Concerning the common boundary of two domains.

By

R. L. Moore (Austin, Texas, U. S. A.).

In an article[1]) published in Volume 5 of this journal, J. R. Kline has shown that in order that a bounded continuum should be a simple closed curve it is necessary and sufficient that it should remain connected[2]) (in the weak sense) on the removal of any one of its connected proper subsets. In the same volume[3]) C. Kuratowski has shown that in order that a bounded continuum should be a simple closed curve it is necessary and sufficient that it should remain connected in the *strong* sense on the removal of any one of its connected proper subsets which is *closed*. In the present paper I will show, among other things, that if a bounded continuum has more than one prime part[4]) and no one of its prime parts separates the plane then in order that it should have just two complementary domains and be the complete boundary of each of them it is necessary and sufficient that it should remain connected in the *weak sense* on the removal of any one of its connected proper subsets which is *closed*.

[1]) Closed connected sets which remain connected upon the removal of certain connected subsets, pp. 3—10.

[2]) Two point sets are said to be *mutually separated* if they have no point in common and neither of them contains a limit point of the other one. A set of points is said to be connected (or connected in the *weak sense*) if it is not the sum of two mutually separated point sets. Cf. N. J. Lennes, Amer. Journ. of Math., vol. 33 (1911), pp. 287—326. A set of points M is said to be connected in the *strong sense* (or *strongly* connected) if every two points of M lie in some closed and connected subset of M.

[3]) Contribution à l'étude de continus de Jordan, pp. 112—122.

[4]) The continuum M is said to be *connected im kleinen* at the point P if P belongs to M and for every positive number e there exists a positive number d_{Pe} such that every point of M whose distance from P is less than d_{Pe} lies in a

Theorem 1. *If, in a plane S, D_1 and D_2 are two mutually exclusive domains and M, the boundary of D_1, is a subset of the boundary of D_2 then M is the complete boundary of some domain which contains D_2 but which has no point in common with D_1.*

This theorem may be established with the aid of an argument strictly analogous to that used in the proof of Theorem 3 on Page 258 of my paper Concerning continuous curves in the plane [1]).

Theorem 2. *If, in a plane S, the bounded continuum M is the boundary of each of two mutually exclusive domains, D_1, and D_2, then M contains no subcontinuum whose omission disconnects [2]) M.*

Proof. Suppose, on the contrary, that M contains a subcontinuum K such that $S-K$ is the sum of two mutually separated point sets, M_1 and M_2. Then M_1+K and M_2+K are bounded continua whose common part is the continuum K. But $(M_1+K)+(M_2+K)$ separates D_1 from D_2. It follows [3]) that either M_1+K or M_2+K separates D_1 from D_2. But this is contrary to a theorem of Mazurkiewicz's [4]) to the effect that, under the hypothesis of Theorem 2, no subcontinuum of M separates a point of D_1 from a point of D_2.

Theorem 3. *If, in a plane S, the bounded continuum M is the boundary of each of two mutually exclusive domains D_1 and D_2 and S is not disconnected ty the omission of any prime part of M, then $S-M=D_1+D_2$.*

connected subset of M which contains P and is of diameter less than e. A point P of M is said to be a *regular* point of M or an *irregular* point of M according as M is, or is not, connected im kleinen at the point P. Cf. Hans Hahn, Jahresbericht der Mathematische Vereinigung, vol. 23 (1914), p. 319. See also S. Mazurkiewicz, Fund. Math., vol. 1, pp. 166—209. Hahn introduced the notion *prime part of a continuum* in his article Über irreduzible Kontinua, Sitzungsberichte der Akademie der Wissenschaften in Wien, Mathem.-Naturw. Klasse, vol. 130 (1921), pp. 217—250. If P is a point of a continuum M then by the *prime part* K_P (of M) is meant the set of all points $[X]$ belonging to M such that, for every positive number e, there exists a finite set of irregular points (of M) $X_1, X_2, X_3, \dots X_n$, such that the distance between every two successive points in the sequence $P, X_1, X_2, X_3, \dots X_n, X$ is less than e.

