On the dispersion sets of connected point-sets[1].

By

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I shal call a point-set H a *dispersion set* of a connected[2] point-set M, provided that (1) H is a proper[3] subset of M, and (2) $M-H$ is totally disconnected[4]. It in addition to these conditions no proper subset L of H exists such that $M-L$ is totally disconnected, then I shall call H a *primitive* dispersion set of M. If H consists of a finite number of points, I shall call it a *finite* dispersion set of M.

Clearly every connected set contains dispersion sets. For instance, M being a connected set, and P and Q points of M, the set $M-(P+Q)$ is a dispersion set of M. But not every connected set contains finite dispersion sets. The existence of sets of the latter type was shown by Knaster and Kuratowski[5] in 1921.

I propose in the present paper to investigate some of the properties of connected sets that have finite dispersion sets.

Theorem 1. *Let M be a connected set, and G a dispersion set of M. If G is finite, then some subset of G is a primitive dispersion set of M. If G is not finite, then it may or may not contain a primitive dispersion set of M.*

[1]) Presented to the American Mathematical Society, Dec. 29, 1923.

[2]) In this paper a point-set will be called *connected* if it contains more than one point, and if no matter how it be divided into two subsets, one of these contains a limit point of the other. A point-set will be called *connected in the general sense* if it is *connected*, or if it contains only one point.

[3]) A set H is a *proper* subset of a set M provided that H is a subset of M and $M-H$ is not vacuous.

[4]) A set is said to be *totally disconnected* if it contains no connected subset.

[5]) Cf. B. Knaster and C. Kuratowski. *Sur les ensembles connexes*, Fund. Math. II (1921), pp. 206—255.

Proof. If G is finite, let n be the number of its points. If G is not itself primitive, there exists a proper subset G_1 of G, such that $M-G_1$ is totally disconnected. If G_1 is not primitive, there exists a proper subset G_2 of G_1, such that $M-G_2$ is totally disconnected. This process cannot be continued indefinitely, since G_n would be vacuous. There will, then, result a proper subset, G_i $(i<n)$, of G, such that G_i is a primitive dispersion set of M.

That a dispersion set that is not finite may contain a primitive dispersion set is shown by the following example [1]: Let C be a non-dense perfect set on the segment $[(1, 0), (1, 1)]$ (as referred to rectangular X and Y axes). Let c denote the ordinate of each point of C, and $L(c)$ the set of all points (x, y) which satisfy the equation

$$y=c+\frac{1}{x}\sin\frac{\pi}{x}, \qquad 0<x\leqq 1.$$

Let S_1 denote the set of all points on the curves $L(c)$ (where c is an end-point of a complementary interval of C) whose ordinates are rational, and S_2 the set of all points on the curves $L(c)$ (where c is not an end-point of a complementary interval of C) whose ordinates are irrational. Let G denote the set of all points on the Y-axis. Then, if

$$M=G+S_1+S_2,$$

G is a primitive dispersion set of the connected set M.

An example of a dispersion set that is not finite and that contains no primitive dispersion set is furnished if we let M be the set of all points on the interval $[0, 1]$ of the X-axis, and G the set of all rational points in the same interval.

Theorem 2. *If N is a proper connected (in the general sense) subset of a connected set M and $C_1, C_2, \dots C_n$ a set of n composants* [2]) *of $M-N$, then the set $M-(C_1+C_2+\dots+C_n)$ is connected (in the general sense).*

[1]) This example is a modification of an example given by Knaster and Kuratowski of a biconnected set that does not contain any bounded connected subset. Knaster and Kuratowski, *Loc. cit.*, pp. 244—245.

[2]) If M is a point-set and P a point of M, the *composant* of M determined by P is the set of all points of M that lie with P in a connected (in the general sense) subset of M. In other words, it is the maximal connected (in the general sense) of M determined by P. Obviously the composant determined by P might be P itself.

Proof. This theorem is a generalization, and a direct consequence, of a theorem due to Knaster and Kuratowski[1]) to the effect that: If M is a connected set, N a proper connected (in the general sense) subset of M, and C a composant of $M-N$, then $M-C$ is connected (in the general sense).

Theorem 3. *If M is a connected set and P a point of M such that the set $M-P$ is totally disconnected, then there does not exist in $M-P$ a finite set of points, G, such that $M-G$ is totally disconnected.*

Proof. Since $M-P$ is totally disconnected, each point of G is a composant of $M-P$. Hence by Theorem 2 $M-G$ is connected, and cannot therefore be totally disconnected[2]).

