Concerning the sum of a countable number of mutually exclusive continua in the plane [1]).

By

R. L. M o o r e (Austin, Texas, U. S. A.).

In 1918 S i e r p i ń s k i [2]) showed that if the sum of a countably infinite collection of closed point sets is bounded then it is not a continuum. He raised the question whether this theorem remains true if the restriction that the sum should be bounded is removed from the hypothesis. It will be shown, in the present paper, that, *for the case where each point set of the collection in question is itself a continuum,* this question may be answered in the affirmative. I will first establish some auxiliary theorems.

Theorem 1. *If the domains [3]) D_1 and D_2 have no point in common and the boundary of D_1 is a bounded subset of the boundary of D_2 then the boundary of D_1 is connected.*

Proof. Suppose, on the contrary, that B_1, the boundary of D_1, is the sum of two mutually separated [4]) point sets H and K. Each point of H can be enclosed within a circle which neither contains nor encloses any point of K. Since H is closed and bounded it follows, by the Heine-Borel-Lebesgue Theorem, that there exists a finite

---

[1]) *Remarque de la Rédaction.* Cet ouvrage contient des résultats que M. M o o r e a trouvés indépendamment de M. M a z u r k i e w i c z, qui a obtenu les principaux théorèmes de cet ouvrage dans une note du vol. V des *Fund. Math.* (p. 188—205).

[2]) W. S i e r p i ń s k i, Un théorème sur les continus, *Tôhoku Mathematical Journal,* vol. 13, N⁰ 4, June, 1918, pp. 300—303.

[3]) A domain is a connected point set whose complement is closed.

[4]) Two point-sets H and K are said to be mutually separated if neither of them contains a point or a limit point of the other one.

set G of circles such that (a) each point of H is within some circle of the set G, (b) no circle of G either contains or encloses any point of K. Let L denote the point set obtained by adding together the interiors of all the circles of the set G. Let P denote some point of H and let L_P denote the greatest connected subset of L which contains P. It is clear that L_P is a domain and that its boundary T contains no point of B_1. Let Q denote some point of K. Let W denote that complementary domain of $L_P + T$ which contains Q. By a theorem of Brouwer's [1], B_3 the boundary of W is connected. Both P and Q are limit points both of D_1 and of D_2. But Q is in W and P is without W. Hence both D_1 and D_2 contain points in W and points without W and, therefore, since they are connected, each of them contains a point of B_3, the boundary of W. Thus the connected point set B_3 contains at least one point of D_1 and at least one point which does not belong to D_1. It follows that B_3 contains a point of B_1. But B_3 is a subset of T. Hence T contains a point of B_1. Thus the supposition that B_1 is not connected leads to a contradiction.

Theorem 2. *If the domains D_1 and D_2 have no point in common and B, the boundary of D_1, is a bounded subset of the boundary of D_2 and O is any point of B, then $B — O$ is connected.*

Proof. Suppose, on the contrary, that $B — O$ is the sum of two mutually separated point sets B_1 and B_2. By Theorem 1, B is connected. It follows that $B_1 + O$ and $B_2 + O$ are bounded continua with only the point O in common. Furthermore, if A_1 and A_2 are points of D_1 and D_2 respectively then the sum of the point sets $B_1 + O$ and $B_2 + R$ separates A_1 from A_2. It follows [2] that either $B_1 + O$ or $B_2 + O$ separates A_1 from A_2. Suppose that $B_1 + O$ does and let

[1] L. E. J. Brouwer, Beweis des Jordanschen Kurvensatzes, *Mathematische Annalen*, vol. 69 (1910), p. 170.

