Does every continuum of type $\aleph_0$ contain an indecomposable subcontinuum? Is every continuum of type $\aleph_0$ separated by a countable collection of its subcontinua?

Reference

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UNIVERSITY OF CALIFORNIA, Riverside,
and
UNIVERSITY OF MICHIGAN, Ann Arbor, Michigan

Reçu par la Rédaction le 15. 11. 1965

Open mappings on graphs and manifolds *

by

Richard A. Duke (Seattle, Wash.)

1. Introduction. In [3], pp. 182 and 197, Whyburn has shown that the image of a finite graph under a (light) open mapping is again a finite graph and also that the image under a light, open mapping of a 2-manifold is a 2-manifold. The purpose of this paper is to investigate the conditions under which an open mapping $f(G) = H$ defined on a graph G can be extended to a light, open mapping $F(M) = N$ defined on a 2-manifold M when G and H are imbedded in M and N, respectively.

In section 3 it is shown that if $f(G) = H$ is such an open mapping, then there exist imbeddings of G and H in some orientable 2-manifolds M and N respectively, and an extension of f to a light, open map $F(M) = N$. The imbedding of H may be taken to be any orientable, 2-cell (and hence any minimal) imbedding of H. Further, any light, open map $F(M) = N$ on a closed orientable 2-manifold M can be obtained as an extension of such a map $f(G) = H$ on a graph G minimally imbedded in M. In section 4 it is shown that for each positive integer n there is an open mapping of a planar graph onto some graph of genus n, and this is used to show that the imbedding obtained for G in the above result may necessarily be non-minimal.

2. Background. The term *mapping* will be used to denote a continuous transformation. A mapping $f(X) = Y$ is said to be *open* provided that every open set in X maps onto a set open in Y. If for each y in Y, $f^{-1}(y)$ is totally disconnected, f is said to be a *light* mapping. The term *graph* will denote a finite, connected 1-complex.

If $f(G) = H$ is an open mapping on a graph G, then H is also a graph. Further, f is a light mapping and it is possible to designate certain interior points on the edges of G and H as additional vertices in such a way that f maps each edge of G topologically onto an edge in H. In light of these facts, we shall assume for the remainder that any open mapping on a graph is a simplicial transformation.

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* This research was supported in part by a grant from the National Science Foundation.

It is also known (cf. [3], p. 197) that if M is a 2-manifold and $F(M) = N$ a light, open mapping, then N is also a 2-manifold (though not necessarily orientable). Henceforth the term 2-manifold will refer to a compact, closed, *orientable* 2-manifold. If the genus of such manifold M is denoted by $\gamma(M)$, then the genus, $\gamma(G)$, of a graph G is the smallest of the numbers $\gamma(N)$ for 2-manifolds N in which G can be imbedded. An imbedding of G in M is called *minimal* if $\gamma(G) = \gamma(M)$. When each component of the complement of G in M is an open 2-cell, the imbedding of G in M is a *2-cell imbedding*. In [5], J. W. T. Youngs has shown that each minimal imbedding is a 2-cell imbedding. The notation $G(M)$ will denote an imbedding of the graph G in the 2-manifold M as well as the geometric realization of G in M. For a vertex x of G, the *adjacency set* $V(x)$ will be the set of all vertices of G which are adjacent (i.e. joined by an edge) to x in G.

Extensive use will be made of a technique for obtaining all 2-cell imbeddings of a given graph G due to J. R. Edmonds and described in detail in [5]. Briefly, for each vertex x of G one chooses a cyclic permutation T_x on the set $V(x)$. (For simplicity, we assume that G has no points of order one.) Each choice of a set such permutations, one for each vertex of G, determines a 2-cell imbedding of G as follows. A transformation $T(W) = W$ is defined on the set of all *ordered* pairs of adjacent vertices of G by $T\big((x, y)\big) = \big(y, T_y(x)\big)$. The pairs of a given orbit of T can be identified with the edges of some regular polygon. Performing identifications on the edges of the polygons thus associated with T, one obtains a 2-manifold M with the edges of these polygons yielding a copy of G in M. It is a part of the Edmonds result that for a given 2-cell imbedding $G(M)$, each orientation on M induces a set of permutations which determine the imbedding in question.

