Point transitive flows, algebras of functions and the Bebutov system

by

Jozeph Auslander and Frank Hahn *
(College Park, Maryland and New Haven, Connecticut)

Introduction. In this paper we will be concerned with certain kinds of dynamical systems and algebras of complex valued functions associated with them. By a *dynamical system* or *flow* we mean an action of the additive group of real numbers T as a transformation group on a topological space X. More precisely, there is a continuous map $\pi\colon T\times X\to X$ satisfying $\pi(0, x) = x$, and $\pi(t_1, \pi(t_2, x)) = \pi(t_1+t_2, x)$ $(x \in X,\ t_1, t_2 \in T)$.

We write tx instead of $\pi(t, x)$. If $x \in X$, the *orbit of* x is the set $\gamma(x) = [tx| -\infty < t < \infty]$. In the dynamical systems studied here, we suppose that every orbit closure $\overline{\gamma(x)}$ is compact.

As general references for dynamical systems, consult [11] and [14].

Now, let us explain the three objects mentioned in our title.

(1) The *point transitive flows*—those dynamical systems with compact phase space which contain a dense orbit.

(2) The algebra $\mathfrak{A}$ of bounded uniformly continuous complex valued functions of a real variable and this subalgebras which are closed under uniform convergence, complex conjugation and translation.

(3) The *Bebutov dynamical system*—the bounded uniformly continuous functions from the real to the complex numbers, provided with the compact open topology and made into a dynamical system by translation of functions.

We are going to study these and the ways in which they are interrelated. The basic connection between (1) and (2) is that the point transitive flows are precisely those dynamical systems which can be obtained as maximal ideal spaces of the sub-algebras described in (2). This is studied in detail in section I. The "shift operators" of a subalgebra $\mathfrak{A}'$ (the homomorphisms of $\mathfrak{A}'$ of norm one which commute with translation) are shown to correspond to the endomorphisms of the dynamical system (P', T) in

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the maximal ideal space. These in turn correspond to a certain subsemigroup of the enveloping semigroup of (P', T) (theorem I.1). (The enveloping semigroup is a kind of compactification of the real line T. T is regarded as a subset of the space $P'^{P'}$, the enveloping semigroup is the closure of T in this space.) The maximal ideal space of $\mathfrak{A}$ is the universal point transitive flow (every such flow is a homomorphic image of it) (theorem I.2). It is shown (theorem I.4) that all shift operators of subalgebras are restrictions of shift operators of $\mathfrak{A}$, and that the shift invariant subalgebras are those whose maximal ideal spaces are isomorphic to their own enveloping semigroups (theorem I.5).

In section II, the Bebutov system (B, T) is studied. By virtue of theorems of Bebutov and Kakutani, (B, T) is "universal" in the sense that a large class of compact metric flows may be embedded in it. Its enveloping semigroup coincides (as a set) with the shift operators of $\mathfrak{A}$, and is shown to be isomorphic with the enveloping semigroup of the universal point transitive flow. The maximal ideal space of a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is embeddable in (B, T) if and only if $\mathfrak{B}$ is generated by the translates of a single element (corollary II.5). Moreover, two functions have isomorphic orbit closures in (B, T) if and only if they generate the same algebra (corollary II.7).

The minimal functions are considered in section III. These are the functions whose orbit closures in the Bebutov system are minimal sets. A subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ consists of minimal functions if and only if its maximal ideal space is a minimal dynamical system (theorem III.4). Two such "minimal" subalgebras which we study are the distal and weakly distal functions.

In section IV, we consider the (Bohr) almost periodic functions. We obtain a necessary and sufficient condition for the orbit closures of two almost periodic functions to be isomorphic, in terms of the supports of their Fourier transforms (theorem IV.2).

Some of our results are similar to those obtained by other authors. In [10], Ellis considers subalgebras of the continuous functions on the Stone-Čech compactification of a discrete group. Knapp, [13], studies shift operators as well as minimal and distal functions; we show that our definitions are equivalent to his. Some aspects of the Bebutov system were studied in [3]. In order to make this paper self-contained, and also since our viewpoint is somewhat different, there will be some duplication with the above work here. We have tried to indicate each such occurence.

Much of what we do here is evidently capable of generalization to transformation groups (X, T), where T is an arbitrary topological group. We have restricted ourselves to the case of the real numbers, since we wish to make use of the Bebutov-Kakutani theorem. Extensions of our work depend upon generalizations of this theorem.

I. The subalgebras of $\mathfrak{A}$ and their maximal ideal spaces. Let $\mathfrak{A}$ be the algebra of all bounded uniformly continuous functions from the real numbers T into the complex numbers $\boldsymbol{C}$. We make $\mathfrak{A}$ a Banach $*$-algebra by defining

$$\|f\| = \sup_{t\in T} |f(t)| \quad \text{and} \quad f^*(t) = \overline{f(t)}.$$

By a $*$, closed, invariant subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ we mean a subalgebra which is closed in the norm topology, contains the complex conjugates of each of its elements and is invariant under translation by elements of T. That is, if $f \in \mathfrak{A}'$, then $f_t = tf \in \mathfrak{A}'$ where $tf(s) = f_t(s) = f(s+t)$.

Let $\mathfrak{A}'$ be a $*$, closed, invariant subalgebra of $\mathfrak{A}$, and let P' be the maximal ideal space of $\mathfrak{A}'$. P' is constructed as follows. Let $\mathfrak{A}'^*$ be the dual space of $\mathfrak{A}'$, and, for each $t \in T$, let $p'_t \in \mathfrak{A}'^*$ be defined by $p'_t(f) = f(t)$ $(f \in \mathfrak{A}')$. Then P' is the weak star closure of the set $\{p'_t|\ t \in T\}$; P' is known to be compact ([15]). If $t \in T$, then t induces a self homeomorphism of P', which we again denote by t, defined by $t\mu(f) = \mu(f_t)$ $(\mu \in P', f \in \mathfrak{A}')$. Since the action of T on $\mathfrak{A}'$ is continuous in both variables, the same is true of the action of T on P'. Thus (P', T) is a dynamical system. It is point transitive, since by definition $\gamma(p'_0) = \{tp'_0|\ t \in T\} = \{p'_t|\ t \in T\}$ is dense in P'. We will call p'_0 a *distinguished point* of P'.

For each $f \in \mathfrak{A}'$, there corresponds a unique $\hat{f} \in C(P')$ such that $f(t) = \hat{f}(p'_t) = \hat{f}(tp'_0)$; $\hat{f}$ is the *Gelfand transform* of f. The mapping $f \to \hat{f}$ is an isometric isomorphism of $\mathfrak{A}'$ onto $C(P')$ ([15]).

Conversely, if (P', T) is a point transitive dynamical system with P' compact, and $p'_0 \in P'$ has a dense orbit, let $\mathfrak{A}' = [f\colon T \to \boldsymbol{C}|\ f(t) = F(tp'_0)$, for some $F \in C(P')]$. Then $\mathfrak{A}'$ is a $*$, closed, invariant subalgebra of $\mathfrak{A}$. It is easy to see that its maximal ideal space is homeomorphic with P', and, indeed, the action of T on P' as defined above is identical with the original action. Thus there is a one-to-one correspondence between $*$, closed, invariant subalgebras $\mathfrak{A}'$ of $\mathfrak{A}$, and point transitive flows (P', T) with P' compact, and $p'_0 \in P'$ a distinguished point with dense orbit.

If $\mathfrak{A}'$ is a $*$, closed, invariant subalgebra of $\mathfrak{A}$, a *shift operator* is a homomorphism $\xi\colon \mathfrak{A}' \to \mathfrak{A}'$ such that $\|\xi\| = 1$, $\xi t = t\xi$, for each $t \in T$, and $\xi(f^*) = (\xi f)^*$ for all $f \in \mathfrak{A}'$. The set of shift operators of $\mathfrak{A}'$ will be denoted by $S(\mathfrak{A}')$; clearly $S(\mathfrak{A}')$ is a semigroup under composition.

Closely related to the shift operators are the endomorphisms $\mathcal{E}(P')$ of the dynamical system (P', T); that is, the continuous maps $\pi\colon P' \to P'$ such that $\pi t = t\pi$. Note that $\mathcal{E}(P')$ is also a semigroup under composition, and that each $\pi \in \mathcal{E}(P')$ is completely determined by its value at p'_0.

If $\xi \in S(\mathfrak{A}')$, let H_ξ denote the homomorphism of $C(P')$ defined by $H_\xi\hat{f}(p'_t) = \xi f(t)$. Then ([6], p. 278), there is a unique $\pi_\xi \in \mathcal{E}(P')$ such that $\xi f(t) = H_\xi\hat{f}(p'_t) = \hat{f}(\pi_\xi p'_t)$. The map $\xi \to \pi_\xi$ is one to one onto, and, since

$H_\xi \hat{f} = \xi \hat{f}$, it follows from a routine computation that $\pi_{\xi\eta} = \pi_\eta \pi_\xi$. Thus $S(\mathfrak{A}')$ and $\mathcal{E}(P')$ are anti-isomorphic. If $\pi \in \mathcal{E}(P')$, we may regard it as a shift operator as follows: $\pi f(t) = \hat{f}(\pi p_t')$.