[2] See Theorems A and B on Page 129 of an article by S. Straszewicz, *Fundamenta Mathematicae*, t. 4. These theorems are referred to by Straszewicz as having been established by Janiszewski in an article titled: Sur les coupures du plan faites par des continus, *Prace matem.-fizyczne*, tom XXVI, 1913. A proposition which is a logical consequence of these theorems of Janiszewski's has been recently established by Miss Anna M. Mullikin in her Doctor's dissertation, which will appear soon in the Transactions of the American Mathematical Society. This paper had gone to the printers before either Miss Mullikin or I was aware that the proposition had already been proved. Apparently Janiszewski's paper is printed in Polish.

K denote that complementary domain of $B_1 + O$ which contains A_1. Clearly K contains D_1. But every point of B_2 is a limit point of D_1. It follows that K contains B_2. But every point of B_2 is a limit point of D_2. Hence K contains at least one point of D_2. But D_2 is connected and it contains no point $B_1 + O$. It follows that K contains D_2. Therefore $B_1 + O$ does not separate A_1 from A_2. Thus the supposition that Theorem 2 is false leads to a contradiction.

Theorem 3. *The outer* [1]) *boundary of a bounded domain is connected and it is not disconnected by the ommision of any one of its points.*

Theorem 3 is a consequence of Theorem 2 and either Theorem 1 or the above mentioned theorem of Brouwer.

Theorem 4. *If G is a collection of point sets and M is the point set obtained by adding together all the point sets of the collection G and N is the sum of the boundaries of the point sets of the collection G and the point P is a limit point of M, then P either belongs to $M + N$ or is a limit point of N.*

Theorem 5. *If A and B are two distinct points and M is a closed point set which is the sum of a countable collection of mutually exclusive closed point sets $M_1, M_2, M_3, \ldots$ no one of which separates A from B, then M does not separate A from B.*

Proof. Suppose, on the contrary, that M does separate A from B. Let G denote the sequence of point sets $M_1, M_2, M_3, \ldots$ Let $M_{1,1}$ denote the point set obtained by adding to M_1 all of its complementary domains which do not contain A and B. Let M_2^* denote the set composed of all those points of M_2 that do not belong to $M_{1,1}$. Let $M_{2,2}$ denote the point set obtained by adding to M_2^* all of its complementary domains which do not contain A and B. Let $M_{1,2}$ denote the set of all those points of $M_{1,1}$ which do not belong to $M_{2,2}$. Let M_3^* denote the set of all those points of M_3 which do not belong to $M_{1,2} + M_{2,2}$. Let $M_{3,3}$ denote the point set obtained by adding to M_3^* all of its complementary domains which do not contain A and B. For each positive integer $i (1 \leqq i \leqq 2)$ let $M_{i,3}$ denote the set of all those points of $M_{i,2}$ which do not belong to $M_{3,3}$. This process may be continued. Thus we have a sequence $M_1^*, M_2^*, M_3^*, \ldots$ and an array:

---

[1]) If D is a bounded domain and B is its boundary and C is a circle which encloses both B and D, then the *outer boundary* of D is the boundary of that complementary domain of B which contains C.

$$M_{1,1}$$
$$M_{1,2}, \quad M_{2,2}$$
$$M_{1,3}, \quad M_{2,3}, \quad M_{3,3}$$
$$M_{1,4}, \quad M_{2,4}, \quad M_{3,4}, \quad M_{4,4}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

such that $M_1^* = M_1$ and such that, for every n, $M_{n,n}$ is the point set obtained by adding to M_n^* all of its complementary domains which do not contain A and B and M_{n+1}^* is the set of all those points of M_{n+1} which do not belong to $M_{1,n} + M_{2,n} + \cdots + M_{n,n}$ and, for every m and n, $(m \geqq n)$, $M_{n,m+1}$ is the set of all those points of $M_{n,n}$ which do not belong to $M_{m+1,m+1}$.