Theorem 4. *If M is a connected set, and G a finite primitive dispersion set of M, then G is the only primitive dispersion set of M.*

Proof. Suppose M contains a primitive dispersion set H which is not identical with G. Clearly H cannot be a subset of G. Let P be a point of H which does not belong to G. Then $M-(H-P)$ contains a connected set N which must contain P, and of which P is a primitive dispersion set. Now N contains points of G, and these form a finite dispersion set of N. Then N contains a point P whose omission totally disconnects N, and a finite dispersion set which is a subset of $M-P$. This is a contradiction of Theorem 3.

As a result of Theorem 4 it follows that if a connected set M contains two primitive dispersion sets which are not identical, these sets must each contain infinitely many points. This suggests the interesting problem: *Can a connected set M contain two primitive dispersion sets G and H, each of which contains infinitely many points, and such that $G \equiv\!\!\!\!/\; H$?* I can give an example to show that in three dimensions two such sets G and H can exist, but I have not settled the problem for two dimensions[3]).

Theorem 5. *If G is a finite dispersion set of a connected set M and g a primitive dispersion set of M such that* (1) *g is a subset*

[1]) Loc. cit., p. 214, Th. X.

[2]) Theorem 3 is a generalization of a theorem proved by J. R. Kline to the effect that if a connected set M contains a point P whose omission totally disconnects M, then P is the only point whose omission totally disconnects M. See J. R. Kline, *A theorem concerning connected pointsets*, Fund. Math. III (1922), pp. 238—239.

[3]) This problem is settled by Knaster. His example will appear in the next volume of *Fund. Math.*

and (2) $M-(G-g)$ *is connected, then* g *is also a primitive dispersion set of the set* $M-(G-g)$.

Proof. The theorem is clearly true if $G-g$ is vacuous. Suppose that $G-g$ is not vacuous, and that the theorem is not true in this case.

g is a dispersion set of the set $M-(G-g)$, because G is a dispersion set of M and

$$M-(G-g)-g=M-G.$$

Let g_1 be a subset of g which is a primitive dispersion set of the set $M-(G-g)$, and let x be a point of $g-g_1$.

Since g is a primitive dispersion set of M, $M-(g-x)$ contains a connected set, N, which must contain x. Let

$$(G-g)\times N=g' \text{ [1]}.$$

(1) If g' is vacuous, then, since N is a subset of $M-(g-x)$, it follows that N is a subset of the set $M-(G-g)-(g-x)$. But g_1 is a subset of $g-x$. Hence N is a subset of the set $M-(G-g)-g_1$. But this is a contradiction of the fact that g_1 is a dispersion set of the set $M-(G-g)$.

(2) If g' is not vacuous, there are two possibilities: (a) $N-g'$ may contain a connected set. In this case a contradiction results if a method of argument similar to that used in (1) be followed. (b) $N-g'$ may be totally disconnected. In this case N is a connected set which is totally disconnected by the omission of a point x, and by a finite set of points g' which is a subset of $N-x$. This is a contradiction of Theorem 3.

The supposition that there exists a point x of $g-g_1$ has therefore led to a contradiction. It follows, then, that $g=g_1$; i. e., that g is a primitive dispersion set of the set $M-(G-g)$.

Theorem 6. *If* G *is a finite primitive dispersion set of a connected set* M, P *a point of* G, *and the set* $M-P$ *is connected, then the set* $G-P=g$ *is a primitive dispersion set of the set* $M-P$.

Proof. Suppose g is not a primitive dispersion set of the set $M-P$. Then g contains a set g' which is a primitive dispersion set of $M-P$. Let x be a point of $g-g'$.

[1]) M and N being any two point-sets, $M\times N$ denotes the set of points common to M and N.

The set $M-(G-x)$ contains a connected set, N, which contains x. But the set $M-(G-x)$ is a subset of the set $M-P-g'$, which is totally disconnected. Thus the supposition that g is not a primitive dispersion set of $M-P$ leads to a contradiction.