1) Mathematische Zeitschrift, vol 15 (1922), pp. 254—260.

2) If K is a subset of the connected point set M, the omission of K is said to disconnect M if $M-K$ is the sum of two mutually separated point sets.

3) Cf. S. Janiszewski, Sur les coupures du plan faites par des continus, Prace mat.-fizyczne, tom XXVI, 1913.

4) S. Mazurkiewicz, Les continus plans non bornés, Fund. Math., vol. 5, (1924), p. 193. Lemme 1.

Proof. If a simple closed curve is defined [1]) as a bounded continuum which is disconnected by the omission of any two of its points the above mentioned theorem of Kuratowski's holds true, not only in ordinary space, but in any space satisfying Axioms 1 and 4 and Theorem 4 of my paper On the foundations of plane analysis situs [2]). But, in my paper On the prime parts of a continuum [3]) I have shown that if M is a bounded continuum in a plane S and the word „point", as used in F. A., is interpreted to mean „prime part of M" and $\overline{S}$ denotes the set of all such „points" and the word „region", as used in F. A., is interpreted to mean a certain sort of collection of „points", then, in the space $\overline{S}$, Axioms 1 and 4 and Theorem 4 of F. A. all hold true. Furthermore, if $\overline{K}$ is a continuum of „points" of $\overline{S}$ then point set obtained by adding together the points (in the ordinary sense) of all the „points" (prime parts of M) that compose $\overline{K}$ is a continuum of ordinary points. Hence, by Theorem 2, if $\overline{S}-\overline{K}$ is not vacuous, $M-K$ is a connected set of points and therefore $\overline{S}-\overline{K}$ is a connected set of „points". But $\overline{K}$ is a closed set of „points". Hence $\overline{S}-\overline{K}$ is a „domain" in the sense of F. A. But it was proved in F. A. that every two „points" of a „domain" are the extremities of a „simple continuous arc" which lies wholly in that „domain". Thus every „domain" is *strongly* connected. Hence, by an extension, to the space $\overline{S}$, of the above mentioned theorem of Kuratowski's, $\overline{S}$ is disconnected by the omission of any two of its „points" and therefore M is disconnected by the omission of any two of its prime parts. But, in a paper recently submitted for publication in the Proceedings of the National Academy of Sciences [4]), I have shown that if a bounded continuum M has this property and M has more than one prime part and no one of its prime parts separates S, then $S-M$ is the sum of two mutually exclusive domains. Clearly, in the present case, these domains must be D_1 and D_2.

[1]) See my paper Concerning simple continuous curves, Trans. Amer. Math. Soc., vol, 21 (1920), pp. 333—347.

[2]) Trans. Amer. Math. Soc., vol. 17 (1916), pp. 131—164. This paper will be referred to as F. A.

[3]) This paper has been submitted for publication in Mathematische Zeitschrift. For a short abstract see Bull. Amer. Math. Soc., vol. 29 (1923), p. 438.

[4]) The title of this paper is Concerning the prime parts of certain continua which separate the plane.

Theorem 3 does not remain true on the removal of the stipulation that no prime part of M shall separate S. Indeed, for any preassigned positive integer n, an example, modelled largely after one designed for another purpose by K. Yoneyama[1]), may be given of n mutually exclusive bounded domains all having the same boundary.

Definition. A connected point set M which is connected im kleinen is said to be a continuous curve *relatively to* (or *with respect to*) the domain D if M is a subset of D and every limit point of M belongs either to M or to the boundary of D.

Theorem 4. *If M is a continuous curve with respect to some domain then every two points of M are the extremities of a simple continuous arc which is a subset of M.*

Theorem 4 may be proved by an argument similar to that employed, for the case of an ordinary curve, in my paper A theorem concerning continuous curves[2]).