Another approach to shift operators, essentially the one developed by Knapp in [13], is by means of certain nets in T. Let $\{t_n\}$ be a net in T such that $g(s) = \lim_n f(t_n + s)$ exists pointwise for each $f \in \mathfrak{A}'$, and such that the limit function g is in $\mathfrak{A}'$. Then the net $\{t_n\}$ defines a map $\xi\colon \mathfrak{A}' \to \mathfrak{A}'$ and it is easy to see that $\xi \in S(\mathfrak{A}')$. Conversely, if $\xi \in S(\mathfrak{A}')$ we show that there is a net $\{t_n\}$ in T such that $f(t_n + s) \to \xi f(s)$. For, let $\pi_\xi \in \mathcal{E}(P')$, where P' is the maximal ideal space of $\mathfrak{A}'$, satisfying $\xi f(t) = \hat{f}(\pi_\xi p_t')$. Let $\{t_n\}$ be a net in T such that $t_n p_0' \to \pi_\xi p_0'$. Then $t_n f(s) = f(t_n + s) = \hat{f}((t_n + s) p_0')$ $= \hat{f}(t_n p_s') \to f(\pi_\xi p_s') = \xi f(s)$.

We call a net $\{t_n\}$ in T *admissible* if it defines a shift operator ξ in the manner just described.

There is still another semigroup which will concern us. This is the *enveloping semigroup* of the dynamical system (P', T) ([8]) which is defined as follows. Regard T as a group of maps from P' to itself, and let $E(P')$ be the closure of T in $P'^{P'}$. (Then, if $\{\xi_n\}$ is a net in $E(P')$, $\xi_n \to \xi$ if and only if $\xi_n(p') \to \xi(p')$, for all $p' \in P'$.) Since $P'^{P'}$ is compact, so is $E(P')$, and T acts on $E(P')$ by $(t\xi)(p') = t(\xi(p'))$. Thus $(E(P'), T)$ is a dynamical system, and since e (= the identity map of P') has a dense orbit, it is point transitive. Moreover, it is not difficult to show that if $\xi, \eta \in E(P')$; then $\xi\eta \in E(P')$ (where $\xi\eta(p') = \xi(\eta(p'))$), so that $E(P')$ is indeed a semigroup. We remark that although the elements of T define homeomorphisms of P' onto itself, the elements of $E(P')$ are, in general, not continuous, onto or one to one. Note also that $\xi t = t\xi$, for $\xi \in E(P')$, $t \in T$; however, $E(P')$ is not in general commutative.

Next, we show how $\mathcal{E}(P')$ and $S(\mathfrak{A}')$ are related to a certain subsemigroup of $E(P')$.

Firstly, let $P_0' = [p' \in P' \mid p' = \pi(p_0'),\ \text{some } \pi \in \mathcal{E}(P')]$, and then let

$$E_0(P') = [\xi \in E(P') \mid \xi p_0' \in P_0'] = [\xi \in E(P') \mid \xi p_0' = \pi(p_0'),\ \text{ some } \pi \in \mathcal{E}(P')].$$

Now $E_0(P')$ is a subsemigroup of $E(P')$. To show this, first note that, if $\pi \in \mathcal{E}(P')$, and $\eta \in E(P')$, then $\pi\eta = \eta\pi$. Let $\xi, \eta \in E_0(P')$ and let $\pi_\xi, \pi_\eta \in \mathcal{E}(P')$ such that $\pi_\xi(p_0') = \xi p_0'$ and $\pi_\eta(p_0') = \eta p_0'$. Then $\eta\xi p_0' = \eta\pi_\xi(p_0') = \pi_\xi(\eta p_0')$ $= \pi_\xi \pi_\eta(p_0')$. That is, $\eta\xi \in E_0(P')$.

Since the elements of $\mathcal{E}(P')$ are determined by their values at the point p_0', we may define a map $E_0(P')$ onto $\mathcal{E}(P')$ by $\xi \to \pi_\xi$. The argument in the preceding paragraph shows that $\pi_\xi \pi_\eta = \pi_{\eta\xi}$, so the map $\xi \to \pi_\xi$ is an antihomomorphism.

We remark further that if $\pi_\xi = \pi_\eta$, then ξ agrees with η on P_0', and conversely. For, if $\pi_\xi = \pi_\eta$, then $\xi p_0' = \eta p_0'$. If $p' \in P_0'$, then there is a $\theta \in \mathcal{E}(P')$ for which $\theta(p_0') = p'$, and then $\xi p' = \xi\theta(p_0') = \theta(\xi p_0') = \theta(\eta p_0')$ $= \eta\theta(p_0') = \eta p'$.

Now, if $P_0' = P'$, then $E_0(P') = E(P')$. Thus, if $\pi_\xi = \pi_\eta$, $\xi = \eta$ on $P_0' = P'$—that is to say $\xi = \eta$, and the map $\xi \to \pi_\xi$ is one to one.

Thus we have proved

THEOREM I.1. *Let $\mathfrak{A}'$ be a $*$, closed, invariant subalgebra of $\mathfrak{A}$, and let P' be the maximal ideal space of $\mathfrak{A}'$. Let P_0', $E_0(P')$, $S(\mathfrak{A}')$ and $\mathcal{E}(P')$ be defined as above. Then there is an antihomomorphism $\xi \to \pi_\xi$ of $E_0(P')$ onto $\mathcal{E}(P')$ and an antiisomorphism of $\mathcal{E}(P')$ onto $S(\mathfrak{A}')$. Let $\sigma\colon E_0(P') \to S(\mathfrak{A}')$ be the homomorphism obtained by the composition of these maps. Then, if we identify notationally $\sigma(\xi)$ with ξ, we may write*

$$\hat{f}(\xi p_t') = \hat{f}(\pi_\xi p_t') = \xi f(t).$$

If $P_0' = P'$, then $\sigma\colon E(P') \to S(\mathfrak{A}')$ is an isomorphism.

Now, we want to consider a point transitive flow, which, in a sense, "encompasses" all such flows. We say that a point transitive dynamical system (P, T) with a distinguished point p_0 is a *universal* point transitive system if, for any point transitive dynamical system (X, T), with dense orbit $\gamma(x_0)$, there is a homomorphism (that is, a continuous map which commutes with the action of T) $\pi\colon (P, T) \to (X, T)$ such that $\pi(p_0) = x_0$.

If such a system exists, it is unique up to isomorphism. For, suppose that (P, T) and (Q, T) were both universal point transitive systems with distinguished points p_0 and q_0, respectively. Then, let $\pi\colon (P, T) \to (Q, T)$ and $\theta\colon (Q, T) \to (P, T)$ be homomorphisms such that $\pi(p_0) = q_0$ and $\theta(q_0) = p_0$. We see that $\pi\theta$ and $\theta\pi$ are the identity maps on Q and P, respectively, so (P, T) and (Q, T) are isomorphic and the isomorphism preserves the distinguished points. The next theorem proves the existence of universal point transitive dynamical systems.

THEOREM I.2. *Let P be the maximal ideal space of $\mathfrak{A}$. Then (P, T) is the univarsal point transitive system with distinguished point p_0.*

Proof. If (X, T) is point transitive and $\{tx_0\colon t \in T\}$ is dense, let $\mathfrak{A}' = [f \in \mathfrak{A} \mid f(t) = F(tx_0),\ \text{some } F \in C(X)]$. We see that $\mathfrak{A}'$ is isometrically isomorphic to $C(X)$. Since $\mathfrak{A}' \subset \mathfrak{A}$, the inclusion map induces an isometry of $C(X)$ into $C(P)$. This isometry induces a homomorphism $\pi\colon (P, T) \to (X, T)$. Since p_0 and x_0 correspond to evaluation at $t = 0$, we see that $\pi(p_0) = x_0$. This concludes the proof.

THEOREM I.3. *$P_0 = P$; that is, if $q \in P$ then there is a $\pi \in \mathcal{E}(P)$ such that $\pi(p_0) = q$.*

Proof. Let Q = closure of $\{tq\colon t \in T\}$. Then (Q, T) is point transitive with dense orbit $\{tq\colon t \in T\}$. From the definition of universal point transitive dynamical system (P, T) there is a $\pi\colon (P, T) \to (Q, T) \subset (P, T)$ for which $\pi(p_0) = q$. The proof is completed.

Referring to theorem I.1 we see that $E(P)$ is antiisomorphic with $\mathcal{E}(P)$. Thus $E(P)$ is isomorphic with $S(\mathfrak{A})$ and each element $\xi \in E(P)$ can be regarded as being in $S(\mathfrak{A})$ as follows:

$$\xi f(t) = \hat{f}(\xi p_0) = f(t\xi p_0)\,.$$

The next theorem shows that it suffices always to work with $S(\mathfrak{A})$; that is, the shift operators on subalgebras are restrictions of elements of $S(\mathfrak{A})$.