Theorem 7. *Let M be a connected set which contains a primitive dispersion set G consisting of a finite number of points, and let H be any finite set of points of M, such that $G\times H=0$. Then $M-H$ is the sum of a finite number of mutually separated*[1]) *connected sets, each of which contains a primitive dispersion set consisting of those points of G that belong to it.*

Proof. Order the points of H in a sequence $P_1, P_2, \ldots, P_n$. Consider the set $M-P_1$. If the latter set is connected, G is a primitive dispersion set of it by Theorem 5. If $M-P_1$ is not connected, order the points of G in a sequence $x_1, x_2, \ldots x_k$, and consider the point-set

$$S=C_1+C_2+\ldots+C_k,$$

where for every value of i, $(i=1, 2, \ldots k)$, C_i is that composant of $M-P_1$ determined by x_i. (These composants may or may not be all distinct).

Every point of $M-P_1$ belongs to S. For suppose y is a point of $M-P_1$ that does not belong to S. By Theorem 2, $M-S$ is connected. That is, there exists a connected subset of M which must contain both y and P_1, but which contains no point of G. Clearly this is a contradiction of the hypothesis that G is a primitive dispersion set of M.

For no value of i is $x_i\equiv C_i$. For suppose such a situation existed. Since

$$M=S+P_1,$$

as shown above, x_i must be a limit point of S, and hence a limit point of at least one composant, C_j, say, which does not contain x_i. Then must

$$C_i\equiv C_j,$$

and since $x_i \not\equiv C_j$ a contradiction results.

[1]) Let C be any collection of point-sets. Then the point-sets of C are said to be *mutually exclusive* if no one of them contains a point in common with any other. The point-sets of C are said to be *mutually separated* if they are mutually exclusive and no one of them contains a limit point of any other.

For every i, let

$$G_i = G \times C_i.$$

Then G_i is a primitive dispersion set of C_i. For suppose not. Since G_i is necessarily a dispersion set of C_i, it contains a primitive dispersion set, g_i of C_i. Let t be a point of $G_i - g_i$. The set $M-(G-t)$ contains a connected set N, which contains t by virtue of the hypothesis that G is a primitive dispersion set of M. Now N cannot be a subset of C_i, as it contains no points of g_i. It therefore contains points of $M-C_i$, and it follows that N contains P_1, and that P_1 disconnects N. If Q is that set of points of N that belong to C_i, it follows that $Q+P_1$ is a connected set. Q cannot be connected or contain a connected set, since it is a subset of $C_i - g_i$. Hence P_1 is a dispersion set of $Q+P_1$. But t is itself a dispersion set of $Q+P_1$. This is impossible, by Theorem 3. Hence the supposition that G_i is not a primitive dispersion set of C_i leads to a contradiction.

In any case, then $M-P_1$ is the sum of a finite number of mutually separated connected sets, $K_1, K_2, \ldots, K_m$ each of which contains a primitive dispersion set consisting of that set of points that it has in common with G.

Let that composant of $M-P_1$ to which P_2 belongs be K_m. The sets $K_1, K_2, \ldots K_{m-1}$ are not, then affected by the omission of P_2 from the set $M-P_1$. It can be shown that the set $K_m - P_2$ is the sum of a finite number, $j+1$, of mutually separated sets, which we can denote by the symbols $K'_m, K_{m+1}, \ldots K_{m+j}$, each of which contains a primitive dispersion set consisting of that set of points that it has in common with G. Hence $M-P_1-P_2$ is the sum of $m+j$ mutually separated connected sets $K_1, K_2, \ldots K_{m-1}, K'_m, K_{m+1}, \ldots K_{m+j}$ each of which contains a primitive dispersion set consisting of that set of points that it has in common with G.

The remainder of the proof should be obvious.

Theorem 7. *Let* (1) *M be a connected set that contains a primitive dispersion set G consisting of a finite set of points $P_1, P_2, \ldots P_n$ $(n>1)$;* (2) *G_1 be a subset of G such that $G_1 = P_1 + P_2 + \ldots + P_i$ $(1<i\leq n)$, $M - G_1 = M_1 + M_2$, where M_1 and M_2 are mutually separated sets, and such that for every point P of G_1, M_1+P and M_2+P are connected sets. Then there exist two sets H_1 and H_2, (where H_2 may be vacuous) such that $H_1+H_2=G_1$, and such that M_1+H_1*

is a connected set having the set $G \times (M_1 + H_1)$ *as a primitive dispersion set, and* $M_2 + H_2$ *is the sum of a finite number of mutually separated connected sets each of which contains a primitive dispersion set consisting of the set of all points of* G *that belongs to it.*

Proof. I shall divide the proof into two cases.