Theorem 5. *If M is a continuous curve with respect to some domain then every open*[3]) *subset of M which is connected in the weak sense is also connected in the strong sense and, indeed, if K is an open subset of M and A and B are two points which lie in a subset of K which is connected in the weak sense, then A and B can be joined by a simple continuous arc which lies wholly in K.*

Theorem 5 may be proved by an argument strictly analogous to that used in the proof of Theorem 1 on Page 255 of my paper Concerning continuous curves in the plane[4]).

Theorem 6. *If the closed point set B is a proper subset of the continuum M and $M-B$ is connected and C is a circle whose interior I contains a point Y belonging to $M-B$ and M is connected im kleinen at every one of its points which belongs to I, and K_Y denotes the greatest connected subset of $M-B$ which contains Y and lies within C then K_Y is a continuous curve relatively to D_Y, that complementary domain of $B+C$ which contains Y; and, if $(M-B)-K_Y$ is not vacuous, C contains at least one limit point of K_Y; and, furthermore, no point of K_Y is a limit point of $M-K_Y$.*

1) Tohoku Mathematical Journal, vol. 12 (1917), pp. 60—62.

2) Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 233—236. Cf. also S. Mazurkiewicz, Fund. Math., vol. 1. pp. 166—209.

3) The point set K is said to be an open subset of the point set M if K is a subset of M and $M-K$ is either vacuous or closed.

4) Loc. cit.

Proof. Clearly K_Y contains all of its limit points which do not belong to the boundary of D_Y.

That K_Y is connected im kleinen may be proved as follows. Let X denote any point of K_Y and let C_1 denote a circle with center at X. Let C_2 denote a circle with center at X and such that the interior of C_2 is a subset both of D_Y and of the interior of C_1. Since M is connected im kleinen at the point X there exists a circle C_3 with center at X such that every point of M which lies within C_3 can be joined to X by a connected subset of M which lies within C_2. Let P denote a point of M which lies wihin C_3. There exists, within C_2, a connected point set T which contains X and P and is a subset of M. Since the connected point sets T and K_Y have the point X in common, the set $T+K_Y$ is a *connected* subset both of D_Y and of $M-B$. Hence T is a subset of K_Y. It follows that K_Y is connected im kleinen at the point X, and also that no point of K_Y is a limit point of $M-K_Y$.

Finally, if $(M-B)-K_Y$ is not vacuous, C contains at least one limit point of K_Y. For if this were not the case then the two point sets $(M-B)-K_Y$ and K_y would be mutually separated, contrary to the hypothesis that $M-B$ is connected.

Theorem 7. *If, in a plane S, M is a bounded continuum which has more than one prime part and no one of its prime parts separates S, then in order that $S-M$ should be the sum of two mutually exclusive domains such that M is the complete boundary of each of them it is necessary and sufficient that M should contain no subcontinuum whose omission disconnects M.*

The necessity of this condition has been proved under Theorem 2. That it is sufficient may be proved as follows. By hypothesis M contains no subcontinuum whose omission disconnects M. Hence, by an argument which forms a part of the above proof of Theorem 3, $S-M$ is the sum of two mutually exclusive domains. One of these domains is bounded and the other one is unbounded. Let D_1 denote the former and D_2 the latter. Let M_1 and M_2 denote the boundaries of D_1 and D_2 respectively. Suppose that $M-M_1$ is not vacuous. By a theorem of Brouwer's [1]) M_1 is a continuum. Hence $M-M_1$ is connected. It follows that U, the set of points composed of $M-M_1$ together with all of its limit points, is a con-

[1]) Math. Ann., vol. 69.

tinuum. Let K denote the set of points common to M_1 and U. There are two cases.

Case 1. Suppose that K is a proper subset of M_1. Then if K were connected $M-K$ would be connected. But $M-K$ is the sum of two mutually separated point sets $M-M_1$ and M_1-K. It follows that K is not connected. Thus the common part of the two continua M_1+D_1 and U is not connected. It follows that the sum of these continua separates S[1]. But their sum is D_1+M and $S-(D_1+M)$ is the connected point set D_2. Thus the supposition that K is a proper subset of M_1 has led to a contradiction.

