Does every continuum of type \( \kappa \) contain an indecomposable subcontinuum? Is every continuum of type \( \kappa \) separated by a countable collection of its subcontinua?

Reference

[1] E. S. Thomas, Jr., Monotone decompositions of irreducible continua, Rozprawy Mat. 50 (1965).

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Open mappings on graphs and manifolds*

by

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1. Introduction. In [3], pp. 182 and 197, Whyburn has shown that the image of a finite graph under a (light) open mapping is again a finite graph and also that the image under a light, open mapping of a 2-manifold is a 2-manifold. The purpose of this paper is to investigate the conditions under which an open mapping \( f(G) = H \) defined on a graph \( G \) can be extended to a light, open mapping \( F(M) = N \) defined on a 2-manifold \( M \) when \( G \) and \( H \) are imbedded in \( M \) and \( N \), respectively.

In section 3 it is shown that if \( f(G) = H \) is such an open mapping, then there exist imbeddings of \( G \) and \( H \) in some orientable 2-manifolds \( M \) and \( N \) respectively, and an extension of \( f \) to a light, open map \( F(M) = N \). The imbedding of \( H \) may be taken to be any orientable, 2-cell (and hence any minimal) imbedding of \( H \). Further, any light, open map \( F(M) = N \) on a closed orientable 2-manifold \( M \) can be obtained as an extension of such a map \( f(G) = H \) on a graph \( G \) minimally imbedded in \( M \). In section 4 it is shown that for each positive integer \( n \) there is an open mapping of a planar graph onto some graph of genus \( n \), and this is used to show that the imbedding obtained for \( G \) in the above result may necessarily be non-minimal.

2. Background. The term mapping will be used to denote a continuous transformation. A mapping \( f(X) = Y \) is said to be open provided that every open set in \( X \) maps onto a set open in \( Y \). If for each \( y \) in \( Y \), \( f^{-1}(y) \) is totally disconnected, \( f \) is said to be a light mapping. The term graph will denote a finite, connected 1-complex.

If \( f(G) = H \) is an open mapping on a graph \( G \), then \( H \) is also a graph. Further, \( f \) is a light mapping and it is possible to designate certain interior points on the edges of \( G \) and \( H \) as additional vertices in such a way that \( f \) maps each edge of \( G \) topologically onto an edge in \( H \). In light of these facts, we shall assume for the remainder that any open mapping on a graph is a simplicial transformation.

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It is also known (cf. [3], p. 197) that if \( M \) is a 2-manifold and \( F(M) = N \) a light, open mapping, then \( N \) is also a 2-manifold (though not necessarily orientable). Henceforth the term 2-manifold will refer to a compact, closed, orientable 2-manifold. If the genus of such a manifold \( M \) is denoted by \( \gamma(M) \), then the genus, \( \gamma(G) \), of a graph \( G \) is the smallest of the numbers \( \gamma(N) \) for 2-manifolds \( N \) in which \( G \) can be imbedded. An imbedding of \( G \) in \( M \) is called minimal if \( \gamma(G) = \gamma(M) \). When each component of the complement of \( G \) in \( M \) is an open 2-cell, the imbedding of \( G \) in \( M \) is a 2-cell imbedding. In [5], J. W. T. Youngs has shown that each minimal imbedding is a 2-cell imbedding. The notation \( G(M) \) will denote an imbedding of the graph \( G \) in the 2-manifold \( M \) as well as the geometric realization of \( G \) in \( M \). For a vertex \( x \) of \( G \), the adjacency set \( V(x) \) will be the set of all vertices of \( G \) which are adjacent (i.e. joined by an edge) to \( x \) in \( G \).

Extensive use will be made of a technique for obtaining all 2-cell imbeddings of a given graph \( G \) due to J. R. Edmonds and described in detail in [5]. Briefly, for each vertex \( x \) of \( G \) one chooses a cyclic permutation \( T_x \) on the set \( V(x) \). (For simplicity, we assume that \( G \) has no points of order one.) Each choice of a set such permutations, one for each vertex of \( G \), determines a 2-cell imbedding of \( G \) as follows. A transformation \( T(W) = W \) is defined on the set of all ordered pairs of adjacent vertices of \( G \) by \( T((x, y)) = (y, T_x(y)) \). The pairs of a given orbit of \( T \) can be identified with the edges of some regular polygon. Performing identifications on the edges of the polygons thus associated with \( T \), one obtains a 2-manifold \( M \) with the edges of these polygons yielding a copy of \( G \) in \( M \). It is a part of the Edmonds result that for a given 2-cell imbedding \( G(M) \), each orientation on \( M \) induces a set of permutations which determine the imbedding in question.