Case (a). Neither $G \times M_1$ nor $G \times M_2$ vacuous.

As a result of the hypothesis, the sets $M_1 + P_1$ and $M_2 + P_3 + \ldots + P_i$ are connected

$$G \times (M_1 + P_1) = g.$$

Either (1) g is a primitive dispersion set of the set $M_1 + P_1$, or (2) it is not.

(1) If g is a primitive dispersion set of $M_1 + P_1$, consider the set $M_2 + P_2 + \ldots + P_i$. For simplicity of notation, let

$$M_2 + P_2 + \ldots + P = K,$$

and let

$$G \times K = g$$

If g' is a primitive dispersion set of K, the theorem is proved. If g' is not a primitive dispersion set of K, it contains, by Theorem 1, a set g_2 which is a primitive despersion set of K.

Now g_2 contains all points of the set $G \times M_2$. For suppose P is a point of $G \times M_2$ not belonging to g_2. Let

$$g_2' = g' - P.$$

Then g_2' is a dispersion set of the set K, as it contains g_2. The set $M - (G - P)$ is totally disconnected, since

$$M - (G - P) = (M_1 - g) + (M_2 - g_2'),$$

and the sum of two mutually separated totally disconnected sets is totally disconnected. But this is a contradiction of the hypothesis that G is a primitive disper*on set of M. Hence g_2 must contain all points of $G \times M_2$.

Let

$$g' - g_2 = Q.$$

Then Q is a subset of G_1, and the set $M_1 + P_1 + Q$ is connected. Also let

$$g + Q = g_1$$

and

$$M_1 + P_1 + Q = L.$$

I shall show that g_1 is a primitive dispersion set of L.

Suppose that g_1 is not a primitive dispersion set of L. Then by Theorem 1 it contains a primitive dispersion set, g_1' which can be shown to contain all points of the set $G \times M_1$. Also, g_1' contains P_1. For if not, the set $M_1 + P_1$ is totally disconnected by the omission of the set $G \times M_1$, which is a contradiction of the case under consideration namely, that g is a primitive dispersion set of $M_1 + P_1$. Hence g_1' contains g, and the set $g_1 - g_1'$ is a subset of Q.

Let x be any $(g_1 - g_2')$. By hypothesis the set $M-(G-x)$ is not totally disconnected. It contains therefore a connected set N, which must contain x. The set N must also contain a connected subset which belongs to one of the sets M_1+x, M_2+x. To show this, let

$$N_1 = N \times M_1,$$

and

$$N_2 = N \times M_2.$$

If one of these sets, say N_1, is vacuous, clearly N_2+x $(=N)$ is connected. If neither of them is vacuous, then both N_1+x and N_2+x are connected. For

$$N-x = N_1 + N_2,$$

where N_1 and N_2 are mutually separated sets. If N_2+x, say, is not connected,

$$N_2 + x = R_1 + R_2,$$

where R_1 and R_2 are mutually separated, and R_2 contains x. Then

$$N = R_1 + (R_2 + N_1),$$

where R_1 and R_2+N_1 are mutually separated. But this is a contradiction of the fact that N is connected. Hence N_2+x is connected, and similarly, N_1+x is connected. In any case then, $M-(G-x)$ contains a connected subset which is a subset of M_1+x or of M_2+x.

Since $L-g_1'$ cannot contain any connected subset, it follows that $M-(G-x)$ contains a connected subset, R, which is a subset of M_2+x. But M_2+x is a subset of K, and $g_2 \times R = 0$. Hence R is a connected subset of $K-g_2$. But this is impossible, since g_2 is a dispersion set of K. Hence such a point as x cannot exist, and therefore

$$g_1 = g_1$$

That is, g_1 is a primitive dispersion set of L

If now we let

$$H_1 = P_1 + Q,$$

and

$$H_2 = G_1 - H_1,$$

the theorem is proved for Case (a) (1). For, if H_2 is non-vacuous, $M_2 + H_2$ is connected, and contains the primitive dispersion set $g_2 = G \times (M_2 + H_2)$. If H_2 is vacuous, then the set M_2 is by Theorem 7, the sum of a finite number of mutually separated connected sets, each of which has a primitive dispersion set consisting of all those points of g_2 that belong to it.