3. Extension theorem. Let $f(G) = H$ be an open mapping on a graph G. f is said to be *locally homogeneous* provided that for each vertex x of H and each $x_i \in f^{-1}(x)$, the number of edges joining x_i to points of $f^{-1}(z)$ is equal to the number of edges joining x_i to $f^{-1}(z)$ for any two vertices y and z in $V(x)$.

Concerning such mappings we have the following lemma, for which a detailed proof is given at the end of this section.

3.1. Lemma. *If $f(G) = H$ is an open mapping on a graph G, there exists a graph G' containing G and a locally homogeneous mapping $f'(G) = H$ which extends f.*

G' is formed by adding to G new edges joining existing vertices of G. To derive an algorithm for forming G', use is made of the incidence matrix A of G. The entries of A are altered forming a new matrix A' whose entries are non-negative integers. A' is used to specify G' and f'.

3.2. Theorem. *Let $f(G) = H$ be an open mapping on a graph G and let $H(N)$ be any 2-cell imbedding of H in a 2-manifold N. There exists an imbedding of G in a 2-manifold M and a light, open mapping $F(M) = N$ which extends f.*

Proof. By 3.1 there is a graph G' containing G with a locally homogeneous map $f'(G') = H$ extending f. Let $H(N)$ be any 2-cell imbedding of H in a 2-manifold with given orientation τ, and let $L_x, L_y, \ldots$ be the cyclic permutations determined by $H(N)$ and τ on the adjacency sets $V(x), V(y), \ldots$ of H as described in section 2. Since f' is open, $b \in V(a)$ implies $f'(b) \in V\big(f'(a)\big)$. By the local homogeneity of f' we may choose cyclic permutations $T_a, T_b, \ldots$ on the adjacency sets of G', so that for any adjacent vertices $a \in f'^{-1}(x)$ and $b \in f'^{-1}(y)$ we have

$$T_b(a) \in V(b) \cap f'^{-1}\big(L_y(x)\big) . \tag{1}$$

This choice of permutations for G' determines an orientable, 2-cell imbedding $G'(M)$. Let W be the set of all ordered pairs of adjacent vertices of G' and $T(W) = W$ the transformation determined by $T_a, T_b, \ldots$ Let L be the transformation for $H(N)$ determined by $L_x, L_y, \ldots$ If for (a, b) in W we have $f'(a) = x$ and $f'(b) = y$, then by (1), $T\big((a, b)\big) = (b, c)$ for some c in $f'^{-1}(z)$ where $L\big((x, y)\big) = (y, z)$. Under the map f' each orbit of T, thought of as the boundary of an open 2-cell, is mapped onto an orbit of L by a map which (ignoring identifications) is topologically equivalent to the mapping $w = z^k$ on $|z| = 1$ in the complex z-plane for k some positive integer.

Let $D_1, \ldots, D_r$ be the orbits of T and $E_1, \ldots, E_r$ the set of open 2-cells bounded by $D_1, \ldots, D_r$, respectively. Let $R_1, \ldots, R_t$ and $S_1, \ldots, S_t$ be the orbits and 2-cells associated with L. $F(M) = N$ is the mapping which agrees with f' on $G'(M)$ (and hence on $G(M)$) and sends the 2-cell E_i onto the 2-cell S_j bounded by $R_j = F(D_i)$ by a map topologically equivalent to $w = z^k$ on $|z| < 1$, where k is the power of $F(D_i) = R_j$.

F is open on $E_1, \ldots, E_r$. Since each edge of G' occurs once with each possible orientation in the collection $D_1, \ldots, D_r$, and since the two copies of a given edge have distinct oppositely oriented images in the collection $R_1, \ldots, R_t$, it follows that $F(M) = N$ is open.

The selection of the permutations for G' in 3.2, and hence the imbedding $G'(M)$, is not unique as is shown by the following example in which $G' = G$.