THEOREM I.4. *Let $\mathfrak{A}'$ be $*$, closed, invariant in $\mathfrak{A}$. Let $\xi'\colon \mathfrak{A}'\to\mathfrak{A}$ be a $*$ homomorphism of norm 1 such that $\xi' t = t\xi'$ for each $t \in T$. Then there is a $\xi \in S(\mathfrak{A})$ for which $\xi f = \xi' f$ for each $f \in \mathfrak{A}'$.*

Proof. Let P' be the maximal ideal space of $\mathfrak{A}$ and let $\pi\colon P\to P'$ be the homomorphism such that $(\xi' f)(p) = f(\pi(p))$ $(f \in \mathfrak{A}' = C(P'), p \in P)$. If $g \in C(P) = \mathfrak{A}$, it is easy to see that $g \in$ image ξ' if and only if g is constant on all sets of the form $\pi^{-1}(p')$ $(p' \in P')$. From this it follows that $\mathfrak{A}'' = \xi'(\mathfrak{A}')$ is a $*$, closed, invariant subalgebra of $\mathfrak{A}$. Let P'' be the maximal ideal space of $\mathfrak{A}''$, with distinguished point p_0''. Then ξ' induces a homomorphism $\pi'\colon (P'', T)\to(P', T)$ for which $\pi'(p_0'') = p'$. Let $\nu\colon (P, T)\to(P'', T)$ and $\theta\colon (P, T)\to(P', T)$ such that $\nu(p_0) = p_0''$ and $\theta(p_0) = p_0'$ (ν and θ are onto). We have the diagram:

$$\begin{array}{ccc} (P, T) & & (P, T) \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\theta} \\ (P'', T) & \underset{\pi'}{\to} & (P', T) \end{array}$$

We must show that there is a homomorphism $\pi\colon (P, T)\to(P, T)$ which completes the diagram. There is a $p \in P$ for which $\theta(p) = p' = \pi'\nu(p_0)$. Choose π such that $\pi(p_0) = p$. Then $\theta\pi(p_0) = \theta(p) = \pi'\nu(p_0)$. Since all maps are continuous, we have $\theta\pi = \pi'\nu$. We define $\xi f(t) = \hat{f}(\pi t p_0)$ for $f \in \mathfrak{A}$. We wish to show that $\xi' f = \xi f$ for $f \in \mathfrak{A}'$. If $f \in \mathfrak{A}'$ then there is an $F \in C(P')$ such that $f(t) = F(tp_0')$ and $\xi' f(t) = F(\pi' t p_0'')$. We observe that since $f(t) = \hat{f}(tp_0)$ and $f(t) = F(tp_0') = F(\theta t p_0)$, that $f(p) = F(\theta p)$ for $p \in P$. Thus we have

$$\xi f(t) = \hat{f}(\pi t p_0) = F(\theta\pi t p_0) = F(\pi'\nu t p_0) = F(\pi' t \nu p_0) = F(\pi' t p_0'') = \xi' f(t)\,.$$

This completes the proof. (See also [10], lemma 2.)

We say that an algebra $\mathfrak{A}'$ is *shift invariant* if $\xi\mathfrak{A}' \subset \mathfrak{A}'$, for all $\xi \in S(\mathfrak{A})$.

THEOREM I.5. *Let $\mathfrak{A}'$ be a closed, $*$, invariant subalgebra of $\mathfrak{A}$, and let P' be the maximal ideal space of $\mathfrak{A}'$, with distinguished point p_0'. Then the following are equivalent:*

(1) *$\mathfrak{A}'$ is shift invariant.*

(2) $P_0' = P'$.

(3) *(P', T) is isomorphic with $(E(P'), T)$.*

Proof. (1) $\Leftrightarrow$ (2). Let $\nu\colon (P, T)\to(P', T)$ with $\nu(p_0) = p_0'$, where ν is induced by the inclusion $\mathfrak{A}' \subset \mathfrak{A}$. Invariance of $\mathfrak{A}'$ under all shift operators is equivalent to the assertion that, if $\pi \in \mathcal{E}(P)$, then there is a $\pi' \in \mathcal{E}(P')$ such that the following diagram commutes:

$$\begin{array}{ccc} (P, T) & \xrightarrow{\pi} & (P, T) \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu} \\ (P', T) & \xrightarrow{\pi'} & (P', T) \end{array}$$

Now, suppose that such a π' always exists. Let $p' \in P'$ and let $p \in P$ such that $\nu(p) = p'$. Choose $\pi \in \mathcal{E}(P)$ such that $\pi(p_0) = p$, and choose π' to complete the diagram. Then $p' = \nu(p) = \nu\pi(p_0) = \pi'\nu(p_0) = \pi'(p_0')$.

Conversely, suppose $P_0' = P'$, and let $\pi \in \mathcal{E}(P)$. Let $p' = \nu\pi(p_0)$, and choose $\pi' \in \mathcal{E}(P')$ such that $\pi'(p_0') = p'$. Then $\nu\pi(p_0) = p' = \pi'(p_0') = \pi'\nu(p_0)$, and therefore $\nu\pi = \pi'\nu$. The proof is completed.

(3) $\Rightarrow$ (2). If $\xi \in E(P')$, let $\psi_\xi \in \mathcal{E}\big(E(P')\big)$ be defined by $\psi_\xi(\eta) = \eta\xi$ $\big(\eta \in E(P')\big)$. Then, if e is the identity of $E(P')$, $\psi_\xi(e) = \xi$, and (2) is satisfied, since (P', T) and $\big(E(P'), T\big)$ are isomorphic.

(2) $\Rightarrow$ (3). Consider the homomorphism $\sigma\colon \big(E(P'), T\big)\to(P', T)$ defined by $\sigma(\xi) = \xi p_0'$; σ is continuous and onto. Suppose $\sigma(\xi) = \sigma(\eta)$. Then $\pi_\xi(p_0') = \pi_\eta(p_0')$. Since $P_0' = P'$, theorem I.1 tells us $\xi = \eta$, and σ is one-one.

THEOREM I.6. *Let $\mathfrak{A}'$ and P' be as in theorem* I.5, *and let $\mathfrak{A}''$ be the smallest shift invariant algebra containing $\mathfrak{A}'$. Then the maximal ideal space P'' of $\mathfrak{A}''$ is isomorphic with $E(P')$.*

Proof. The idea of the proof is to construct homomorphisms $\psi\colon (P'', T)\to\big(E(P'), T\big)$ and, $\varphi\colon \big(E(P'), T\big)\to(P', T)$ for which $\psi(p_0'') = e$ and $\varphi(e) = p_0'$. If we let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ obtained by restricting continuous functions on $E(P')$ to $\gamma(e)$, then we have $\mathfrak{A}'' \supset \mathfrak{B} \supset \mathfrak{A}'$. To complete the proof we need only observe that $\mathfrak{B}$ is a $*$, closed, shift invariant algebra. From the hypothesis it follows that $\mathfrak{A}'' = \mathfrak{B}$ and thus (P'', T) and $\big(E(P'), T\big)$ are isomorphic. We now fill in the details. The homomorphism φ is given by $\varphi(\xi) = \xi(p_0')$ for $\xi \in E(P')$. The homomorphism ψ is defined as follows: Since $\mathfrak{A}' \subset \mathfrak{A}''$, there is an onto homomorphism $\pi\colon (P'', T)\to(P', T)$. Let $\theta\colon \big(E(P''), T\big)\to\big(E(P'), T\big)$ be the induced semi-group homomorphism defined by $\theta(\xi'')\pi(p'') = \pi(\xi'' p'')$, $\xi'' \in E(P'')$, $p'' \in P''$ ([8]). Since $\mathfrak{A}''$ is $*$, closed, and shift invariant, there is an isomorphism $\lambda\colon (P'', T)\to\big(E(P''), T\big)$. We let $\psi = \theta\circ\lambda$ and observe that $\psi(p_0'') = e$. It is clear that $\mathfrak{B}$, the algebra of restrictions of continuous functions to $\gamma(e)$, is $*$ and closed. To see that it is shift invariant we remark that the maps $\eta\to\xi\eta$ (ξ fixed in $E(P')$) are endomorphisms $\big(E(P'), T\big)$ and we apply theorem I.5. Compare [10] proposition 7.

II. The Bebutov system. Until now, we have been considering the bounded uniformly continuous functions $f\colon T\to C$ provided with the topology of uniform convergence. It is also convenient to provide this set with the compact open topology (or, what is the same thing, the topology of uniform convergence on compact sets). This topology is induced by the metric

$$\varrho(f, g) = \sup_{T>0} \{\min\{ \sup_{|t|\leqslant T} |f(t)-g(t)|, 1/T\} .$$

If we define, as above, $tf(s) = f(s+t)$, then this action of T defines a dynamical system, called the *Bebutov dynamical system*, and denoted by (B, T).

We write $\gamma(f)$ for the orbit of f in B, i.e., the set $[f_t|\ t \in T]$. Since every $f \in B$ is uniformly continuous, it follows that every orbit closure $\overline{\gamma(f)}$ is compact.

The importance of the Bebutov system is a consequence of the following remarkable theorem of Kakutani ([12]) which is a generalization and simplification of a theorem of Bebutov ([14], p. 33): every dynamical system (X, T), where the phase space X is compact metric and whose set of stationary points is either empty or homeomorphic to a subset of T, may be isomorphically embedded in the Bebutov system (B, T). We shall refer to this theorem as the Bebutov-Kakutani theorem.

Kakutani actually used the space C of all continuous functions on T with the above metric. This space is complete whereas B is not. The Kakutani-Bebutov theorem however applies to B for the following reason: A function f in C has compact orbit closure if and only if it is bounded and uniformly continuous ([11], p. 10). In other words if X is compact, then any homomorphic image of it in (C, T) is contained in (B, T).

Let us reiterate that $\mathfrak{A}$ and B consist of the same collection of functions—the bounded uniformly continuous functions from T to C, and that the action of T by translation gives rise to a dynamical system in both cases. Each of $\mathfrak{A}$ and B has advantages and disadvantages. $\mathfrak{A}$ is more useful when algebraic questions are considered. In the dynamical system $(\mathfrak{A}, T)$, T acts as a group of isometries, and orbit closures are not, in general, compact. Thus $(\mathfrak{A}, T)$ is of limited interest as a dynamical system. On the other hand, all orbit closures in (B, T) are compact, but, as we shall see in Section III, it is badly behaved algebraically.

