\sigma = \sigma$. Each E_σ is a non-empty, totally unordered, finite subset of $L(B)$, by Corollaries 2 and 3 of Theorem 10. Moreover, $\{E_\sigma: \sigma \in \Delta\}$ is precisely the M-decomposition of the pwo-set $\bigcup \{E_\sigma: \sigma \in \Delta\}$. Hence by Theorem 9 we may choose exactly one W'_σ from each E_σ such that $\{W'_\sigma: \sigma \in \Delta\}$ is linearly ordered by inclusion. Let $A' = \bigcup \{W'_\sigma: \sigma \in \Delta\}$. Also, for each $\sigma \in \Delta$, let V_σ be the member of D with $\tau V_\sigma = \sigma$. Then V_σ is isomorphic to W'_σ for all $\sigma \in \Delta$, and hence by Theorem 8, A is isomorphic to A'. But $A' \subseteq B$, and so $\tau A \leqslant \tau B$. Hence $\tau A = \text{l.u.b. } \Delta$.

COROLLARY. *If P is a normal* pwo-*set, then every chain in $\Gamma(P)$ has a least upper bound in $\Gamma(P)$.*

Proof. If Δ is any chain in $\Gamma(P)$, then, as in the proof of Theorem 11, we may use Theorem 9 to select a chain D in $L(P)$ whose image under the mapping τ is precisely Δ. Then $\tau(\bigcup\{W\colon W \in D\}) = \text{l.u.b.}\,\Delta$.

We now prove another theorem describing the structure of $\Gamma(P)$.

THEOREM 12. *Let P be a normal* pwo-*set. If $\sigma_1, \sigma_2 \in \Gamma(P)$, then* (a) *the set of upper bounds in $\Gamma(P)$ of σ_1 and σ_2 has a finite minimal subset, and* (b) *the set of lower bounds of σ_1 and σ_2 has a finite maximal subset.*

Proof. (a) is obvious, since $\Gamma(P)$ is pwo. To prove (b), let Σ be the set of all common lower bounds of σ_1 and σ_2. If Δ is any chain in Σ, then Δ has a least upper bound ξ in $\Gamma(P)$, by Theorem 11. But σ_1 and σ_2 are upper bounds of Δ; hence $\xi \leqslant \sigma_1$, $\xi \leqslant \sigma_2$, and $\xi \in \Sigma$. Thus every chain in Σ has a least upper bound in Σ, and hence by Zorn's Lemma, Σ has a maximal subset, which must be finite.

It is natural to ask whether the class $\mathcal{O}$ of ordinals can be characterized "abstractly" as a certain subclass of $\mathfrak{T}$. It is clear that if $\gamma \in \mathcal{O}$, then $\{\sigma \in \mathfrak{T}\colon \sigma < \gamma\}$ is linearly ordered, but this property is not characteristic of the ordinals. Consider, for example, the pwo-set $(P, \leqslant)$, where $P = \{a, b, c\}$ and $b < a$, $c < a$ (but b and c are incomparable). Since the order ideals $\{b\}$ and $\{c\}$ are isomorphic, $\Gamma(P)$ is linearly ordered. However, we do have the following theorem.

THEOREM 13. *Let γ be a partial ordinal with $\gamma = \tau P$. A necessary and sufficient condition that γ be an ordinal is that P be isomorphic to $\{\sigma \in \mathfrak{T}\colon \sigma < \gamma\}$.*

Proof. The necessity of the condition is clear. To prove the sufficiency, assume that P is isomorphic to $\{\sigma \in \mathfrak{T}\colon \sigma < \gamma\}$, but that γ is not an ordinal. This means that P is not linearly ordered. Let $\{M_\alpha\colon \alpha < \lambda\}$ be the M-decomposition of P, and let f be an isomorphism of P onto $\{\sigma \in \mathfrak{T}\colon \sigma < \gamma\}$. Let $\{M'_\alpha\colon \alpha < \lambda\}$ be the M-decomposition of $\{\sigma \in \mathfrak{T}\colon \sigma < \gamma\}$. Then $f[M_\alpha] = M'_\alpha$ for all $\alpha < \lambda$, by Lemma 4. Since P is not linearly ordered, there is a least ordinal $\beta < \lambda$ such that both M_β and M'_β contain more than one element. Then $\bigcup\{M_\alpha\colon \alpha < \beta\}$ is a chain, say C; and so is $f[C] = \bigcup\{M'_\alpha\colon \alpha < \beta\}$. Let us consider the partial ordinal τC. We assert that $\tau C \in f[C]$. For if this were not so, then $\tau C \in M'_\alpha$ for some $\alpha \geqslant \beta$. This would mean that (i) τC is incomparable with some partial ordinal $\sigma < \gamma$, or (ii) τC has a pair of incomparable predecessors in $\{\sigma\colon \sigma < \gamma\}$. But this is impossible, because the order ideal $C \in L(P)$ is comparable with all elements of $L(P)$ and contains no pair of incomparable order ideals.