3.3. Example. Let the graph H be a simple triangle and let G consist of three triangles intersecting in a single common vertex a. $f(G) = H$ is the open map which sends each triangular block of G onto H. By varying the permutation chosen on $V(a)$, the light, open extension $F(M) = N$ of f can be chosen to be a mapping of (i) a 2-sphere M onto a 2-sphere N with two singular points (see [3], p. 194) on M and two on N; (ii) a 2-sphere M onto a 2-sphere N with five singular points on M and three on N; or (iii) a torus M onto a 2-sphere N with three singular points on M and two on N.

Proof of lemma 3.1. The square, symmetric matrix A whose rows and columns correspond to the vertices of G and for which the entry in the x_i-row and y_j-column, denoted by $(x_i y_j)$, is one if $x_i \in V(y_j)$ and zero otherwise, is the *incidence matrix* of G. For the vertex x_i of G and the vertices $y_1, y_2, \ldots, y_Y$ which constitute $f^{-1}(y)$, let the sum $\sum_{j=1}^{Y}(x_i y_j)$ be denoted by $(x_i Y)$. Since $(x_i Y)$ represents the number of edges joining x_i to points of $f^{-1}(y)$, we wish to increase the entries of A so that for a fixed x_i, the sums $(x_i Y)$ which are not zero are made equal to each other and so that the symmetry of the matrix is preserved.

If suffices to consider only graphs having no points of order one.

Suppose that b, d, and e are vertices of H with $f^{-1}(b) = b_1, \ldots, b_B$, $f^{-1}(d) = d_1, \ldots, d_D$ and $f^{-1}(e) = e_1, \ldots, e_E$, and let the first rows of A correspond to $b_1, \ldots, b_B$, $d_1, \ldots, d_D$, $e_1, \ldots, e_E$ in that order. If in the first row of A, the sums $(b_1 Y)$ which are not zero are equal to each other, we pass on to the b_2-row. Otherwise there is a largest such sum, say $(b_1 Z)$, in which case for each y with $(b_1 Z) > (b_1 Y) > 0$, one increases $(b_1 y_1)$ by $\big((b_1 Z) - (b_1 Y)\big)$. One then increases the corresponding entries in the b_1-column as required by symmetry. The rows and columns labeled $b_2, \ldots, b_B$ are treated similarly.

To obtain equality of the non-zero sums $(d_1 Y)$, one increases the appropriate $(d_1 y_1)$ entries. If $(d_1 b_1)$ is increased, one must increase $(b_1 d_1)$ by the same amount, adding this amount also to the entries $(b_1 y_1)$ for which $(b_1 Y) \neq 0$, $y \neq b, d$, and to the corresponding entries of b_1-column. The rows and columns labeled $d_1, \ldots, d_D$ are treated according to the same rules.

If $(e_1 B)$ and $(e_1 D)$ are zero, the e_1-row is treated as was the b_1-row. If, instead, at least one of these sums must be increased, the e_1-row is altered and the b_1 and d_1 rows and columns are adjusted as necessary. Note that if $(e_1 Y) = 0$, then e and y are not adjacent in H. In this case, $(y_1 E) = 0$ and this sum will require no further change. If both $(e_1 B)$ and $(e_1 D)$ are non-zero and only one must be increased or the two must be increased by different amounts, it may be necessary to use the e_2-column to avoid cyclical difficulties. For example, an increase $(e_1 B)$ may require a like increase in $(b_1 E)$ and hence in $(b_1 D)$, $(d_1 B)$ and $(d_1 E)$. To avoid changing $(e_1 D)$ it may be necessary to add to $(d_1 e_2)$ rather than to $(d_1 e_1)$. The rows and columns labeled $e_2, \ldots, e_{E-1}$ are handled similarly.