Next, we want to relate the enveloping semigroup $E(B)$ of (B, T) with the shift operators $S(\mathfrak{A})$ defined in the preceding section. Now, the space B is obviously not compact; indeed, it contains a subspace homeomorphic with the real line (namely, the constant real valued functions.) However, every orbit closure is compact, and this is all that is needed for the enveloping semigroup to be defined. For, we may regard T as a subset of $\prod_{f\in B} \overline{\gamma(f)}$, which is compact. Then define $E(B)$ to be the closure of T in $\prod_{f\in B} \overline{\gamma(f)}$. $E(B)$ is, as before, a compact semigroup. We show that the enveloping semigroup $E(B)$ of (B, T) coincides with the set of shift operators $S(\mathfrak{A})$. For, let $\xi \in S(\mathfrak{A})$. Then, as we have seen, there is an admissible net $\{t_n\}$ in T which determines ξ. That is, for every $f \in B$ and $s \in T$, $t_nf(s) = f(t_n+s) \to \xi f(s)$. Now, since the family $\{t_nf\}$ is equiuniformly continuous, and uniformly bounded, this convergence is actually uniform on compact sets. Covnersely, if $t_n \to \xi \in E(B)$, then $t_nf \to \xi f$ uniformly on compact sets. Therefore, certainly $f(t_n+s) \to \xi f(s)$ for each $s \in T$, and ξ is a shift operator.

Thus $S(\mathfrak{A})$ and $E(B)$ are identical as sets. However, we find it convenient to continue to distinguish between them conceptually. An element ξ of $S(\mathfrak{A})$ is to be regarded as a map of $\mathfrak{A}$ to itself, and with respect to the norm toplogy of $\mathfrak{A}$, ξ is continuous. If we regard ξ as in $E(B)$, it is not in general continuous. To compensate somewhat for this, $E(B)$ has a compact topology, the topology of pointwise convergence. (This is the topology for shift operators defined in [13].)

However, if $\xi_n \to \xi$ in $E(B)$, it is not in general true that $\xi_n f \to \xi f$ in $\mathfrak{A}$. For, let $f \in B$ be any function which is not Bohr almost periodic, so $(\overline{\gamma(f)}, T)$ is not equicontinuous. (These notions are defined in Section IV.) Then there is a sequence $f_n \in \overline{\gamma(f)}$ such that $\varrho(f, f_n) \to 0$, but $\varrho(t_nf_n, t_nf) \geqslant \varepsilon > 0$, for some sequence $t_n \in T$. Since $f_n \in \overline{\gamma(f)}$, there is a $\xi_n \in E(B)$ such that $\xi_n f = f_n$. By choosing a subnet if necessary suppose $\xi_n \to \xi \in E(B)$; then, since $\xi_n f = f_n \to f$, $\xi f = f$. However, $\xi_n f$ does not approach ξf in $\mathfrak{A}$, (i.e. uniformly) since, as we have observed, $\varrho(t_nf_n, t_nf) \geqslant \varepsilon > 0$.

Now, let (P, T) denote the universal point transitive dynamical system with distinguished point p_0. We show that its enveloping semigroup $E(P)$ is isomorphic with $E(B)$. We have just observed that $E(B) = S(\mathfrak{A})$. Moreover, from section I we know, since $P_0 = P$, that each $\xi \in E(P)$ determines a unique element of $E(B)$, which we still call ξ, such that $\xi f(t) = \hat{f}(\xi p_t)$, and that this correspondence is an (algebraic) isomorphism.

It is only necessary to show that this mapping from $E(P)$ to $E(B)$ is bicontinuous. Since $E(P)$ and $E(B)$ are compact, and the mapping is one-to-one onto, it is sufficient to show that the inverse mapping from $E(B)$ to $E(P)$ is continuous. Suppose, then, that $\xi_n \to \xi$ in $E(B)$. Considering ξ_n and ξ as in $E(P)$, we must show that $\xi_n p \to \xi p$, for each $p \in P$. For all $f \in \mathfrak{A} = B$ we have $\xi_n f \to \xi f$, and therefore $\hat{f}(t\xi_n p_0) \to \hat{f}(t\xi p_0)$. If $p \in P$, there is a $\pi \in \mathcal{E}(P)$ for which $\pi(p_0) = p$, and since $\xi_n p = \xi_n \pi(p_0) = \pi(\xi_n p_0)$, $\xi p = \xi\pi(p_0) = \pi(\xi p_0)$, and we must show that $\pi\xi_n p_0 \to \pi\xi p_0$. This is equivalent to showing that $\hat{g}(\pi\xi_n p_0) \to \hat{g}(\pi\xi p_0)$, for each $\hat{g} \in C(P)$. If we let $\hat{g}^0\pi = \hat{f}$, our previous argument applies, and the assertion follows.

The further study of the Bebutov system is facilitated by the notion of a function coming from a flow. Let (X, T) be a dynamical system, let $x \in X$, and let $f\colon T \to C$ be defined by restricting F to the orbit of x; that is, $f(t) = F(tx)$. It is easy to see that $f \in \mathfrak{A}$; we say that f *comes from* (X, T) *at* x. This notion was defined and studied extensively in [3].

We denote by $\mathfrak{A}_x$ the set of $f \in \mathfrak{A}$ which come from (X, T) at x. $\mathfrak{A}_x$ is a $*$, closed, invariant subalgebra of $\mathfrak{A}$. For the most part, we only consider $\mathfrak{A}_x$ when $\overline{\gamma(x)} = X$; in this case $\mathfrak{A}_x$ is isomorphic with $C(X)$. In particular, if $g \in B$, $\mathfrak{A}_g$ will mean the set of $f \in \mathfrak{A}$ which come from $(\overline{\gamma(g)}, T)$ at g.

Let (P, T) be the universal point transitive flow with distinguished point p_0, and let $f \in \mathfrak{A}$. Then, as we have seen, the Gelfand transform $\hat{f} \in C(P)$ satisfies $\hat{f}(tp_0) = f(t)$. That is, $\mathfrak{A}_{p_0} = \mathfrak{A}$.

LEMMA II.1. *Let (X, T), (Y, T) be point transitive flows with $x \in X$, $y \in Y$ having dense orbits. Then $\mathfrak{A}_y \subset \mathfrak{A}_x$ if and only if there is a homomorphism $\pi\colon (X, T) \to (Y, T)$ such that $\pi(x) = y$.*

Proof. Suppose that such a homomorphism exists. Let $f \in \mathfrak{A}_y$, so $f(t) = F(ty)$ for some $F \in C(Y)$. Let $G = F^{\circ}\pi \in C(X)$. Then $f(t) = G(tx)$, and $f \in \mathfrak{A}_x$.

If $\mathfrak{A}_x \subset \mathfrak{A}_y$, let i denote the inclusion map. Then the maximal ideal spaces of $\mathfrak{A}_x$ and $\mathfrak{A}_y$ are, respectively, X and Y, and the homomorphism π induced by i maps x to y.

The next lemma is an immediate consequence of the Stone-Weierstrass theorem.

LEMMA II.2. *Let (X, T) be a point transitive flow, $x \in X$ a point with dense orbit, and $\mathfrak{B}$ a $*$, closed invariant subalgebra of $\mathfrak{A}_x$. Suppose that $\hat{\mathfrak{B}} = [F \in C(X) \mid F(tx) = f(t)$, some $f \in B] = [\hat{f} \mid f \in \mathfrak{B}]$ separates points of X. Then $\mathfrak{B} = \mathfrak{A}_x$.*

If $K \subset \mathfrak{A}$, by $\mathfrak{A}(K)$ we mean the $*$, closed, invariant, algebra generated by K. If $f \in \mathfrak{A}$, we write $\mathfrak{A}(f)$ instead of $\mathfrak{A}(\{f\})$.

If (X, T) is a dynamical system, and I an arbitrary set, then (X^I, T) is the dynamical system defined by the coordinatewise action of T; i.e., if $\bar{x} = (x_j)$, then the jth coordinate of $t\bar{x}$ is tx_j $(j \in I)$. This is in conformity with our definition of the dynamical system $(E(X), T)$, where $E(X)$ is regarded as a subspace of $\prod_{x \in X} \overline{\gamma(x)}$

THEOREM II.3. *Let $K = \{f_a \mid a \in I\}$ be a family of functions in $\mathfrak{A}$. Let f^* be the point of B^I whose a-th coordinate is f_a, and let $X = \overline{\gamma(f^*)} \subset B^I$. Then $\mathfrak{A}_{f^*} = \mathfrak{A}(K)$.*

Proof. Clearly, for each $a \in I$, $f_a \in \mathfrak{A}_{f^*}$, and since $\mathfrak{A}_{f^*}$ is a $*$, closed, invariant algebra $\mathfrak{A}(f_a) \subset \mathfrak{A}_{f^*}$, and therefore $\mathfrak{A}(K) \subset \mathfrak{A}_{f^*}$.