For every n let K_n denote the set of points common to all the point sets of the infinite sequence $M_{n,n}, M_{n,n+1}, M_{n,n+2}, \ldots$ Some of the point sets of the sequence $K_1, K_2, K_3, \ldots$ may be vacuous. But the boundary of that complementary domain of M which contains A is clearly a subset of $K_1 + K_2 + K_3 + \cdots$ It follows that $K_1 + K_2 + K_3 + \cdots$ separates A from B. But the point sets of the sequence $K_1, K_2, K_3, \ldots$ are mutually exclusive and clearly they are all closed and no one of them separates space. This contradicts a theorem of Miss Mullikin's [1]) to the effect that *if M is the sum of a countable number of closed, mutually exclusive point sets $M_1, M_2, M_3 \ldots$ no one of which disconnects a plane S, then M does not disconnect S.* The truth of Theorem 5 is therefore established.

Theorem 6. *If A, B and Q are three distinct points and M is a closed and bounded point set which is the sum of a countable collection G of closed point sets no one of which separates A from B and no two of which have, in common, any point except Q, then M does not separate A from B.*

Proof. Let $M_1, M_2, M_3, \ldots$ denote the point sets of the collection G. Subject the plane to an inversion about Q and let $\overline{A}, \overline{B}, \overline{M}, \overline{M_1}, \overline{M_2}, \ldots$ respectively denote the images of A, B, M, M_1, M_2, under this inversion. Let $\overline{G}$ denote the sequence of point sets $\overline{M_1}, \overline{M_2}, \ldots$ It is easy to see that no point set of the sequence G separates $\overline{A}$ from $\overline{B}$. But the point sets of this sequence are mutually

---

[1]) Cf. Theorem 3 of her thesis, loc. cit.

exclusive and, since $\overline{M}$ is closed and bounded, M is closed. Likewise, the point sets of $\overline{G}$ are all closed. It follows, by Theorem 5, that $\overline{M}$ does not separate $\overline{A}$ from $\overline{B}$. Hence $\overline{M} + O$ does not separate $\overline{A}$ from $\overline{B}$. Hence M does not separate A from B.

Theorm 7. *There do not exist two bounded domains D_1 and D_2 a closed and bounded point set K which does not separate every point of $D_1 + D_2$ from infinity* [1]) *such that each of the domains D_1 and D_2 contains every point of the boundary of the other one which does not belong K.*

Proof. Suppose, on the contrary, that there do exist two bounded domains D_1 and D_2, with boundaries B_1 and B_2 respectively, and a closed and bounded point set K and a point O belonging to $D_1 + D_2$ such that (1) K does not separate O from infinity, and (2) $(B_1 + K) - K$ is a subset of D_2 and $(B_2 + K) - K$ is a subset of D_1. Let C denote a circle which encloses $D_1 + D_2 + B_1 + B_2 + K$ and let X denote a point on C. There exists a simple continuous arc OX which contains no point of K. In the order from O to X on the arc OX, there exists a point Z which is the last point that this arc has in common with $B_1 + B_2$. Since Z belongs either to B_1 or to B_2, it therefore belongs either to D_2 or to D_1. Hence, since D_1 and D_2 are domains, the boundary of one of them contains a point which lies on OX between Z and X. Hence Z is not the last point of $B_1 + B_2$ on the arc OX. Thus the supposition that Theorem 7 is false leads to a contradiction.

Theorem 8. *If the boundary of a bounded domain D_2 is a subset of a simply connected bounded domain D_1 then D_2 is itself a subset of D_1.*

Proof Suppose, on the contrary, that D_2 is not a subset of D_1. Since the boundary of D_2 is a subset of D_1 there is at least one point of D_2 in D_1]. Thus D_2 contains at least one point of D_1 and at least one point which does not belong to D_1. But D_2 is connected. Hence it contains at least one point of B_1, the boundary of D_1. But, since D_1 is simply connected, B_1 is connected. Thus the connected point set B_1 contains a point of D_2 and therefore, since it contains no point of B_2, the boundary of D_2, B_1 must be a subset of D_2. But, by Theorem 7, this is impossible. Thus the supposition that Theorem 8 is false leads to a contradiction.

---

[1]) A bounded and closed point set K is said to separate the point P from infinity if P lies in a bounded complementary domain of K.