3. Extension theorem. Let \( f(G) = H \) be an open mapping on a graph \( G \), and let \( H(N) = \text{any 2-cell imbedding of} \ H \) in a 2-manifold \( N \). Theorem 3.2 exists an imbedding of \( G \) in a 2-manifold \( M \) and a light, open mapping \( F(M) = N \) which extends \( f \).

Proof. By 3.1 there is a graph \( G' \) containing \( G \) with a locally homogeneous map \( f'(G') = H \) extending \( f \). Let \( H(N) = \text{any 2-cell imbedding of} \ H \) in a 2-manifold with given orientation \( r \), and let \( T_{i1}, T_{i2}, \ldots \) be the cyclic permutations determined by \( H(N) \) and \( r \) on the adjacency sets \( V(x), V(y), \ldots \) of \( H \) as described in section 2. Since \( f' \) is open, \( b \in V(a) \) implies \( f(b) \in V(f'(a)) \). By the local homogeneity of \( f' \) we may choose cyclic permutations \( T_{i1}, T_{i2}, \ldots \) on the adjacency sets of \( G' \), so that for any adjacent vertices \( a \in f'^{−1}(x) \) and \( b \in f'^{−1}(y) \) we have

\[
T_{i1}(a) \in V(b) \iff f'^{−1}(T_{i1}(a)) \in V(b) \tag{1}
\]

This choice of permutations for \( G' \) determines an orientable, 2-cell imbedding \( G'(M) \). Let \( W \) be the set of all ordered pairs of adjacent vertices of \( G' \) and \( T(W) = W \) the transformation determined by \( T_{i1}, T_{i2}, \ldots \). Let \( L \) be the transformation for \( H(N) \) determined by \( I_{i1}, I_{i2}, \ldots \). If \( (a, b) \in W \) then we have \( f'(a) = x \) and \( f'(b) = y \), then by (1), \( T((a, b)) = (b, c) \) for some \( c \in f'^{−1}(x) \) where \( L((x, y)) = (y, c) \). Under the map \( f' \) each orbit of \( T \), thought of as the boundary of an open 2-cell, is mapped onto an orbit of \( L \) by a map which (ignoring identifications) is topologically equivalent to the mapping \( w = x^k \) on \( |z| = 1 \) in the complex \( z \)-plane for \( k \) some positive integer.

Let \( D_1, \ldots, D_n \) be the orbits of \( T \) and \( E_1, \ldots, E_t \), the set of open 2-cells bounded by \( D_1, \ldots, D_n \), respectively. Let \( E_1, \ldots, E_t \) and \( S_1, \ldots, S_t \) be the orbi-2-cells associated with \( L \). \( F(M) = N \) is the mapping which agrees with \( f' \) on \( G'(M) \) (and hence on \( G(M) \)) and sends the 2-cell \( E_i \) onto the 2-cell \( S_i \) bounded by \( E_i = F(D_i) \) by a map topologically equivalent to \( w = x^k \) on \( |z| < 1 \), where \( k \) is the power of \( F(D_i) = R_i \). \( F \) is open on \( E_1, \ldots, E_t \). Since each edge of \( G' \) occurs once with each possible orientation in the collection \( D_1, \ldots, D_n \), and since the two copies of a given edge have distinct opposite oriented images in the collection \( E_1, \ldots, E_t \), it follows that \( F(M) = N \) is open.

The selection of the permutations for \( G' \) in 3.2, and hence the imbedding \( G'(M) \), is not unique as is shown by the following example in which \( G' = G \).

3.3. Example. Let the graph \( H \) be a simple triangle and let \( G \) consist of three triangles intersecting in a single common vertex \( x, f(G) = H \) is the open map which sends each triangular block of \( G \) onto \( H \). By varying the permutation chosen on \( V(a) \), the light, open extension \( F(M) = N \) of \( f \) can be chosen to be a mapping of \((i)\) a 2-sphere \( M \) onto
a 2-sphere $N$ with two singular points (see [3], p. 194) on $M$ and two
on $N$; (ii) a 2-sphere $M$ onto a 2-sphere $N$ with five singular points on $M$
and three on $N$; or (iii) a torus $M$ onto a 2-sphere $N$ with three singular
points on $M$ and two on $N$.