Having altered the b and d rows and columns one has $(b_i D) = (b_i E)$ and $(d_j B) = (d_j E)$ for $i = 1, 2, \ldots, B$ and $j = 1, 2, \ldots, D$. By symmetry one then has

$$\sum_{i=1}^{B}(b_i D) = \sum_{j=1}^{D}(d_j B)\,,$$

and hence

$$\sum_{i=1}^{E}(e_i B) = \sum_{i=1}^{E}(e_i D)\,.$$

These relations are preserved by the operations on the rows $e_1, \ldots, e_{E-1}$. $(e_i B) = (e_i D)$ for $i = 1, 2, \ldots, E-1$. Thus $(e_E B) = (e_E D)$. If either of these sums must be changed, both must be increased and by the same amount. This can be done as above without an additional e-row and column.

A similar procedure may now be used to treat the remaining rows and columns.

Let A' be the matrix so obtained. G' is taken to be the graph which has the same vertices as G with x_i and y_i joined by a number of edges equal to the $(x_i y_i)$ entry of A'. If $(x_i y_i) > 1$, interior points of the edges of H and G' are chosen as new vertices. f' is defined to be f on G and such that it maps an edge $x_i y_i$ in $G' - G$ onto the edge joining $f(x_i)$ and $f(y_i)$ in H.

4. Open mappings on planar graphs. The imbedding $H(N)$ in 3.2 may be taken to be any 2-cell imbedding, and hence any minimal imbedding, of H. $G(M)$ need not be minimal, however, as seen in (iii) of 3.3. In fact, there may be no minimal imbedding $G(M)$ for which a light, open extension of $f(G) = H$ exists. This is shown by the next theorem in conjunction with the fact that, except for homeomorphisms, there exist no light, open mappings of a 2-manifold M onto a 2-manifold N when $\gamma(M) < 2\gamma(N) - 1$. This last fact follows from Whyburn's "characteristic equation" for light, open mappings on 2-manifolds (cf. [3], p. 202). In particular, there exists no light, open mapping of a 2-sphere onto a 2-manifold of positive genus.

4.1. THEOREM. *For each positive integer n, there exists an open mapping (in fact a local homeomorphism) $f(R_n) = G_n$ of a planar graph R_n onto a graph G_n of genus n.*

Proof. Whyburn has given an example ([3], p. 189) of an open mapping of a planar graph onto one of the Kuratowski skew curves (of genus one). There is also an open mapping of a planar graph onto the second Kuratowski curve.

For the general result, a family of open maps $f_n(R_n) = G_n$ is described where for each $n \geqslant 2$, R_n is a planar graph and G_n is the graph of genus n constructed in [1] and shown to be irreducible in [2]. Adopting the notation of paper [2], the graph G_n consists of n concentric circles, $C^1, \ldots, C^n$,

$4n$ radial lines $B_1, \ldots, B_{4n}$ reaching from the innermost circle C^1 to the outermost C^n, and $2n$ edges, $A_1, \ldots, A_{2n}$, where for each $i = 1, 2, \ldots, 2n$, A_i joins the points b_i and b_{2n+i} at which C^n intersects B_i and B_{2n+i}, respectively.

Let H_n denote the subgraph of G_n obtained by deleting the (open) edges A_i, $i = 1, 2, \ldots, 2n$. The planar graph R_n consists of two disjoint copies, H_n^1 and H_n^2, of H_n together with $4n$ additional edges $D_1, \ldots, D_{4n}$. For each $i = 1, 2, \ldots, 4n$, the edge D_i joins the copies of b_i on H_n^1 and H_n^2.

To see that this graph is planar, consider the imbedding obtained by locating H_n^1 and H_n^2 on opposite ends of a closed cylinder with the edges $D_1, \ldots, D_{4n}$ being drawn along the length of the cylinder from H_n^1 to H_n^2.

The open mapping $f_n(R_n) = G_n$ sends H_n^1 and H_n^2 homeomorphically onto H_n in such a way that the endpoints of D_i are mapped onto the endpoints of A_i in G_n. D_i is mapped homeomorphically onto A_i for each $i = 1, 2, \ldots, 4n$. f_n is a 2-to-1 local homeomorphism.