Theorem 9. *If D is a bounded domain, the boundary of D does not contain two closed subsets B_1 and B_2 such that (a) either B_1 and B_2 have no point in common or $B_1 \times B_2$[1]) is a point set which does not separate D from infinity, (b) each of the point sets B_1 and B_2 separates D from infinity.*

Proof. Suppose, on the contrary, that there exists a bounded domain D whose boundary contains two such closed subsets B_1 and B_2. Let D_1 and D_2 denote those complementary domains of B_1 and B_2 respectively which contain D. Let T_1 and T_2 respectively denote the boundaries of D_1 and D_2. Clearly $T_1 - T_1 \times T_2$ is a subset of D_2 and $T_2 - T_1 \times T_2$ is a subset of D_1, and $T_1 \times T_2$ does not separate D from infinity. But this is contrary to Theorem 7.

Theorem 10. *There do not exist, in a plane S, a bounded continuum M and point O such that $M - O$ is connected and such that M is the sum of a countable collection G of continua $M_1, M_2, M_3, \ldots$ each of which contains O, no two of which have, in common, any point except O, and no one of which separates the plane S or is itself disconnected by the omission of the point O.*

Proof. Suppose, on the contrary, that there exists such a set. Let C denote some definite circle which encloses M. A point set M_n of the set G will be said to be of class 1 if there exists an arc XY which contains only the point X in common with M and only the point Y in common with C, the point X being a point of M_n distinct from O. Under these conditions the arc XY will be said to join M_n simply to C. If a, b, c and d are four distinct sets of the collection G and a and c are simply joined to C by the mutually exclusive arcs $X_1 Y_1$ and $X_3 Y_3$ respectively and also by the mutually exclusive arcs $X_1' Y_1'$ and $X_3' Y_3'$ respectively and, furthermore, of the two domains into which the point set $(a + X_1 Y_1 + c + X_3 Y_3) - (Y_1 + Y_3)$ separates[2]) the interior of C, one contains $b - O$ and

---

[1]) If M and N are two point sets, with at least one point in common, then $M \times N$ denotes the set of all points which are common to M and N.

[2]) Since neither a nor $X_1 Y_1$ separates the plane S and they have only one point in common, therefore, by a theorem of Janiszewski's mentioned above, their sum does not separate S. Similarly, $c + X_3 Y_3$ does not separate S. But $a + X_1 Y_1$ and $c + X_3 Y_3$ have only O in common. Hence $a + X_1 Y_1 + c + X_3 Y_3$ does not separate S. The circle C is the sum of two arcs $Y_1 A Y_3$ and $Y_1 B Y_3$ which have, in common, only their end points Y_1 and Y_3. The point set $a + X Y + c + X Y + Y A Y$ has only one bounded complementary domain H_1 and $a + X_1 Y_1 + X_3 Y_3 + Y_1 B Y_3$ has only one bounded complementary domain H_2. (Cf. **A. Rosenthal**, Teilung der

the other contains $d—O$ and $X_1'Y_1'+X_3'Y_3'$ has no point in common with $b+d$, then of the two domains into which $(a+X_1'Y_1'+c+X_3'Y_3')—$ $—(Y_1'+Y_3')$ separates the interior of C, one contains $b—O$ and the other contains $d—O$. If a, b, c and d are four distinct point sets of the collection G and there exist two arcs X_1Y_1 and X_3Y_3 satisfying, with respect to a, b, c and d, the conditions stipulated above, then a and c are said to ordinally separate b from d[1]). It can be shown that (1) if a, b, c and d are four distinct point sets of class 1 then two of them ordinally separate the other two from each other and if a and c separate b from d then b and d separate a from c, but a and d do not separate b from c, (2) if a and c are distinct G-sets [2]) of class 1 and b is a G-set (either of class 1 or not) and there exists some G-set which is separated from b by a and c then there exists a G-set of class 1 which is separated from b by a and c.