Proof of Lemma 3.1. The square, symmetric matrix $A$ whose rows and columns correspond to the vertices of $G$ and for which the entry in the $x_i$-row and $y_j$-column, denoted by $(x_i,y_j)$, is one if $x_i \in V(y_j)$ and zero otherwise, is the incidence matrix of $G$. For the vertex $x_i$ of $G$ and
the vertices $y_1, y_2, \ldots, y_r$ which constitute $f^{-1}(y_i)$, let the sum $\sum_{j=1}^r (x_i, y_j)$
be denoted by $(x_i, Y)$. Since $(x_i, Y)$ represents the number of edges joining $x_i$
to points of $f^{-1}(y_i)$, we wish to increase the entries of $A$ so that for a fixed $x_i$, the sums $(x_i, Y)$ which are not zero are made equal to each other and so that
the symmetry of the matrix is preserved.

It suffices to consider only graphs having no points of order one.

Suppose that $b, d, e$ are vertices of $H$ with $f^{-1}(b) = b_1, \ldots, b_r$, \(f^{-1}(d) = d_1, \ldots, d_r\) and $f^{-1}(e) = e_1, \ldots, e_k$, and let the first rows of $A$
correspond to $b_1, \ldots, b_r, d_1, \ldots, d_r, e_1, \ldots, e_k$ in that order. If in the first
row of $A$, the sums $(b_i, Y)$ which are not zero are equal to each other, we
pass on to the $b_i$-row. Otherwise there is a largest such sum, say $(b_i, Z)$,
in which case for each $y$ with $(b, Z) > (b_i, Y)$, one increases $(b_k, y)$ by
$(b, Z) - (b_i, Y)$. One then increases the corresponding entries in the
$b_i$-column as required by symmetry. The rows and columns labeled $b_1, \ldots, b_r$ are treated similarly.

To obtain equality of the non-zero sums $(d_i, Y)$, one increases the
appropriate $(d_i, Y)$ entries. If $(d_i, h)$ is increased, one must increase $(b_i, h)$
by the same amount, adding this amount also to the entries $(b_k, y)$ for
which $(b, Z) \neq 0$, $y \neq b, d, e$, and to the corresponding entries of $b_i$-column.
The rows and columns labeled $d_1, \ldots, d_r$ are treated according to the
same rules.

If $(e_i, B)$ and $(e_i, D)$ are zero, the $e_i$-row is treated as was the $b_i$-row.
If, instead, at least one of these sums must be increased, the $e_i$-row is
altered and the $b_i$ and $d_i$ rows and columns are adjusted as necessary.
Note that if $(e_i, Y) = 0$, then $e$ and $y$ are not adjacent in $H$. In this case,
$(y, E) = 0$ and this sum will require no further change. If both $(e_i, B)$
and $(e_i, D)$ are non-zero and only one must be increased or the two must
be increased by different amounts, it may be necessary to use the $e_i$-column
to avoid cyclical difficulties. For example, an increase $(e_i, B)$ may require
a like increase in $(b_i, E)$ and hence in $(b_i, D)$, $(b_i, B)$ and $(b_i, E)$. To avoid
changing $(e_i, D)$ it may be necessary to add to $(d_i, e_i)$ rather than to $(d_i, e_i)$.
The rows and columns labeled $e_1, \ldots, e_k$ are handled similarly.

Having altered the $b$ and $d$ rows and columns one has $(b_i, D) = (b_i, E)$
and $(d_i, B) = (d_i, E)$ for $i = 1, 2, \ldots, B$ and $j = 1, 2, \ldots, D$. By symmetry
one then has
$$\sum_{i=1}^B (b_i, D) = \sum_{j=1}^D (d_j, B),$$
and hence
$$\sum_{i=1}^B (e_i, B) = \sum_{j=1}^E (e_j, D).$$

These relations are preserved by the operations on the rows $e_1, \ldots, e_k$.
$(e_i, B) = (e_i, D)$ for $i = 1, 2, \ldots, E-1$. Thus $(e_k, B) = (e_k, D)$. If either of
these sums must be changed, both must be increased and by the same
amount. This can be done as above without an additional $e$-row and
column.