By the techniques of 3.2 one can obtain 2-cell imbeddings of R_n and G_n in 2-manifolds M and N respectively (where $\gamma(M) = 2n-1$ and $\gamma(N) = n$) and an extension of f_n to a light, open mapping $F(M) = N$ which sends each open 2-cell of $M - R_n(M)$ topologically onto an open 2-cell of $N - G_n(N)$.

5. Open mappings on manifolds. For $f(G) = H$ an open mapping on a graph G and 2-cell imbeddings $G(M)$ and $H(N)$ of G and H in 2-manifolds M and N, respectively, $G(M)$ is said to have *property* (0) *with respect to f and $H(N)$*, provided that there exist orientations on M and N such that the cyclic permutations induced by these orientations on the adjacency sets of G and H satisfy condition (1) of 3.2.

5.1. THEOREM. *Let $F(M) = N$ be a light, open mapping where M and N are orientable 2-manifolds. Then there exist graphs G and H and minimal imbeddings $G(M)$ and $H(N)$, together with an open mapping $f(G) = H$ such that $G(M)$ has property* (0) *with respect to f and $H(N)$ and so that F extends f.*

Proof. Following Whyburn ([4], p. 99), there exist simplicial subdivisions of M and N into 2-complexes J and K, respectively, so that each simplex of J maps topologically onto a simplex of K under F. The 1-dimensional structures of J and K may be considered as 2-cell imbeddings $G(M)$ and $H(N)$ of some graphs G and H respectively. $G(M)$ and $H(N)$ are minimal imbeddings (cf. [5], p. 309). We shall use the symbols G and H to denote both the graphs and their copies $G(M)$ and $H(N)$.

Let $f \equiv F|G$. Since $G = F^{-1}(H)$, $f(G) = H$ is both light and open (cf. [4]). Let an orientation τ be chosen for N and let x be some vertex of H. τ induces a cyclic permutation L_x on $V(x)$ as follows: For $u \in V(x)$, there is exactly one 2-cell D of K having the edge xu of K as a 1-face and such that the orientation on D induced by τ directs xu from u to x. Let $L_x(u)$ be the remaining 0-face of D. If L is the transformation on the set of all ordered pairs of adjacent vertices of H defined by $L((x, u)) = (u, L_x(u))$, the orbits of L yield the imbedding $H(N)$ by the procedure described in section 2.

Assigning to each 2-simplex of J the orientation of its image in K yields a coherent orientation on M. This orientation may be used to obtain a collection of permutations which determine the imbedding $G(M)$.

Let b be some vertex of G with $f(b) = x$, and suppose that $a \in V(b) \cap f^{-1}(u)$. Then $u \in V(x)$. Let E be the simplex of K having b, a, and $T_b(a)$ as 0-faces. The orientation on E directs the 1-face ba from a to b. If $F(E) = D$, the edge xu of D is directed from u to x by τ. Hence D has x, u, and $L_x(u)$ as 0-faces. Therefore $F(T_b(a)) = L_x(u)$ and $G(M)$ satisfies (1) of 3.2.

The procedure of 3.2 yields an extension of $f(G) = H$ to a light, open mapping $F'(M) = N$. By construction, F' agrees with F on G and is topologically equivalent to F on M.

The above theorem, combined with 3.2, yields the following corollaries.

5.2. COROLLARY. *For 2-manifolds M and N, there exists a light, open mapping $F(M) = N$ if and only if there are graphs G and H together with (minimal) 2-cell imbeddings $G(M)$ and $H(N)$ and an open mapping $f(G) = H$ such that $G(M)$ has property* (0) *with respect to f and $H(N)$.*

5.3. COROLLARY. *Given an open mapping $f(G) = H$ on a graph G and imbeddings $G(M)$ and $H(N)$ in orientable 2-manifold M and N respectively, there exists a light, open mapping $F(M) = N$ extending f if and only if there are graphs G' and H' containing G and H respectively, 2-cell imbeddings $G'(M)$ and $H'(N)$ extending $G(M)$ and $H(N)$, and an extension f' of f over G' such that $G'(M)$ has property* (0) *with respect to f' and $H'(N)$.*

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Reçu par la Rédaction le 16. 12. 1965