With the aid of the fact that $M—O$ is connected it can be easily proved that there exist four distinct G-sets a_1, a_2, a_3 and a_4, all of class 1 and such that a_1 and a_3 are separated from each other by a_2 and a_4. Let us now confine our attention to the collection $\overline{G}$ consisting of a_1 and a_3 together with those G-sets which are separated from a_4 by a_1 and a_3. It can easily be seen that if x and y are two distinct $\overline{G}$-sets and x is of class 1 then y is separated either from a_1 by x and a_3 or from a_3 by x and a_1. In the first case y will be said to follow x. In the second case it will be said to precede x. It is clear that if x and y are two distinct $\overline{G}$-sets, both of class 1, then (a) either x follows y or y follows x, and (b) if x follows y then y precedes [3]) x and does not follow x.

Ebene durch irreduzible Kontinua, *Sitzungsber. der Math.-phys. Klasse der Bayerischen Akad. der Wiss. München*, 1919, p. 102, Theorem 6. For a more general result see a theorem on p. 130 of the above mentioned paper of Straszewicz. But see also Theorem 5 of Miss Mullikin's dissertation. This result of Miss Mullikin's is stated in an abstract on page 349 of The Bulletin of the American Mathematical Society, vol. 27, May 1921). It is easy to prove that if I denotes the interior of the circle C, then $I = a + X_1Y_1 + c + X_3Y_3 + H_1 + H_2$.

[1]) It is to be noted that this does not imply that b and d ordinally separate a from c. The point sets b and d may conceivably not be of class 1.

[2]) By a G-set is meant a point set of the collection G.

[3]) It is to be observed that no meaning is here attached to the statement that x precedes y in case y is not of class 1.

Also if x, y and z are $\overline{G}$-sets and y follows x and x follows z then y follows z. If x, y and z are $\overline{G}$-sets the statement that z is between x and y means that (a) x and y are of class 1, and (b) z either follows x and precedes y or follows y and precedes x. If x and y are two distinct $\overline{G}$-sets of class 1 then by the *segment* xy is meant the collection of all those $\overline{G}$-sets which are between x and y, while by the *interval* xy is meant the collection of $\overline{G}$-sets consisting of x and y and all the $\overline{G}$-sets of the segment xy. It is easy to see that if R is the point set obtained by adding together all the $\overline{G}$-sets of the segment xy, then no point of $R-O$ is a limit point of $M-R$. It is also clear that if a segment is not vacuous, that is to say if it contains at least one $\overline{G}$-set, then it contains at least one $\overline{G}$-set of class 1. There does not exist more than one pair of distinct $\overline{G}$-sets (x, y) such that x and y are both of class 1 and such that there is no G set between them. For suppose there are two such pairs (x, y) and (z, w). Let $\bar{x}$ and $\bar{y}$ denote x and y respectively or y and x respectively according as x precedes y or y precedes x. Let $\bar{z}$ and $\bar{w}$ denote z and w respectively or w and z respectively according as z precedes w or w precedes z.

The point sets $\bar{x}$ and $\bar{z}$ must be distinct. For suppose they were identical. Then, since $(\bar{x}, \bar{y})$ and $(\bar{z}, \bar{w})$ are distinct pairs, either $\bar{y}$ would precede $\bar{w}$ or $\bar{w}$ would precede $\bar{y}$. In the first case $\bar{y}$ would be between z and w and, in the second case, $\bar{w}$ would be between x and y, both of which are contrary to hypothesis. Since $\bar{x}$ and $\bar{z}$ are distinct one of them precedes the other one. If $\bar{x}$ precedes $\bar{z}$ let a and b denote $\bar{x}$ and $\bar{y}$ respectively and let c and d denote $\bar{z}$ and $\bar{w}$ respectively. If $\bar{z}$ precedes $\bar{x}$ let a and b denote $\bar{z}$ and $\bar{w}$ respectively and let c and d denote $\bar{x}$ and $\bar{y}$ respectively. Then b either precedes c or is identical with it. For otherwise c would be between a and b, contrary to hypothesis. Let t denote either the point set b or the set of points obtained by adding together all the point sets of the interval bc, according as b is or is not, identical with c. Then the point sets $t-O$ ane $M-t$ are mutually separated and therefore $M-O$ is not connected. Thus the supposition that there exists more than one vacuous segment leads to a contradiction. It follows that there exist two $\overline{G}$-sets a and b of class 1 such that a precedes b and such that if x and y are two $\overline{G}$-sets of class 1