A similar procedure may now be used to treat the remaining rows and
columns.

Let $A'$ be the matrix so obtained. $G'$ is taken to be the graph which
has the same vertices as $G$ with $x_i$ and $y_i$ joined by a number of edges
equal to the $(x_i, y_i)$ entry of $A'$. If $(x_i, y_i) > 1$, interior points of the edges of $H$
and $G'$ are chosen as new vertices. $f'$ is defined to be $f$ on $G'$ and such that
it maps an edge $x_i y_i$ in $G' - G$ onto the edge joining $f(x_i)$ and $f(y_i)$ in $H$.

4. Open mappings on planar graphs. The imbedding $H(N)$ in 3.2 may be taken to be any 2-cell imbedding, and hence any minimal
imbedding, of $H$. $G(M)$ need not be minimal, however, as seen in (iii)
of 3.3. In fact, there may be no minimal imbedding $G(M)$ for which a
light, open extension of $f(G) = H$ exists. This is shown by the next
theorem in conjunction with the fact that, except for homeomorphisms,
there exist no light, open mappings of a 2-manifold $M$ onto a 2-manifold $N$
when $y(M) \neq 2y(N)-1$. This last fact follows from Whyburn's "characteristic
equation" for light, open mappings on 2-manifolds (cf. [3], p. 202).
In particular, there exists no light, open mapping of a 2-sphere onto a
2-manifold of positive genus.

4.1. Theorem. For each positive integer $n$, there exists an open mapping
(in fact a local homeomorphism) $f(R_n)$ of a planar graph $R_n$ onto a
graph $G_n$ of genus $n$.

Proof. Whyburn has given an example ([3], p. 189) of an open
mapping of a planar graph onto one of the Kuratowski skew curves (of
genus one). There is also an open mapping of a planar graph onto the
second Kuratowski curve.

For the general result, a family of open maps $f_s(R_n) = G_s$ is described
where for each $n \geq 2$, $R_n$ is a planar graph and $G_s$ is the graph of genus $n$
constructed in [1] and shown to be irreducible in [2]. Adopting the
notation of paper [2], the graph $G_s$ consists of $s$ concentric circles, $C_1, \ldots, C_s$. 

Fundamenta Mathematicae, T. LX, 1951
4n radial lines $B_1, ..., B_m$ reaching from the innermost circle $C$ to the outermost $C^*$, and 2n edges, $A_1, ..., A_m$, where for each $i = 1, 2, ..., 2n$, $A_i$ joins the points $b_i$ and $b_{i+1}$ at which $C^*$ intersects $B_i$ and $B_{i+1}$, respectively.

Let $H_k$ denote the subgraph of $G_k$ obtained by deleting the (open) edges $A_k$, $i = 1, 2, ..., 2n$. The planar graph $E_k$ consists of two disjoint copies, $E_k^a$ and $E_k^b$, of $H_k$ together with 4n additional edges $D_1, ..., D_m$. For each $i = 1, 2, ..., 4n$, the edge $D_i$ joins the copies of $b_i$ on $E_k^a$ and $E_k^b$.

To see that this graph is planar, consider the imbedding obtained by locating $H_k^a$ and $H_k^b$ on opposite ends of a closed cylinder with the edges $D_k$ being drawn along the length of the cylinder from $H_k^a$ to $H_k^b$.

The open mapping $f_0(E_k) = G_k$ sends $H_k^a$ and $H_k^b$ homeomorphically onto $E_k$ in such a way that the endpoints of $A_k$ are mapped onto the endpoints of $A_k$ in $G_k$. $D_k$ is mapped homeomorphically onto $A_k$ for each $i = 1, 2, ..., 4n$, $f_0$ is a 2-to-1 local homeomorphism.

By the techniques of 3.2 one can obtain 2-cell imbeddings $E_k$ and $G_k$ in 2-manifolds $M$ and $N$ respectively (where $y(x) = 2n+1$ and $z(x) = x$) and an extension of $f_0$ to a light, open mapping $F(M) = N$ which sends each open 2-cell of $M - E_k^a(M)$ topologically onto an open 2-cell of $N - G_k^a(N)$.