Now let $\Omega = \Gamma(C) = \{\sigma\colon \sigma \leqslant \tau C\}$ and consider two possibilities.

Case 1. $\Omega = f[C]$. Note that $L(C)$ consists of all initial segments of C, together with C itself, partially ordered by inclusion; and furthermore $L(C)$ is isomorphic to Ω. Now if $\Omega = f[C]$, it follows that C is isomorphic to $L(C)$. But this is a contradiction, since C is well ordered.

Case 2. $\Omega \subset f[C]$. In this case τC is not the greatest element of $f[C]$, and hence Ω is an initial segment of $f[C]$. But then we have C isomorphic to $f[C]$ and also to an initial segment of $f[C]$: again a contradiction.

6. Arithmetic of partial ordinals. It may be remarked, in conclusion, that there is an arithmetic of partial ordinals which is a natural generalization of the arithmetic of ordinal numbers. It is easy to verify that the ordinal sum and ordinal product ([1], [2]) of two pwo-sets are also pwo-sets. This enables us to define the sum and product of partial ordinals in a natural way. Of more interest is the possibility of also defining exponentiation for partial ordinals. It is to be noted that Birkhoff's ordinal power ([1], [2]) does not provide us with a satisfactory definition of exponentiation, since if P and Q are pwo-sets, the ordinal power ${}^{Q}P$ may fail to be pwo. A satisfactory definition may be obtained, however, by generalizing the constructive procedure used by Sierpiński [16] in defining exponents for ordinal numbers. The details of this construction will not be given here.

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On a property of sets of positive measure

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1. For any subsets A, B of the real line we put $A+B = \{a+b\colon a \in A,\ b \in B\}$, $A-B = \{a-b\colon a \in A,\ b \in B\}$, $\mathfrak{D}(A, B) = \{|a-b|\colon a \in A,\ b \in B\}$, $D(A) = D(A, A)$. $\mathfrak{D}(A)$ is called the *set of distances* of A.

For every subset E of the real line, $E^{(\xi)}$ denotes the symmetrical reflection of E, ξ being the center of symmetry. It is evident that

$$A+B = A-B^{(0)}, \quad A-B = A+B^{(0)},$$

$$\mathfrak{D}(A, B) = [(A-B) \cup (B-A)] \cap [0, \infty), \quad \mathfrak{D}(A) = (A-A) \cap [0, \infty).$$

If E_ξ denotes the intersection of E and $E^{(\xi)}$ then E_ξ is symmetric with respect to ξ. If ξ is a metric density point for E then ξ is also a metric density point for E_ξ.

mE denotes the Lebesgue measure of E.

We say that a system Σ_n of 3^n congruent and disjoint segments $\Delta_{t_1,t_2,\ldots,t_n}$, where $t_k \in \{0, 1, 2\}$, is *quite symmetric* if for any natural number k, $1 \leqslant k \leqslant n$, it satisfies the following conditions:

(*) *$\Delta_{t_1,\ldots,t_{k-1},t_k,\ldots,t_n}$ and $\Delta_{t_1,\ldots,t'_{k-1},t'_k\ldots,t'_n}$ are symmetric with respect to the center of $\Delta_{t_1,\ldots,t_{k-1},1,1,\ldots,1}$ provided the sequence $t'_k, t'_{k+1}, \ldots, t'_n$ is obtained from the sequence $t_k, t_{k+1}, \ldots, t_n$ by the substitution 0 for 2 and 2 for 0;*

(**) *$\Delta_{t_1,t_2,\ldots,t_{k-1},\nu_1,\nu_2\ldots,\nu_{n-k+1}}$ will coincide with $\Delta_{\theta_1,\theta_2,\ldots,\theta_{k-1},\nu_1,\nu_2,\ldots,\nu_{n-k+1}}$ if the real line will be translated at a distance such that $\Delta_{t_1,\ldots,t_{k-1},1,1,\ldots,1}$ coincides with $\Delta_{\theta_1,\ldots,\theta_{k-1},1,1,\ldots,1}$.*

A perfect set F is called *quite symmetric* if there exists a sequence of quite symmetric systems $\Sigma_1, \Sigma_2, \ldots, \Sigma_n, \ldots$ such that the segment $\Delta_{t_1,t_2,\ldots,t_n}$ of Σ_n contains the segments $\Delta_{t_1,t_2,\ldots,t_n,t}$, $t \in \{0, 1, 2\}$, of Σ_{n+1} the centers of $\Delta_{t_1,t_2,\ldots,t_n}$ and $\Delta_{t_1,t_2,\ldots,t_n,1}$ coincide and $F = \bigcap_{n=1}^{\infty} S_n$, where $S_n = \bigcup_{\Delta \in \Sigma_n} \Delta$.

2. THEOREM 1. *If F is a quite symmetric perfect set, then there exists a perfect set P such that $P+P = F$.*

Proof. We denote the length of $\Delta_{t_1t_2,\ldots,t_n}$ by $2r_n$.

Let be $\Delta_t = [d_t, b_t]$, $t \in \{0, 1, 2\}$.