(2) (In handling this part of case (a) I shall not give all the details of the proof as they are so similar to those of (1)).

If g is not a primitive dispersion set of the set $M_1 + P_1$, let g' be a subset of g which is a primitive dispersion set of $M_1 + P_1$. It can be shown that g' contains all the points of the set $G \times M_1$. Hence

$$g - g' = P_1.$$

Consider the set $M_2 + G_1$. If the set

$$g_2' = G \times (M_2 + G_1)$$

is not a primitive dispersion set of $M_2 + G_1$, let g_2 be a subset of g_2 that is. Let

$$g_2' - g_2 = Q.$$

It can be shown that Q is a subset of G_1, and that the set

$$g_1 = g' + Q$$

is a primitive dispersion set of the set $M_1 + Q$.

Case (b). One, or both of the sets $G \times M_1$, $G \times M_2$, vacuous. Suppose $G \times M_1$ vacuous. Consider the set

$$K = M_2 + G_1.$$

Either G is a primitive dispersion set of the set K or it is not. If G is a primitive dispersion set of K, then, by Theorem 6, $G - P_1$ is a primitive dispersion set of $K - P_1$. If we let $H_1 = P_1$ and $H_2 = G_1 - P_1$, the theorem is proved.

If G is not a primitive dispersion set of K, it contains a primitive subset, g_2. It can be shown that g_2 contains all points of the set $G \times M_2$. Let

$$g_1 = G - g_2.$$

Then g_1 is a subset of G_1, and $M_1 + g_1$ is a connected set.

Now g_1 is a primitive dispersion set of the set M_1+g_1. For suppose not. Then g_1 contains a subset, g, which is a primitive dispersion set of M_1+g_1. Let x be a point of the set g_1-g. It can be shown that the set $M-(G-x)$ contains a connected subset, N, which is a subset of one of the sets M_1+x, $M_2-G\times M_2+x$. But M_1+x is a subset of the set M_1+g_1-g, which is totally disconnected; hence N cannot be a subset of M_1+x. Also, $M_2-G\times M_2+x$ is a subset of $K-g_2$, which is totally disconnected; hence N cannot be subset of $M_2-G\times M_2+x$. It follows that N cannot exist, and that x cannot exist. That is

$$g_1=g,$$

and g_1 is a primitive dispersion set of the set M_1+g_1. If we let $H_1=g_1$, and $H_2=g_2+G\times M_2$, the theorem is proved.

Theorem 9. *If M is a connected set, and G a primitive dispersion set of M consisting of a finite number of points $P_1, P_2, \dots P_n$, then M is the sum of n mutually exclusive connected point-sets, each of which contains one point of G.*

Proof. Clearly the theorem is true for $n=1$. I shall prove it for the general case by mathematical induction, assuming it true where the number of points of G is $2, 3, \dots n-1$.

Since $M-G$ is a totally disconnected set, it can be expressed as the sum of n mutually separated point-sets $L_1, L_2, \dots L_n$. Then either (1) for all values of i and j, $(i, j=1, 2, \dots n)$ L_i+P_j is a connected set, or (2) there exist two numbers h and k, $(1\leqq h\leqq n$, $1\leqq k\leqq n)$ such that L_h+P_k is not connected.

If (1) holds, the theorem is satisfied by the n connected sets $L_1+P_1, L_2+P_2, \dots L_n+P_n$.

If (2) holds, suppose, for simplicity, that $h=1$ and $k=1$; that is, that L_1+P_1 not a connected set.

Then

$$L_1+P_1=H_1+H_2,$$

where H_1 and H_2 are mutually separated sets, and H_2 contains P_1. It follows immediately that $M-(P_2+P_3+\dots+P_n)$ is not connected, since

$$M-(P_2+P_3+\dots+P_n)=H_1+(H_2+L_2+\dots+L_n),$$

and H_1 and $H_2+L_2\dots+L_n$ are mutually separated sets.

Let
$$H_2 + L_2 + \ldots + L_n = Q.$$

There are three possibilities to consider:

(a) There may exist a point of $G - P_1$, say P_2, such that $H_1 + P_2$ is not connected. Then
$$H_1 + P_2 = N_1 + N_2,$$
where N_1 and N_2 are mutually separated sets, and N_2 contains P_2. It follows that $M - (P_3 + P_4 + \ldots + P_n)$ is not a connected set, since it is the sum of the two mutually separated sets N_1 and $N_2 + Q$.