5. Open mappings on manifolds. For $f(G) = H$ an open mapping on a graph $G$ and 2-cell imbeddings $G(M)$ and $H(N)$ of $G$ and $H$ in 2-manifolds $M$ and $N$ respectively, $G(M)$ is said to have property (0) with respect to $f$ and $H(N)$, provided that there exist orientations on $M$ and $N$ such that the cyclic permutations induced by these orientations on the adjacency sets of $G$ and $H$ satisfy condition (1) of 3.2.

5.1. Theorem. Let $F(M) = N$ be a light, open mapping where $M$ and $N$ are orientable 2-manifolds. Then there exist graphs $G$ and $H$ and minimal imbeddings $G(M)$ and $H(N)$, together with an open mapping $f(G) = H$ such that $G(M)$ has property (0) with respect to $f$ and $H(N)$ and so that $F$ extends $f$.

Proof. Following Whyburn ([4], p. 99), there exist simplicial subdivisions of $M$ and $N$ into 2-complexes $J$ and $K$, respectively, so that each simplex of $J$ maps topologically onto a simplex of $K$ under $F$. The 1-dimensional structures of $J$ and $K$ may be considered as 2-cell imbeddings $G(M)$ and $H(N)$ of some graphs $G$ and $H$ respectively. $G(M)$ and $H(N)$ are minimal imbeddings (cf. [5], p. 309). We shall use the symbols $G$ and $H$ to denote both the graphs and their copies $G(M)$ and $H(N)$.

Let $f = F(G)$. Since $G = F^{-1}(H)$, $f(G) = H$ is both light and open (cf. [4]). Let an orientation $\tau$ be chosen for $N$ and let $x$ be some vertex of $H$. $\tau$ induces a cyclic permutation $L_x$ on $V(x)$ as follows: For $x \in V(x)$, there is exactly one 2-cell $D$ of $K$ having the edge $ax$ of $K$ as a 1-face and such that the orientation on $D$ induced by $\tau$ directs $ax$ from $a$ to $x$. Let $L_x(u)$ be the remaining 0-face of $D$. If $L$ is the transformation on the set of all ordered pairs of adjacent vertices of $H$ defined by $L(x,u) = (u, L_x(u))$, the orbits of $V$ yield the imbedding $H(N)$ by the procedure described in section 2.

Assigning to each 2-simplex of $J$ the orientation of its image in $K$ yields a coherent orientation on $M$. This orientation may be used to obtain a collection of permutations which determine the imbedding $G(M)$.

Let $b$ be some vertex of $G$ with $f(b) = x$, and suppose that $a \in V(b) \cap \tau f^{-1}(u)$. Then $x \in V(x)$. Let $F$ be the simplex of $K$ having $a$, $x$, and $T_0(x)$ as 0-faces. The orientation on $E$ directs the 1-face $bx$ from $a$ to $b$. If $F(E) = D$, the edge $ax$ of $D$ is directed from $a$ to $x$. Hence $D$ has $a$, $x$, and $T_0(x)$ as 0-faces. Therefore $F(T_0(x)) = L_x(u)$ and $G(M)$ satisfies (1) of 3.2.

The procedure of 3.2 yields an extension of $f(G) = H$ to a light, open mapping $F(M) = N$. By construction, $F$ agrees with $F$ on $G$ and is topologically equivalent to $F$ on $M$.

The above theorem, combined with 3.2, yields the following corollaries.

5.2. Corollary. For 2-manifolds $M$ and $N$, there exists a light, open mapping $F(M) = N$ if and only if there are graphs $G$ and $H$ together with (minimal) 2-cell imbeddings $G(M)$ and $H(N)$ and an open mapping $f(G) = H$ such that $G(M)$ has property (0) with respect to $f$ and $H(N)$.

5.3. Corollary. Given an open mapping $f(G) = H$ on a graph $G$ and imbeddings $G(M)$ and $H(N)$ in orientable 2-manifolds $M$ and $N$ respectively, there exists a light, open mapping $F(M) = N$ extending $f$ if and only if there are graphs $G'$ and $H'$ containing $G$ and $H$ respectively, 2-cell imbeddings $G'(M)$ and $H'(N)$ extending $G(M)$ and $H(N)$, and an extension $f'$ of $f$ over $G'$ such that $G'(M)$ has property (0) with respect to $f'$ and $H'(N)$.

References


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