Case 2. Suppose that K is identical with M_1. Every point of M belongs either to M_1 or to M_2. For otherwise M would contain a point P which is a limit point neither of D_1 nor of D_2 and if $\overline{P}$ is a point of M distinct from P there would exist a circle, with center at P, which lies wholly in M but neither contains nor encloses $\overline{P}$ and clearly the omission of such a circle would disconnect M, contrary to hypothesis. Let B denote the outer[2] boundary of D_1 and let R_1 and R_2 denote those complementary domains of B which contain D_1 and D_2 respectively. Suppose that M_1-B is not vacuous. Clearly R_1 contains M_1-B. If M_2-B were not vacuous it would be a subset of R_2 and $M-B$ would be the sum of the two mutually separated point sets M_1-B and M_2-B, which is impossible, since, B is a subcontinuum of M. Hence M_2 is identical with B. Therefore $M=M_1$, contrary to the supposition that $M-M_1$ is not vacuous. Thus the supposition that $M-B$ is not vacuous leads to a contradiction. Hence $M_1=B$ and therefore $M=M_2$. Since, by hypothesis, no prime part of M separates S it is clear that B contains a point P which is not a limit point of the set of all irregular points of M. There exists a circle C with center at P such that every point of M which lies within C is a regular point of M, but such that there is at least one point of $M-B$ which lies without C. Let K denote the greatest connected subset of M which contains P and lies within C. With the use of the fact that every point of K is a regular point of M it is easy to see that every point of K is a regular point of K and that no

[1]) Cf. Janiszewski, loc. cit.

[2]) The *outer* boundary of a bounded domain D is the boundary of that complementary domain of the boundary of D which is unbounded.

point of K is a limit point of $M-K$. By Theorem 4 every two points of K are the extremities of a simple continuous arc which lies wholly in K. Let H denote the greatest connected subset of B which contains P and lies within C. Since every point of B is a limit point of $M-B$ and no point of H is a limit point of $M-K$, every point of H is a limit point of the point set T which consists of all those points of $M-B$ which belong to K. Let C_1 and C_2 denote circles concentric with C and lying within C and such that C_2 lies within C_1. Let X denote any point of H which lies within C_2 and let C_X denote any circle with center at X and lying within C_2. Since X is a regular point of K there exists, within C_X, a concentric circle $\overline{C}_X$ such that every point of K within C_X can be joined to X by a simple continuous arc which is a subset of K and which lies wholly within C_X. But there exists, within the circle C_X, a point Y belonging to the set T. The point set K contains an arc YX which lies wholly within C_X. Let Z denote the first [1]) point which the arc YX has in common with H. Let YZ denote that interval of the arc YX whose endpoints are Y and Z. Let K_Y denote the greatest connected subset of $M-B$ which contains Y. By Theorem 6, K_Y is a continuous curve with respect to the domain D_Y, that complementary domain of $B+C$ which contains Y, and furthermore, C contains a limit point of K_Y. Hence there exists a point W which belongs to K_Y and lies without the circle C_1 and by Theorem 4, K_Y contains a simple continuous arc WY. The point set composed of the arcs WY and YZ contains a simple continuous arc WZ. Thus H contains a subset $\overline{H}$ such that every point of H which lies within C_2 is a limit point of H and every point Z of $\overline{H}$ can be joined to some point without C_1 by a simple continuous arc which, except for the point Z, is a subset of $K-H$. There exists an infinite sequence of distinct points $Z_1, Z_2, Z_3, \ldots$, all belonging to $\overline{H}$ and lying within C_2. For each n there exists an arc $W_n Z_n$ which lies wholly in $K-H$ except for the point Z_n, the point W_n being without C_1. Suppose that two of these arcs, $W_n Z_n$ and $W_k Z_k$ have a point in common. Then their sum contains a simple continuous arc $Z_n Z_k$, with endpoints at Z_n and Z_k. Except for its endpoints,

[1]) That such a first point exists may be seen with aid of the fact that H contains all of its limit points which are within C.