(b) There may exist a point of $G - P_1$, say P_2, such that $Q + P_2$ is not connected. As in (a), it follows that the set
$$M - (P_3 + P_4 + \ldots + P_n)$$
is not connected.

(c) If neither of the possibilities (a) and (b) occurs, apply Theorem 8. This theorem shows that M is the sum of a finite number (greater than one) of mutually exclusive connected sets, each of which contains a subset of G as a primitive dispersion set, and to which the present theorem, being true for $n = 1, 2, 3, \ldots, n-1$, applies.

It remains to proceed with the result of (a) and (b), viz., that $M - (P_3 + P_4 + \ldots + P_n)$ is not connected. Then
$$M - (P_3 + P_4 + \ldots + P_n) = A_1 + A_2,$$
where A_1 and A_2 are mutually separated sets.

There are three possibilities:

(d) There may exist a point of the set $G - (P_1 + P_2)$, say P_3, such that $A_1 + P_3$ is not connected. It can be shown, by an argument similar to that used in (a), that $M - (P_4 + \ldots + P_n)$ is not connected.

(e) There may exist a point of the set $G - (P_1 + P_2)$, say P_3, such that $A_2 + P_3$ is not connected. It follows that $M - (P_4 + \ldots + P_n)$ is not connected.

(f) If neither of the possibilities (d) and (e) occurs, apply Theorem 8 as in (c) above.

If now the possibilities occurring in the case where
$$M - (P_4 + \ldots + P_n)$$
is not connected, and so on, we arrive finally to the only remaining

case, viz., the set $M - P_n$ not connected. This case requires a special proof, since Theorem 8 cannot apply.

If $M - P_n$ is not connected, it follows that

$$M - P_n = N_1 + N_2,$$

where N_1 and N_2 are mutually separated sets.

For every value of $i (i=1, 2, \dots n-1)$, let K_i be the composant of $M - P_n$ determined by P_i. Some of these composants may be identical.

Let

$$M - P_n - (K_1 + K_2 + \dots + K_{n-1}) = S.$$

There are two cases to be considered:

(1) S a vacuous set.

In this case it is impossible that

$$K_1 = K_2 = \dots = K_{n-1},$$

since no connected subset of $M - P_n$ can have points in common with both N_1 and N_2. Hence each of the sets N_1 and N_2 contains at least one of the composants $K_1, K_2, \dots K_{n-1}$, and consequently at least one point of the set $G - P_n$.

The set $N_1 + P_n$ is connected. For if it is not connected,

$$N_1 + P_n = B_1 + B_2,$$

where B_1 and B_2 are mutually separated sets, and B_2 contains P_n. But since

$$M = B_1 + (B_2 + N_2)$$

and B_1 and $B_2 + N_2$ are mutually separated, it would follow that M is not connected. This is a contradiction of the hypothesis. Hence $N_1 + P_n$ is a connected set, and similarly $N_2 + P_n$ is a connected set.

Let

$$G_1 = G \times (N_1 + P_n),$$

and

$$G_2 = G \times (N_2 + P_n).$$

If G_1 and G_2 are primitive dispersion sets of $N_1 + P_n$ and $N_2 + P_n$, respectively, let the number of points in G_1 be f and in G_2 be m. Then $N_1 + P_n$ is the sum of f, and $N_2 + P_n$ the sum of m mutually exclusive connected sets, each of which contains a point of G. As those two of these $f + m$ sets that contain P_n may be combined to form one connected set, the result is $f+m-1=n$ connected sets satisfying the theorem.

But suppose G_1 is not a primitive dispersion set of N_1+P_n. Let g_1 be that proper subset of G_1 which is a primitive dispersion set of N_1+P_n. It can be shown that g_1 contains all points of G_1+P_n, and hence

$$G_1-g_1=P_n.$$

If the number of points in G_1 is f, then by application of Theorem 7, and of the fact that the present theorem is assumed true for $n=1, 2, \ldots, n-1$, N_1 is the sum of $f-1$ mutually exclusive connected sets each of which contains one and only one point of G_1-P_n. Furthermore, G_2 is a primitive dispersion set of N_2+P_n. For $M-(G-P_n)$ contains a connected set, R, which contains P_n. If

$$R\times N_1=R_1,$$

and

$$R\times N_2=R_2,$$

R_1+P_n is connected in the general sense. Since $N_1+P_n-g_1$ is totally disconnected, R_1 is vacuous. Then R_2+P_n is connected. If g_2 is a subset of G_2 and a primitive dispersion set of N_2+P_n, then g_2 must contain P_n, else $N_2+P_n-g_2$ would contain a connected set R_2+P_n. Since g_2 must also contain all points of G_2-P_n, it follows that $G_2=g_2$ and is a primitive dispersion set of N_2+P_n. If m is the number of points in G_2, N_2+P_n is the sum of m mutually exclusive connected sets each of which contains a point of G_2. The remainder of the proof for this case should be obvious.