Now, let $\xi, \eta \in E(B)$ (which we identify with $E(B^I)$) such that $\xi f^* \neq \eta f^*$. For $a \in I$, $\tau \in T$, let $H_{a,\tau} \in C(B^I)$ be defined by $H_{a,\tau}\big((g_\beta)_{\beta \in I}\big) = (\tau g_a)(0) = g_a(\tau)$. Then $H_{a,\tau}(tf^*) = \tau f_a(t)$. Now $\tau f_a \in \mathfrak{A}(f_a) \subset \mathfrak{A}(K)$, so $H_{a,\tau}$ (or strictly speaking, its restriction to X) is in $\hat{\mathfrak{A}}(K)$. Now, if $\xi f^* \neq \eta f^*$, there is an $a \in I$ and a $\tau \in T$ such that $\xi f_a(\tau) \neq \eta f_a(\tau)$; that is, $H_{a,\tau}(\xi f^*) \neq H_{a,\tau}(\eta f^*)$. Therefore, $\hat{\mathfrak{A}}(K)$ separates points of X, and by lemma II.2, $\mathfrak{A}(K) = \mathfrak{A}_{f^*}$.

From this theorem, we may deduce a number of corollaries rather easily.

COROLLARY II.4. *Let $f \in \mathfrak{A}$ and let $X = \overline{\gamma(f)}$. Then $\mathfrak{A}_f = \mathfrak{A}(f)$.*

Since $\mathfrak{A}$ is isomorphic with $C(\overline{\gamma(f)})$, the maximal ideal space of $\mathfrak{A}_f$ is $\overline{\gamma(f)}$. Using this fact, and the corollary just proved, we can characterize the maximal ideal spaces of those algebras which are generated by a single element of $\mathfrak{A}$.

COROLLARY II.5. *Let $\mathfrak{B}$ be a $*$, closed, invariant algebra, and let X be the maximal ideal space of $\mathfrak{B}$. Then (X, T) is embeddable in the Bebutov system (B, T) if and only if $\mathfrak{B} = \mathfrak{A}(f)$, for some $f \in \mathfrak{B}$.*

COROLLARY II.6. *Let (X, T) be point transitive, let $x \in X$ such that $\overline{\gamma(x)} = X$, and let $f \in \mathfrak{A}$. Then $f \in \mathfrak{A}_x$ if and only if there is a homomorphism $\pi\colon (X, T) \to (\overline{\gamma(f)}, T)$ with $\pi(x) = f$.*

Proof. Such a homomorphism exists if and only if $\mathfrak{A}(f) \subset \mathfrak{A}_x$, by lemma II.1 and corollary II.4. But $\mathfrak{A}_x$ is a $*$, invariant, closed algebra, so this is equivalent to $f \in \mathfrak{A}_x$.

Of course, it is possible for two distinct functions f and g to have isomorphic orbit closure in (B, T). Our next result, which follows immediately from corollaries II.4 and II.6 tells us when this occurs.

COROLLARY II.7. *Let $f, g \in \mathfrak{A}$, $X = \overline{\gamma(f)}$, $Y = \overline{\gamma(g)}$. Then there is a homomorphism $\pi\colon (X, T) \to (Y, T)$ with $\pi(f) = g$ if and only if $g \in \mathfrak{A}(f)$. Hence (X, T) and (Y, T) are isomorphic, with f corresponding to g, if and only if $\mathfrak{A}(f) = \mathfrak{A}(g)$.*

COROLLARY II.8. *Let $f \in \mathfrak{A}$, and let $\mathfrak{A}_S(f)$ denote the smallest shift invariant, $*$, closed algebra containing f. Then the maximal ideal space of $\mathfrak{A}_S(f)$ is $E(\overline{\gamma(f)})$.*

Proof. The maximal ideal space of $\mathfrak{A}(f) = \mathfrak{A}_f$ is $\overline{\gamma(f)}$. Now apply theorem I.6.

COROLLARY II.9. *Let $f, g \in \mathfrak{A}$. Then $\mathfrak{A}_S(f) = \mathfrak{A}_S(g)$ if and only if $E(\overline{\gamma(f)})$ is isomorphic with $E(\overline{\gamma(g)})$.*

The final result in this section says that, under certain conditions, a countably generated subalgebra of $\mathfrak{A}$ is generated by a single element.

THEOREM II.10. *Let K be a finite or countable subset of B, and suppose that there is an h in K such that $\overline{\gamma(h)}$ has no stationary points. Then there is a $g \in \mathfrak{A}$ such that $\mathfrak{A}(K) = \mathfrak{A}(g)$.*

Proof. By corollary II.5, it is sufficient to show that the maximal ideal space of $\mathfrak{A}(K)$ is embeddable in the Bebutov system. Let $K = \{f_1, f_2, \ldots\}$, and let $f^* = (f_j) \in B^Z$. By theorem II.3, $\mathfrak{A}(K) = \mathfrak{A}_{f^*}$, so the maximal ideal space of $\mathfrak{A}(K)$ is $\gamma(f^*) \subset B^Z$, which is metrizable. Moreover, the set of stationary points of $\overline{\gamma(f^*)}$ is empty. Therefore $\overline{\gamma(f^*)}$ is embeddable in (B, T), and the proof is completed.

III. Minimal functions and their subalgebras. Let (X, T) be a dynamical system. Recall that a *minimal set* in (X, T) is a non-empty, closed invariant set M which has no proper subset with the same properties ([11], 2.12). Equivalently, a non-empty set M is minimal if $\overline{\gamma(x)} = M$, for all $x \in M$. (Hence (M, T) is certainly point transitive.) Any compact invariant subset of X contains a minimal set; this is proved by a simple application of Zorn's lemma.

If $x \in X$, $\overline{\gamma(x)}$ is a minimal set if and only if x is an *almost periodic point*—that is, if U is a neighborhood of x, the set $A = [t \in T|\ tx \in U]$ is a relatively dense subset of T ([11], 4.05 and 4.07).

In this section, we consider the subset $\mathfrak{M}$ of $\mathfrak{A}$ consisting of those functions f whose orbit closures $\overline{\gamma(f)}$ are minimal subsets of the Bebutov system (B, T).

The first theorem in this section will characterize $\mathfrak{M}$ in several ways. In order to state this theorem, it is necessary to discuss the enveloping semigroup further.

Let (X, T) be any dynamical system, with every $\overline{\gamma(x)}$ compact. A *left ideal* in $E = E(X)$ is a subset I such that $EI \subset I$. A left ideal I is called *minimal* if it contains no proper non-empty subset which is also a left ideal. Ellis has shown ([8], lemma 1) that the minimal left ideals in E coincide with the minimal sets of the dynamical system (E, T). Since E is compact, this assures us that there is at least one minimal left ideal in E, and that the minimal left ideals are closed. It can also be shown that any minimal left ideal I contains idempotents, and that, if x is an almost periodic point of X, there is at least one idempotent $u \in I$ such that $ux = x$ ([8], lemma 2 and theorem 1).

Now we can state and prove our theorem concerning the equivalent characterizations of the minimal functions.

THEOREM III.1. *Let $f \in \mathfrak{A}$. Then the following are equivalent:*

(1) *$f \in \mathfrak{M}$.*

(2) *If ξ a shift operator, then there is a shift operator η such that $\eta\xi f = f$.*

(3) *f comes from some minimal set (X, T).*

(4) *If I is a minimal left ideal in $E(B)$, then $f \in I\mathfrak{A} = [\xi g|\ \xi \in I, g \in \mathfrak{A}]$.*

(5) *If I is a minimal left ideal in $E(B)$, and J is the set of idempotents in I, then $f \in J\mathfrak{A}$.*

((2) is the definition of minimal function in [13].)

Proof. If (X, T) is any dynamical system and $x \in X$, the orbit closure $\overline{\gamma(x)} = [\xi x|\ \xi \in E(X)]$. The equivalence of (1) and (2) follows from this and the fact, observed earlier, that the shift operators of $\mathfrak{A}$ are the elements of $E(B)$.

The equivalence of (1) with (4) and (5) is proved in [8] (theorem 1).

If $f \in \mathfrak{M}$, then f comes from the minimal set $(\overline{\gamma(f)}, T)$. If f comes from a minimal set (X, T), then by lemma II.1 and corollary II.4, $(\overline{\gamma(f)}, T)$ is a homomorphic image of (X, T) and is therefore a minimal set. This shows that (1) and (3) are equivalent. (Another proof is in [3], theorem 3.7.)

COROLLARY III.2. *If $f \in \mathfrak{M}$, then $\mathfrak{A}(f) \subset \mathfrak{M}$.*

Proof. If $g \in \mathfrak{A}(f)$, then $g \in \mathfrak{A}_f$, by corollary II.4. The result now follows from theorem III.1, (3).

THEOREM III.3. *$\mathfrak{M}$ is shift invariant, $*$ closed, and uniformly closed.*

Proof. The shift invariance of $\mathfrak{M}$ is a consequence of (2) in theorem III.1. It is immediate that $\mathfrak{M}$ is $*$ closed. Now, let f_n be a sequence of minimal functions converging uniformly to f. Let $\varepsilon > 0$, $\tau > 0$ and let n be chosen so that $\|f - f_n\| < \varepsilon/3$. Let A be a relatively dense subset of T such that $|f_n(s) - tf_n(s)| < \varepsilon/3$, for $t \in A$ and $|s| \leqslant \tau$. Then for such t and s,

$$|f(s) - tf(s)| = |f(s) - f(s+t)| \leqslant |f(s) - f_n(s)| + |f_n(s) - f_n(t+s)| + \\ + |f_n(t+s) - f(t+s)| < \varepsilon .$$

The proof is completed.