which belong to B, this arc is a subset of $M-B$ and therefore of R_2. But R_2 is a simply connected domain. It follows [1]) that the point set $R_2-[Z_h\ Z_k-(Z_h+Z_k)]$ is the sum of two mutually exclusive domains and therefore that M has more than two complementary domains. But M has only two complementary domains, D_1 and D_2. Thus the supposition that there exist two arcs of the sequence $W_1\ Z_1$, $W_2\ Z_2$, $W_3\ Z_3, \ldots$ which have a point in common leads to a contradiction. But, for every n, the arc $W_n\ Z_n$ contains an interval $\overline{W}_n\ \overline{Z}_n$ which lies entirely between the circles C_1 and C_2, except for its endpoints, $\overline{W}_n$ and $\overline{Z}_n$, which lie on C_1 and C_2 respectively. Let N denote the point set consisting of the circles C_1 and C_2 together with all those points which lie between C_1 and C_2, that is to say, without C_2 but within C_1. Suppose that M contains a connected subset L which lies wholly in N and contains two points P_h and P_k which belong respectively to $\overline{W}_h\ \overline{Z}_h$ and to $\overline{W}_k\ \overline{Z}_k$, where h and k are distinct positive integers. The point set K is a continuous curve with respect to the interior of C and the set of points $K-(Z_h+Z_k)$ is an open subset of K. Since the points P_h and P_k lie in the connected subset L of the set $K-(Z_h+Z_k)$, it follows, by Theorem 5, that they can be joined by a simple continuous arc $P_h\ P_k$ which lies wholly in $K-(Z_h+Z_k)$. The point set obtained by adding together the points of the arc $W_h\ Z_h$, $W_k\ Z_k$ and $P_h\ P_k$ clearly contains as a subset an arc which has its endpoints on B and which lies, except for its endpoints, wholly in $M-B$ and therefore in R_2. By an argument similar to one employed above this leads to a contradiction. Hence there is no connected subset of M which lies in N and contains points of two distinct arcs of the set $\overline{W}_1\ \overline{Z}_1$, $\overline{W}_2\ \overline{Z}_2, \ldots$ It follows [2]) that N contains

[1]) Cf. A. Rosenthal, Teilung der Ebene durch irreduzible Kontinua, Sitzungsberichte der Bayerischen Akademie der Wissenschaften zu München, Math.—Physik. Klasse, (1919), pp. 91—109.

[2]) It is easy to see that K would not be connected im kleinen at any point which lies between C_1 and C_2 and is a limit point of an infinite sequence of points F_1, F_2, $F_3, \ldots$ such that, for each n, F_n belongs to the arc $\overline{W}_n\ \overline{Z}_n$. Cf. the theorem of § 3 (p. 296 and 297) of my Report on continuous curves from the viewpoint of analysis situs, Bull. Amer. Math. Soc, vol. 29 (1923). This theorem remains true if what I call parts (3) and (4) are omitted. I take this opportunity of calling attention to the fact that (4) is not properly worded. The words „is common to $\overline{M}$ and $\overline{H}$" are to be replaced by „contains $\overline{M}$ and is a subset of H".

some points of K at which K is not connected im kleinen. Thus, in Case 2 as well as in Case 1, the supposition that $M - M_1$ is not vacuous has led to a contradiction. Hence M_1 is identical with M. With the aid of a similar argument and an inversion of the plane about a circle with center at some point in D_1, it may be shown that M_2 is identical with M. Therefore M is the complete boundary both of D_1 and of D_2.

Theorem 8. *Every cut point*[1] *of a bounded continuum is a cut point of the boundary of some complementary domain of that continuum.*