(2) S non-vacuous.

I shall first prove that $S+P_n$ is a connected set.

Let

$$S\times N_1=S_1,$$

and

$$S\times N_2=S_2.$$

Then

$$S=S_1+S_2,$$

and S_1 and S_2 are mutually separated sets, if neither is vacuous.

The set S_1+P_n is connected (in the general sense). For suppose not. Then

$$S_1+P_n=R_1+R_2,$$

where R_1 and R_2 are mutually separated sets, and R_2 contains P_n.

Let $K_1, K_2, \ldots, K_s$ be those composants of the set $K_1, K_2, \ldots K_i, K_{i+1}, \ldots K_{n-1}$ that belong to N_1.

As R_1 is a subset of S_1, K_1 cannot contain any points of it. Hence

$$R_1 + K_1 = L_1^1 + L_2^1,$$

where L_1^1 and L_2^1 are mutually separated sets, and L_2^1 contains K_1. Likewise, as L_2^1 is a subset of S_1,

$$L_1^1 + K_2 = L_1^2 + L_2^2,$$

where L_1^2 and L_2^2 are mutually separated sets, and L_2^2 contains K_2.

Continuing in this way, we get a set L_1^s, such that L_1^s and $M - L_1^s$ are mutually separated sets a contradiction of the hypothesis that M is a connected set. It follows that $S_1 + P_n$ is a connected set.

It can be shown by a similar argument that the set $S_2 + P_n$ is connected (in the general sense). As the sum of two connected sets that have a point in common is connected, it follows that the set $S + P_n$ is connected.

Let

$$G \times K_i = G_i \qquad (i = 1, 2, \ldots n-1).$$

For every value of i, G_i is a primitive dispersion set of K_i. For suppose, for instance, that G_1 is not a primitive dispersion set of K_1. Let g_1 be that subset of G_1 which, by Theorem 1, is a primitive dispersion set of K_1. Then the set

$$M - P_n - g_1 - G_2 - G_3 - \ldots - G_{n-1} \text{ }^{1)}$$

is totally disconnected. For if it contains any connected subset N, then N could not lie wholly in the totally disconnected set $K_1 - g_1$, nor could it contain any points not belonging to K_1, as the latter set is a composant of $M - P_n$. Hence no such set as N can exist. But $G - (G_1 - g_1)$ cannot be a dispersion set of M, by hypothesis. Therefore the supposition that G_i is not a primitive dispersion set of K_i leads to a contradiction.

The remainder of the proof should be obvious.

The following theorem is a direct consequence of Theorem 9:

[1]) Since there is no loss of generality in so doing, I am supposing here that no two of the composants $K_1, K_2, \ldots K_{n-1}$ are identical.

Theorem 10. *If a connected set M contains a primitive dispersion set G which consists of more than one point, then M is the sum of two mutually exclusive connected sets.*

Proof. If x is a point of G, the set $M-(G-x)$ contains a connected set, N_1, which contains x. Let y be a point of G distinct from x. Then $M-(G-y)$ contains a connected subset, N_2, which contains y.

If N_1 and N_2 have no points in common, then N_2 is a subset of a composant C of $M-N_1$, and the set $M-C$ is connected, by Theorem 2. Then M is the sum of the two mutually exclusive connected sets, C and $M-C$.

If N_1 and N_2 have points in common, let

$$N = N_1 + N_2.$$

Then N is a connected subset of $M-(G-x-y)$, and the set $x+y$ is a primitive dispersion set of N. Hence, by Theorem 9, N is the sum of two mutually exclusive connected sets, L_1 and L_2. L_2 is a subset of some composant, C, of $M-L_1$, and the set $M-C$ is connected. Then M is the sum of the two mutually exclusive connected sets, C and $M-C$.