The set $\mathfrak{M}$ is not an algebra. In [3], Hahn and L. Auslander show, by an example, that the sum of two minimal functions need not be a minimal function. We show this later, by a different method.

However, $\mathfrak{M}$ does contain some interesing subalgebras. By a *minimal algebra*, we shall mean a $*$, closed, invariant algebra $\mathfrak{B}$ such that $\mathfrak{B} \subset \mathfrak{M}$. Now, theorem III.1 tells us that $\mathfrak{M} = J\mathfrak{A} = \bigcup [u\mathfrak{A}|\ u \in J]$, where J is the set of idempotents in any minimal left ideal I of $E(B)$. Since all shift operators are continuous in the uniform topology, and commute with translations, $u\mathfrak{A}$ is a minimal algebra. Thus $\mathfrak{M}$ is a union of minimal algebras. Our next result establishes the connection between minimal algebras and minimal sets.

THEOREM III.4. *Let $\mathfrak{B}$ be a $*$, closed, invariant subalgebra of $\mathfrak{A}$. Then $\mathfrak{B}$ is a minimal algebra if and only if there is a minimal set (X, T) and an $x \in X$ such that $\mathfrak{B} = \mathfrak{A}_x$. Moreover, $\mathfrak{B}$ determines (X, T) up to isomorphism.*

Proof. If (X, T) is minimal, and $x \in X$, then $\mathfrak{A}_x$ is $*$, closed, and invariant, and, by theorem III.1 (3), $\mathfrak{A}_x \subset \mathfrak{M}$.

Now, suppose that $\mathfrak{B}$ is a minimal algebra. Let X denote the maximal ideal space of $\mathfrak{B}$. We know that (X, T) is point transitive, and that $\mathfrak{B} = \mathfrak{A}_x$, for some $x \in X$ with $\overline{\gamma(x)} = X$. The minimality of (X, T) is a consequence of the following lemma.

LEMMA III.5. *Let (X, T) be point transitive and not minimal. Let $q \in X$ with $\overline{\gamma(q)} = X$. Then there is an $f\colon T \to \boldsymbol{C}$ which comes from q such that f is not a minimal function.*

Proof. Since (X, T) is not minimal, q is not an almost periodic point. Therefore, there is a neighborhood V of q, and a sequence $t_n \to +\infty$ such that $tq \notin V$ for $t_n - n < t < t_n + n$. Let U be a neighborhood of q such that $\overline{U} \subset V$, and let $F \in C(X)$ such that $0 \leqslant F(r) \leqslant 1$, $F(r) = 0$ for $r \in U$ and $F(r) = 1$, for $r \notin V$. Let $f\colon T \to \boldsymbol{C}$ be defined by $f(t) = F(tq)$. Then, if ϱ is the metric of the Bebutov system,

$$\varrho(f, \tau f) \geqslant \sup_{|t| \leqslant 1} |f(t) - f(\tau + t)| \geqslant |f(0) - f(\tau)|,$$

which equals 1 if $t_n - n \leqslant \tau \leqslant t_n + n$. But then the set $A = [\tau \in T|\ \varrho(f, \tau f) < 1]$ is not relatively dense and f is not an almost periodic point of (B, T).

This completes the proof of the lemma, and it follows that (X, T) is minimal. The uniqueness of (X, T) follows from the facts that an algebra determines its maximal ideal space uniquely, and may be identified with the continuous functions on that space.

COROLLARY III.6. *If (X, T) is point transitive and not minimal, then (X, T) has a metrizable homomorphic image with the same property.*

A *universal minimal set for T* is a minimal set (M, T) such that every minimal set (X, T) is a homomorphic image of (M, T). It is a consequence of the work of Ellis [9] and Chu [5] that a universal minimal set exists and is unique up to isomorphism.

Now, let (P, T) be the universal point transitive flow, with distinguished point p_0, and let (X, T) be any minimal set. Then, since (X, T) is point transitive, there is a homomorphism π of (P, T) onto (X, T) (and, indeed, π may be chosen so that p_0 is mapped to any preassigned point of X).

Let M be any minimal subset of P, and let π' be the restriction of π to M. Then π' maps M onto X (since a homomorphism of a compact invariant set into a minimal set is always onto). That is, (X, T) is a homomorphic image of (M, T), and (M, T) is a universal minimal set. This argument also shows that all the minimal subsets of P are isomorphic.

Now, let M be a minimal set in (P, T), let $m \in M$, let I be a minimal left ideal in $E(P)$, and let u be the (unique) idempotent in I such that $um = m$, [8]. Suppose that $f \in \mathfrak{A}$ comes from M at m. Then there is an $F \in C(M)$ such that $f(t) = F(tm)$. Let $\{s_n\}$ be a net in T such that $s_n \to u \in E(B) = E(P)$. Since $s_n f(t) = f(t + s_n) = F(ts_n m)$, $s_n f(t) \to uf(t)$, and $F(ts_n m) \to F(tum) = F(tm) = f(t)$, it follows that $uf = f$. That is, if $f \in \mathfrak{A}$ comes from M at m, then $f \in u\mathfrak{A}$. Conversely, if $f \in u\mathfrak{A}$, let $m' = up_0 \in M$. Since $um = m$, there is an automorphism φ of (M, T) such that $\varphi(m) = m'$ ([2], theorem 4). Let $F = \hat{f}|M$, and let $G = F^\circ\varphi \in C(M)$. Then

$$G(tm) = F(\varphi(tm)) = F(tm') = \hat{f}(tm') = \hat{f}(tup_0) = uf(t) = f(t),$$

since $f \in u\mathfrak{A}$. That is, $f \in \mathfrak{A}_m$.

Thus we have proved:

THEOREM III.7. *Let M be a minimal subset of (P, T). Then M is a universal minimal set. If $m \in M$, and I is a minimal left ideal in $E(P)$, then $\mathfrak{A}_m = u\mathfrak{A}$, where u is the idempotent in I such that $um = m$. Therefore, the maximal ideal space of the minimal algebra $u\mathfrak{A}$ is the universal minimal set.*

Now we consider two related minimal algebras. In order to define these, we need the notion of a proximal pair of points.

Let (X, T) be a dynamical system. The points x and y are said to be *proximal* if there is a $z \in X$ and a net $\{t_n\}$ in T such that $t_n x \to z$ and $t_n y \to z$. If x and y are not proximal, they are called *distal*. A point $x \in X$ is said to be a *distal point*, if, for every $y \in X$ with $y \neq x$, x and y are distal. The dynamical system (X, T) is distal if every $x \in X$ is a distal point.

If x is a distal point of X, then x is an almost periodic point ([1], lemma 2). It follows that if (X, T) is distal, then (X, T) is pointwise almost periodic (every orbit closure is minimal.).

A function $f \in \mathfrak{A}$ is said to be *weakly distal* if f comes from a distal point of some dynamical system (X, T), and is said to be *distal* if f comes from a point of some distal dynamical system. Let $\mathfrak{D}$ and $\mathfrak{W}$ denote the distal and weakly distal functions. Obviously $\mathfrak{D} \subset \mathfrak{W}$; we shall see later that the inclusion is proper. From the remarks in the preceding paragraph, a weakly distal function is minimal.

THEOREM III.8. *Let $f \in \mathfrak{A}$. Then the following are equivalent.*

(1) *$f \in \mathfrak{W}$.*

(2) *f is a distal point of $(\overline{\gamma(f)}, T)$.*

(3) *$uf = f$, for all idempotent shift operators u.*

Proof. (1) $\Rightarrow$ (3). Let (X, T) be minimal, let $x_0 \in X$ be a distal point, and let $F \in C(X)$ such that $F(tx_0) = f(t)$. Let $\psi\colon (P, T) \to (X, T)$

be a homomorphism such that $\psi(p_0) = x_0$. Let $G = F^0\psi \in C(P)$. Then $G(tp_0) = F(\psi(tp_0)) = F(tx_0) = f(t)$, so $G = \hat{f}$. Now, if u is an idempotent in $E(P)$, up_0 is proximal to p_0, and $\psi(up_0)$ is proximal to $\psi(p_0) = x_0$, so $\psi(up_0) = x_0$. Then $uf(t) = \hat{f}(tup_0) = G(tup_0) = F(t\psi(up_0)) = F(tx_0) = f(t)$.

(3) $\Rightarrow$ (2). Let I be a minimal left ideal in $E(B)$, and let u be an idempotent in I. Then $f = uf \in I\mathfrak{A} \subset \mathfrak{M}$. Now, if $g \in \overline{\gamma(f)}$ is proximal to f, then $g = uf$ for some idempotent u in $E(B)$ ([1], theorem 4). Since $uf = f$, for all idempotents u, f is a distal point of $(\overline{\gamma(f)}, T)$.

(2) $\Rightarrow$ (1). f comes from itself in $(\overline{\gamma(f)}, T)$.

THEOREM III.9. *Let $f \in \mathfrak{A}$. Then the following are equivalent.*

(1) $f \in \mathfrak{D}$.

(2) $(\overline{\gamma(f)}, T)$ *is a distal minimal set.*

(3) *If $\xi, \eta_1, \eta_2 \in S(\mathfrak{A})$ such that $\xi\eta_1 f = \xi\eta_2 f$, then $\eta_1 f = \eta_2 f$.*

(4) *If $u, \xi \in S(\mathfrak{A})$ and u is idempotent, then $u\xi f = \xi f$.*

(Condition (3) is Knapp's definition of distal function ([13]) and condition (4) is similar to his theorem 1.)