Proof. Suppose the point P is a cut point of the bounded continuum M. Then $M - P$ is the sum of two mutually separated point sets H and N. Let E and F denote points belonging to H and N respectively and let α denote a circle which encloses M. By a theorem of Knaster and Kuratowski's[2] there exists a continuum K which separates E from F but which contains no point of $\alpha + H + N$. Clearly K is bounded. Let B_E denote the boundary of that complementary domain of K which contains E and let B denote the boundary of that complementary domain of B_E which contains F. The continuum B separates E from F and contains no point of $H + N$. But $(H + N) + P$ is a continuum. It follows that B contains P. The point set $B - P$ is[3] connected. Hence it is a subset of some complementary domain D of the continuum M. Let D_E and D_F denote those complementary domains of B which contain E and F respectively. The point set $B - P$ contains a point X which can be joined to E by an arc XE which lies, except for the point X, entirely in D_E. Let Y denote the first point of M on the arc XE in the order from X to E. The interval XY of the arc XE has, in common with M, only the point Y. It follows that Y belongs to C, the boundary of the domain D. Thus $C - P$ contains a point of D_E. In a similar way it may be shown that it contains a point of D_F. Hence if $C - P$ were connected it would contain a point of B. But this is impossible. Therefore $C - P$ is not connected. In other words, P is a cut point of the boundary of D.

[1] A *cut point* of a connected point set M is a point of M whose omission disconnects M.

[2] B. Knaster et C. Kuratowski, Sur les ensembles connexes, Fund. Math., vol. 2 (1921), p. 233.

[3] See my paper Concerning the sum of a countable infinity of continua in the plane, Fund. Math., vol. 6.

Corollary. *In order that a bounded continuum should have no cut point it is sufficient that the boundary of each of its complementary domains should be a simple closed curve.*

Theorem 9. *In order that a bounded continuous curve should have no cut point it is necessary and sufficient that the boundary of each of its complementary domains should be a simple closed curve.*

Proof. The condition of Theorem 8 is sufficient according to the above corollary. I will show that it is necessary. Suppose that K is the boundary of a complementary domain D of a continuous curve M which has no cut point. Suppose first that D is bounded. Then the outer boundary of D is[1]) a simple closed curve J. Let I denote the interior of J. Suppose that J is not identical with K. Then D is a proper subset of I. Let X denote a point of $I-D$. If A is a point of J, M contains a simple continuous arc XA, with endpoints at X and A. Let $\overline{A}$ denote the first point of J which lies on the arc XA in the order from X to A. Let $X\overline{A}$ denote that interval of the arc XA whose endpoints are X and $\overline{A}$. Let B denote some point of J distinct from $\overline{A}$. By hypothesis $M-\overline{A}$ is connected. Hence, since M is a continuous curve, it contains[2]) a simple continuous arc from X to B which does not contain $\overline{A}$. This arc contains as a subset an arc $X\overline{B}$ which has, in common with J, only the point $\overline{B}$. The point set $X\overline{A}+X\overline{B}$ contains as a subset an arc $\overline{A}Y\overline{B}$ which lies, except for its end points, entirely in I. The point set J is the sum of two arcs $\overline{A}Z\overline{B}$ and $\overline{A}W\overline{B}$ which have only their endpoints in common. Let I_z denote the interior of the simple closed curve formed by the arcs $\overline{A}Y\overline{B}$ and $\overline{A}Z\overline{B}$ and let I_w denote the interior of the one formed by the arcs $\overline{A}Y\overline{B}$ and $\overline{A}W\overline{B}$. We have

$$I=I_z+I_w+\overline{A}Y\overline{B}-(\overline{A}+\overline{B})$$

Now Z is a limit point of D but not of I_w or of $\overline{A}Y\overline{B}$ and D is a subset of I. It follows that D contains points of I_z. For a similar reason it contains points of I_w. Furthermore it contains no point of $\overline{A}Y\overline{B}$ and therefore it is a subset of I_z+I_w. But I_z and I_w are

[1]) See my paper Concerning continuous curves in the plane, loc. cit.

[2]) R. L. Moore, loc. cit.

mutually separated. It follows that D is not connected. Thus the supposition that J is not identical with K leads to a contradiction. It follows that K, the boundary of D, is the simple closed curve J. With the help of an inversion the case where D is unbounded may be reduced to the case where it is bounded.

Corollary. *If the boundary of a simply connected domain is a continuous curve then in order that it should be a simple closed urve it is necessary and sufficient that it should have no cut point.*