and belonging to the interval ab then there is a $\overline{G}$-set of class 1 between x and y.

Let e denote a $\overline{G}$-set of class 1 which lies between a and b. Let x_1 denote the point set of lowest subscript in the sequence $M_1, M_2, M_3, \ldots$ which belongs to the segment ae, is of class 1 and is distinct from M_1. The intervals ax_1 an ex_1 do not both contain M_1. If ax_1 contains M_1 let y_1 and z_1 denote x_1 and e respectively. If ax_1 does not contain M_1 let y_1 and z_1 denote a and x_1 respectively. In either case y_1 precedes z_1, the interval $y_1 z_1$ is a subset of the interval ae and the interval $y_1 z_1$ does not contain M_1. Similarly, the interval $y_1 z_1$ contains two G-sets y_2 and z_2 of class 1 and such that y_2 precedes z_2 and the interval $y_2 z_2$ does not contain M_2. This process may be continued. Thus there exist two infinite sequences $y_1, y_2, y_3, \ldots$ and $z_1, z_2, z_3, \ldots$ of $\overline{G}$-sets of class 1, all belonging to the interval ae and such that, for every n, (1) y_n precedes z_n, (2) y_n either precedes, or is identical with, y_{n+1} and z_{n+1} either precedes or is identical with z_n (3) the interval $y_n z_n$ does not contain M_2. In a similar way it may be shown that there exist two sequences $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ of $\overline{G}$-sets of class 1, all belonging to the interval eb and such that, for every n, (1) u_n precedes v_n, (2) u_n either precedes or is identical with u_{n+1} and v_{n+1} either precedes or is identical with v_n, (3) the interval $u_n v_n$ does not contain M_n.

Let L denote the point set obtained by adding together all the intervals $z_1 u_1, z_2 u_2, z_3 u_3, \ldots$ The point sets $L - O$ and $L - M$ are mutually separated and therefore $M - O$ is not connected. Thus the supposition that Theorem 10 is false led to a contradiction.

Theorem 11. *There does not exist, in a plane S, an unbounded continuum which is the sum of a countable collection G of mutually exclusive continua $M_1, M_2, M_3, \ldots$*

Proof. Suppose, on the contrary, that there does exist such a continuum M. Each point set of the collection G is unbounded. For suppose that one of them (call it M_i) is bounded. Then there exists a cirle J which encloses M_1. Let X denote a point of M_1. Since the continuum M contains the point X within J and also contains a point without J, therefore M contains a sub-continuum $\overline{M}$ which contains X and at least one point on J, but no point without J. Since $\overline{M}$ contains a point on J it is not a subset of M_i