Proof. (1) $\Rightarrow$ (2). Let (X, T) be distal minimal, and let f come from $x \in X$. Then $(\overline{\gamma(f)}, T)$ is a homomorphic image of (X, T), and is therefore distal and minimal. (This follows from the fact that a flow (Y, T) is distal if and only if the product flow $(Y \times Y, T)$ is pointwise almost periodic.) Alternately, it can easily be shown that, for every $\xi \in S(\mathfrak{A}) = E(B)$, $\xi f \in \mathfrak{D} \subset \mathfrak{W}$, and then (2) of the preceding theorem may be applied.

(2) $\Rightarrow$ (1). f comes from $(\overline{\gamma(f)}, T)$.

(2) $\Rightarrow$ (4). $f \in \mathfrak{D}$, $\xi f \in \mathfrak{D} \subset \mathfrak{W}$. By (3) of the preceding theorem, $u\xi f = \xi f$.

(4) $\Rightarrow$ (2). By (3) of the preceding theorem $\xi f \in \mathfrak{W}$, for all $\xi \in S(\mathfrak{A}) = E(B)$. Then ξf is a distal point of $(\overline{\gamma(\xi f)}, T) = (\overline{\gamma(f)}, T)$; that is, $(\overline{\gamma(f)}, T)$ is distal.

(2) $\Rightarrow$ (3). In any dynamical system (X, T), x and y are proximal if and only if $\xi x = \xi y$ for some $\xi \in E(X)$, ([8]). Now, $\xi\eta_1 f = \xi\eta_2 f$ implies $\eta_1 f$ and $\eta_2 f$ are proximal and therefore $\eta_1 f = \eta_2 f$.

(3) $\Rightarrow$ (2). Let $g, h \in \overline{\gamma(f)}$, and suppose that they are proximal. There exist η_1 and $\eta_2 \in S(\mathfrak{A})$ such that $\eta_1 f = g$ and $\eta_2 f = h$. Since g and h are proximal, there is a $\xi \in S(\mathfrak{A})$ for which $\xi\eta_1 f = \xi g = \xi h = \xi\eta_2 f$. By (3), $g = \eta_1 f = \eta_2 f = h$. This proves that $(\overline{\gamma(f)}, T)$ is distal.

The two theorems just proved tell us that $\mathfrak{W}$ and $\mathfrak{D}$ are $*$, closed, invariant subalgebras of $\mathfrak{M}$, and that $\mathfrak{D}$ is also shift invariant. By (3) of theorem III.8, we may write $\mathfrak{W} = \bigcap [u\mathfrak{A} | u \text{ an idempotent in } S(\mathfrak{A})]$.

Also, since $\overline{\gamma(f)} = [\xi f | \xi \in S(\mathfrak{A})]$, we can say that $f \in \mathfrak{D}$ if and only if $\xi f \in \mathfrak{W}$ for all $\xi \in S(\mathfrak{A})$.

We can use theorem II.9 to show that $\mathfrak{D}$ is a proper subset of $\mathfrak{W}$. Let (X, T) be a non-distal minimal set, where X is metrizable and contains a distal point x_0 ([11], 12.56). By the Bebutov-Kakutani theorem, there is an $f \in \mathfrak{A}$ such that $(\overline{\gamma(f)}, T)$ is isomorphic with (X, T) and f corresponds to x_0 under the isomorphism. Then $f \in \mathfrak{W}$. However, $f \notin \mathfrak{D}$, for, if it were, $(\overline{\gamma(f)}, T)$ (and therefore (X, T)) would be distal.

Let W and D be the maximal ideal spaces of $\mathfrak{W}$ and $\mathfrak{D}$ respectively. Since $\mathfrak{W}$ and $\mathfrak{D}$ are subalgebras of $\mathfrak{M}$, (W, T) and (D, T) are minimal sets. Let $w_0 \in W$ be the image of $0 \in T$ under the natural maps; i.e. $w_0(f) = \hat{f}(w_0) = f(0)$. Now, since $f \in \mathfrak{W}$ implies $uf = f$ for all idempotents in $E(B)$, it follows that $\hat{f}(uw_0) = \hat{f}(w_0)$, for all $f \in \mathfrak{W}$, and all idempotents u in $E(W)$. Since $[\hat{f} | f \in \mathfrak{W}] = C(W)$, we have $uw_0 = w_0$, and w_0 is a distal point. Similarly, if $x_0 \in D$ is the image of 0, and if $x = \xi x_0 \in D$, from $u\xi f = \xi f$, for all $f \in \mathfrak{D}$, we obtain $u\xi x_0 = \xi x_0$, or $ux = x$, for all $x \in D$ and idempotents u in $E(D)$. That is, every point of D is distal, so (D, T) is distal. If $\mathfrak{W}' \subset \mathfrak{W}$ and $\mathfrak{D}' \subset \mathfrak{D}$, then the same argument shows that the maximal ideal spaces (W', T) and (D', T) have the properties, respectively, of possessing a distal point, and being distal. Finally, if (W'', T) is a minimal set with distal point w_0'' and $\mathfrak{W}''$ is the algebra of functions coming from w_0'', then theorem III.8 tells us $\mathfrak{W}'' \subset \mathfrak{W}$, and thus there is a homomorphism from (W, T) onto (W'', T) taking w_0 to w_0''. Similarly, any distal minimal set is a homomorphic image of (D, T). This proves most of

THEOREM III.10 (1) *(W, T) is a minimal set with a distal point w_0, which is universal in the following sense: any minimal set containing a distal point is a homomorphic image of (W, T) and the homomorphism can be chosen so that w_0 is mapped into any distal point. These properties determine (W, T) up to isomorphism.*

(2) *(D, T) is the universal distal minimal set. That is, (D, T) is distal, and any distal minimal set is a homomorphic image of (D, T). These properties determine (D, T) up to isomorphism.*

Proof. It is only necessary to prove the uniqueness of (D, T). This is known ([3], theorem 4.6), but we give another proof, using the algebra $\mathfrak{D}$. We show that if $\xi \in S(\mathfrak{A})$, then ξ maps $\mathfrak{D}$ onto itself. This will imply that H_ξ maps $C(D)$ onto itself, and it follows easily that π_ξ is one to one. Thus (D, T) is *coalescent* (every endomorphism is an automorphism), and is therefore unique up to isomorphism. Now $\xi\mathfrak{D} \subset \mathfrak{D}$; to prove $\xi\mathfrak{D} = \mathfrak{D}$, we regard ξ as an element of $E(B)$. Let $F = [\eta\xi | \eta \in E(B)]$. Then F is closed in $E(B)$ and $F^2 \subset F$. Thus F contains an idempotent ([7], lemma 1). Let $\eta \in E(B)$ such that $u = \eta\xi$ is an idempotent. Let $f \in \mathfrak{D}$. Then $\xi\eta\xi\eta f = \xi u \eta f = \xi\eta f$ (since $u\eta f = \eta f$, by theorem III.9 (4)). Then by theorem III.9

(3), $\xi\eta f = f$. That is, if $f \in \mathfrak{D}$ and $g = \eta f$, $\xi g = f$. The proof is completed.

Now $\mathfrak{D}$ is a proper subset of $\mathfrak{W}$, so there is a homomorphism of (W, T) onto (D, T) which is not one-one. Since (D, T) is coalescent, it follows easily that (D, T) and (W, T) are not isomorphic.

We do not know whether (W, T) is coalescent. It is known that there are minimal sets which contain distal point and are not coalescent, [2].

THEOREM III.11. *Let* $\mathfrak{B}$ *be a* $*$*, closed, shift invariant subalgebra of* $\mathfrak{A}$ *such that* $\mathfrak{B} \subset \mathfrak{M}$. *Then* $\mathfrak{B} \subset \mathfrak{D}$.

Proof. Let X be the maximal ideal space of $\mathfrak{B}$. It is sufficient to show that (X, T) is distal. By theorem I.5 (3), (X, T) is isomorphic with $(E(X), T)$ and, since $\mathfrak{B} \subset \mathfrak{M}$, $(E(X), T)$ is minimal. It follows easily from minimality that if $\xi \in E(X)$, there is an $\eta \in E(X)$ such that $\eta\xi = e$, the identity of $E(X)$. Now, if $x, y \in X$ are proximal, there is a $\xi \in E(X)$ such that $\xi x = \xi y$. Then, if $\eta \in E(X)$ such that $\eta\xi = e$, $x = \eta\xi x = \eta\xi y = y$. That is, (X, T) is distal.

COROLLARY III.12. *Let* $f \in \mathfrak{M}$ *such that* $\mathfrak{A}_S(f) \subset \mathfrak{M}$. *Then* $f \in \mathfrak{D}$.

We conclude this section by showing, as promised earlier, that the sum of two minimal functions need not be minimal. For, suppose $f+g \in \mathfrak{M}$ whenever f and g are in $\mathfrak{M}$. Now, the square of a minimal function is always minimal, so, if $f, g \in \mathfrak{M}$, then $fg = \frac{1}{2}\big((f+g)^2 - f^2 - g^2\big) \in \mathfrak{M}$. Since theorem III.3 tells us that $\mathfrak{M}$ is $*$, closed, and shift invariant, this would say that $\mathfrak{M}$ is a $*$, closed, shift invariant algebra. In particular $\mathfrak{A}_S(f) \subset \mathfrak{M}$, whenever $f \in \mathfrak{M}$. But this contradicts corollary III.12, since there are minimal sets which are not distal ([11], 12.56), and therefore (by the Bebutov-Kakutani theorem and theorem III.10) minimal functions which are not distal.

IV. Equicontinuos flows and almost periodic functions.

In section II we gave a criterion for $(\overline{\gamma(f)}, T)$ and $(\overline{\gamma(g)}, T)$ to be isomorphic, if $f, g \in B$. In this section we show that if f and g are almost periodic functions, this question can be answerd in terms of their Fourier transforms. The proof uses standard methods of harmonic analysis applied to equicontinuous flows.

The dynamical system (X, T) is said to be *equicontinuous* if T, regarded as a family of maps of X to X, forms an equicontinuous family. That is, if $\mathfrak{U}$ denotes the uniformity of X, and $\alpha \in \mathfrak{U}$, then there is a $\delta \in \mathfrak{U}$ such that $(x, y) \in \delta$ implies $(tx, ty) \in \alpha$, for all $t \in T$. An equicontinuous flow is obviously distal; the converse is not true, even if the flow is minimal ([4], chapter IV).

Let us summarize, as theorem IV.1 some known result on equicontinuous flows.

THEOREM IV.1. (1) *The dynamical system* (X, T) *is equicontinuous if and only if it is almost periodic; that is, for every* $\alpha \in \mathfrak{U}$, *there is a relatively dense* $A \subset T$ *such that* $(x, tx) \in \alpha$, *for all* $x \in X$, *and* $t \in A$.

(2) *The flow* (X, T) *is equicontinuous and minimal if and only if it is a solenoidal topological group. That is,* X *can be given the structure of a compact abelian group, and there is a continuous homomorphism* α: $T \Rightarrow X$ *such that* $\overline{\alpha(T)} = X$. *Any* $x \in X$ *may be chosen as the identity, and then the group structure is unique.*

Proof. [11], 4.38 and 4.44.

Note that (1) tells us that every $x \in X$ is an almost periodic point, and therefore X is a union of minimal sets. (2) reduces the study of equicontinuous minimal sets to solenoidal groups.

Now, let $f \in B$. By theorem IV.1, $(\overline{\gamma(f)}, T)$ is equicontinuous if and only if, for any $\varepsilon > 0$, there is a relatively dense $A \subset T$ such that $\varrho(tsf, sf) < \varepsilon$, for all $t \in A$, and $s \in T$. This is equivalent to $|f(t+s) - f(s)| < \varepsilon$, for all $t \in A$ and $s \in T$. This is the definition of a (*Bohr*) *almost periodic function*, [3].

The almost periodic functions may also be defined as those f whose orbit closure in $(\mathfrak{A}, T)$ is compact, [3]. (Recall, here we mean compact in the uniform topology.) The equivalence of these two definitions may be proved using theorem IV.1 (1). From this it also follows that, if f is almost periodic, the closure of $\gamma(f)$ in $\mathfrak{A}$ is equal to the closure of $\gamma(f)$ in B.

Using this latter characterization, it is easy to show that the almost periodic functions form a $*$, closed invariant subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. Since, for $f \in \mathfrak{B}$, $(\overline{\gamma(f)}, T)$ is an equicontinuous minimal set, $\mathfrak{B}$ is a minimal algebra.

Let β denote the maximal ideal space of $\mathfrak{B}$. Then (β, T) is a minimal set; indeed, Chu ([5], lemma 9) has shown that (β, T) is the *universal equicontinuous minimal* set (that is, (β, T) is equicontinuous, and every equicontinuous minimal set is a homomorphic image of it). β is called the *Bohr compactification* of T; its defining property is that every almost periodic function f may be extended to $f \in C(\beta)$. As a general reference consult [15].

Now we recall some elementary facts concerning the character groups of a solenoidal group. Let T_d denote the real numbers with the discrete topology. A compact abelian group G is a solenoid if and only if its character group G^* is isomorphic with a subgroup Γ of T_d. (In particular, β^* is isomorphic with T_d.) If Γ is a subgroup of T_d, and $G = \Gamma^*$, the dense one parameter subgroup given by the homomorphism α: $T \to G$, is obtained as the dual map of the injection α^*: $\Gamma \to T$, ([11], 4.54).

Let f be an almost periodic function, and let $\hat{f}$ be its extension to β, so $f(t) = \hat{f}(\theta(t))$ (where θ: $T \to \beta$ is a continuous homomorphism with

$\overline{\theta(T)} = \beta$.) For $s \in T_d$, $\lambda_s \in C(\beta)$ is defined by $\lambda_s(t) = e^{ist}$ $(t \in T)$. (Thus $\lambda_s \in \beta^*$.) The *Fourier transform* $\tilde{f}$ of f is defined by

$$\tilde{f}(s) = \int_\beta \overline{(\lambda_s, x)}\hat{f}(x)\,dx\,,$$

where dx is normalized Haar measure on β. We may also write

$$\tilde{f}(s) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)e^{-ist}dt\,.$$

Let

$$\operatorname{car}(\tilde{f}) = [s \in T_d|\ \tilde{f}(s) \neq 0]\,,$$

and let $\Gamma(\tilde{f})$ be the subgroup of T_d generated by $\operatorname{car}(\tilde{f})$. The purpose of this section is to prove the following theorem.

THEOREM IV.2. *Let f and g be almost periodic functions. Then the equicontinuous minimal sets $(\overline{\gamma(f)}, T)$ and $(\overline{\gamma(g)}, T)$ are isomorphic if and only if $\Gamma(\tilde{f}) = \Gamma(\tilde{g})$.*

The proof is preceded by a sequence of lemmas.

LEMMA IV.3. *Let f be an almost periodic function and let Γ be a subgroup of T_d such that $\Gamma \supset \Gamma(\tilde{f})$. Let $G = \Gamma^*$, and let a: $T \to G$ be the dual of the injection a^*: $\Gamma \to T$. Then there is an $f' \in C(G)$ such that $f(t) = f'(a(t))$.*

Proof. Let π: $\beta \to G$ be the homomorphism such that $\pi(\theta(t)) = \varphi(t)$, and let $\hat{f} \in C(\beta)$ satisfying $f(t) = \hat{f}(\theta(t))$. Thus $G = \beta/\ker\pi$. Since $\operatorname{car}(\tilde{f}) \subset \Gamma$, it follows readily, using the uniqueness of the Fourier transform, that $\hat{f}$ is constant on cosets of β by the subgroup $\Gamma^\perp$ = annihilator of $\Gamma = \ker\pi$. Define $f' \in C(G)$ by $f'(x+\Gamma^\perp) = f(x)$. The proof is completed.

The next lemma is a converse.

LEMMA IV.4. *Let f be an almost periodic function. Let G be a solenoidal group such that $f(t) = f'(a(t))$, for some $f' \in C(G)$. Then $G^* = \Gamma \supset \Gamma(\tilde{f})$.*

Proof. Again, let π: $\beta \to G$ such that $\pi(\theta(t)) = a(t)$. Now $f'(\pi(\theta(t))) = f'(a(t)) = f(t) = \hat{f}(\theta(t))$. Since $\hat{f} = f'\circ\pi$ on $\theta(T)$, $\hat{f} = f'\circ\pi$, and f is constant on cosets of β mod $\ker\pi = \Gamma^\perp$. Again it follows that $\operatorname{car}(\tilde{f}) \subset \Gamma$. Since Γ is a subgroup, we have $\Gamma(\tilde{f}) \subset \Gamma$.

Let G be a solenoidal group, and now regard (G, T) as an equicontinuous minimal set, $tx = a(t)x$ $(t \in T, x \in G)$. Let $e = a(0)$, the identity of G. Then using the terminology of section II, lemmas IV.3 and IV.4 say: $f \in \mathfrak{A}_e$ if and only if $\Gamma(f) \subset G^*$.

If f is almost periodic, we write X_f for $\overline{\gamma(f)}$, regarded as a solenoidal group, where ψ: $T \to X_f$ is defined by $\psi(t) = tf = f_t$. Thus f is the identity of the group X_f.

LEMMA IV.5. $X_f^* = \Gamma(\tilde{f})$.

Proof. $f = \psi(0)$ is the identity of the group X_f. Now, $f \in \mathfrak{A}_f$, so $\Gamma(\tilde{f}) \subset X_f^*$. Let $G = \Gamma(\tilde{f})^*$. Then $f \in \mathfrak{A}_e$, where e is the identity of G (lemma IV.3). By corollary II.6, there is a group homomorphism π: $G \Rightarrow X_f$ such that $\pi(e) = f$. Since this homomorphism is onto, we have $\Gamma(\tilde{f}) = G^* \supset X_f^*$.

We can now prove theorem IV.2. Let f and g be almost periodic functions. Then $(\overline{\gamma(f)}, T)$ and $(\overline{\gamma(g)}, T)$ are isomorphic if and only if $g \in \mathfrak{A}(f)$ and $f \in \mathfrak{A}(g)$ (corollary II.7) or, what is the same thing (corollary II.4), $g \in \mathfrak{A}_f$ and $f \in \mathfrak{A}_g$. By the remark following lemma IV.4, this is true if and only if $\Gamma(\tilde{f}) \subset X_g^*$, and $\Gamma(\tilde{g}) \subset X_f^*$, since f and g are the identities, respectively, of the groups X_f and X_g. An application of lemma IV.5 completes the proof.

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UNIVERSITY OF MARYLAND and YALE UNIVERSITY

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