Let M_{i_1}, M_{i_2}, M_{i_3}, ... denote those point sets of the collection G which have points in common with $\overline{M}$. For each n, let $\overline{M}_{i_n}$ denote the set of points common to $\overline{M}$ and M_{i_n}. Since $\overline{M}$ and M_{i_n} are closed so is $\overline{M}_{i_n}$. Thus the *bounded* continuum $\overline{M}$ is the sum of the countable collection of closed point sets $\overline{M}_{i_1}$, $\overline{M}_{i_2}$, $\overline{M}_{i_3}$, ... But this is impossible according to Sierpiński's theorem. Thus the supposition that the point sets of the collection G are not all unbounded leads to a contradiction. Furthermore the set M does not fill up the whole plane. For suppose that it does. Let X denote a point of M_1 and let Y denote a point of M_2. Then the straight line interval XY is a bounded continuum which is the sum of a countable collection of closed point sets $\overline{M}_{i_1}$, $\overline{M}_{i_2}$, $\overline{M}_{i_3}$, ... where, for every n, $\overline{M}_{i_n}$ is the set of points common to the interval XY and the point set M_{i_n}. Thus again a contradiction is reached. Hence there exists, in plane S, a point O which does not belong to M. Let the plane S be subjected to an inversion about some circle with center at O. Let T denote the image of M and, for every n, let T_n denote the image of M_n, under this inversion. Let K denote $T+O$ and, for every n, let K_n denote T_n+O. Let $\overline{G}$ denote the collection of point sets K_1, K_2, K_3,... Clearly $K-O$ is connected and, for every n, K_n-O is connected. Now let c denote a circle which encloses K and has its center at O. One complementary domain of the point set K contains the circle c. Call this domain D_o. If any point set K_n of the collection $\overline{G}$ contains a subset B_n which is the complete boundary of any domain d (other than D_c) which is complementary to K then add all such domains d to the point set K_n and let K_n^* denote the point set constituted by the sum so obtained. In case K_n contains no such subset B_n then let K_n^* denote K_n. We now have a collection G^* of point sets K_1^*, K_2^*, K_3^*,... each of which either is identical, with, or contains, some point set of the collection $\overline{G}$. It is clear that the point sets of the collection G^* are all continua, that every two of them have in common only the point O, that their sum K^* is a bounded continuum lying wholly within the circle c and finally that K^*-O is connected and, for every n, K_n^*-O is connected. No domain, except D_c, which is complementary to K^* has, for its complete boundary, a subset of any one point set of the collection G^*. It is conceivable however that there may be a domain, other than D_c, which is complementary to *some point*

set of the collection G^* (and which is therefore bounded by a subset of that point set). If such a domain exists it must contain points of K^*.

C a s e 1. Suppose there is no domain, other than D_c, which is complementary to K^*. Suppose a point set K_n^* of the collection G^* separates the plane. Then K_n^* has a complementary domain H which contains no point of D_c. The domain H must be a subset of K^*. Let X denote a point of H. There exists a simple continuous arc XY which does contain O but lies wholly in H except for the point Y which lies on the boundary of H. Every point of the bounded continuum XY belongs to some point set of the countable collection G^* and XY contains the point Y of K_n^* and a point X which does not belong to K_n^*. This is contrary to Sierpiński's theorem. Thus the supposition that the plane is separated by any point set of the collection G^* leads to a contradiction. Hence, in Case 1, G^* is a countable collection of continua $K_1^*, K_2^*, K_3^*, \ldots$ such that (a) their sum K^* is a bounded continuum and $K^* - O$ is connected, (b) no two of them have, in common, any point except O, (c) no one of them is disconnected by the omission of O, (d) no one of them separates the plane. But this is contrary to Theorem 10. Thus in Case 1 we have a contradiction.

C a s e 2. Suppose now that there exists a domain D^*, other than D_c, which is complementary to K^*. The boundary k of this domain is not a subset of any one point set of the collection G^*. It follows that k is the sum of a collection g of two or more distinct closed point sets $k_1, k_2, \ldots$ each of which is a subset of some point set of the collection G and no two of which belong to the same point set of this collection. If k_n is a point set of the collection g, one complementary domain d_n of the point set k_n contains the domain D^*. By Theorem 9 there does not exist more than one n such that d_n fails to contain c. By Theorem 6 there does exist one such n. Call it $\bar{n}$. For no n does any complementary domain of k_n, except d_n, contain a point of k. The point set k is a continuum since it is the boundary of a complementary domain of the continuum K^*. That k contains O and that every k_n contains O can be shown by a method similar to that used above to show that every point set of the collection G is unbounded. The point set $k - O$ is connected. For suppose that it is not. Let w denote the outer boundary of D^*. Then $k - O$ is the sum of two mutually separated point sets k' and k'',

where k' contains at least one point of $w - O$. It is clear that w is a subset of k_n^-. By Theorem 3, the point set $w - O$ is connected. Hence $w - O$ is a subset of k'. The point set w is [1] the complete boundary of at least two domains D_1 and D_2 where D_1 contains c and is unbounded while D_2 contains D^* and is bounded. The domain D_2 contains every point of $k - O$ which does not belong to w. If it contains any points of K which do not belong to k, let Z denote the set of all such points. If X is a point of Z then X belongs to a complementary domain of k. Call this domain d_x and call its boundary k_x. Then k_x is a subset of k. By a theorem of Brouwer's mentioned above, k_x is connected and, by Theorem 2, if k_x contains O then $k_x - O$ is connected. Hence, in any case, the point set $(k_x + O) - O$ is a subset either of k' or of k''. Now let K' denote the point set obtained by adding together k', all points of $K - O$, if there are any, which do not belong to $w + D_2$, and all points X of Z such that $(k_x + O) - O$ is a subset of k'. Let K'' denote the set of points obtained by adding together k'' and all points X of Z such that $(k_x + O) - O$ is a subset of k''. With the help of Theorem 4 and the fact that k' and k'' are mutually separated, it is easy to prove that K' and K'' are mutually separated. But clearly $K' + K'' = K - O$. Hence $K - O$ is not connected. Thus the supposition that $k - O$ is not connected has led to a contradiction. Since $k - O$ is connected it easily follows that there are infinitely many point sets in the collection g. Thus $k_1, k_2, \ldots$ is an *infinite* sequence.

If, for any n, k_n has a complementary domain other than d_n then let k_n^* denote the point set obtained by adding to k_n all such complementary domains. If k_n has no complementary domain except d_n, let k_n^* denote k_n. Let g^* denote the countable collection of point sets $k_1, k_2, k_3, \ldots$ and let k^* denote their sum. The point sets of the collection g^* are all closed, no two of them have, in common, any point except O, their sum k^* is closed and connected and, with the aid of the fact that $k - O$ is connected, it can easily be seen that $k^* - O$ is connected. Furthermore, no point set of the collection g^* separates the plane.

If a point P is separated from D^* by a point set k_m of the

[1] Cf. my paper Concerning continuous curves in the plane *Mathematische Zeitschrift*, vol. 15 (1922), pp. 254—260.

collection g and P belongs to a point set of the collection G which contains some point set of the collection g, then P belongs to that point set of the collection G which contains k_m. For suppose that K_n is that point set of the collection G which contains k_m and suppose that P belongs to K_j, where $j \neq n$. Let V denote that complementary domain of k_m to which P belongs. Suppose first that $K_j - O$ is not a subset of V. Then clearly $K_j - O$ is the sum of the mutually separated sets L_1 and L_2, where L_1 consists of all those points of $K_j - O$ which belong to V and L_2 consists of all those which do not belong to V. Hence $K_j - O$ is not connected. Thus, in case $K_j - O$ is not a subset of V, a contradiction is reached. Suppose secondly that $K_j - O$ is a subset of V. Then, since by hypothesis K_j contains some point set of the collection g, therefore V contains a point Y which belongs to k. But Y is a limit point of D^*. Hence V contains a point of D^*, contrary to the supposition that k_m separates P from D^*. Thus again a contradiction is reached. It follows that P belongs to K_n. With the help of this fact it can be shown that, for every n, $k_n^* - O$ is connected. For suppose, on the contrary, that there exists a positive integer m such that $k_m^* - O$ is the sum of two mutually separated point sets H_1 and H_2. Let K_n denote that point set of the collection G which contains k_m. If B is the boundary of a complementary domain of k_m which contains no point of D^*, then, by Theorems 1 and 2, $(B + O) - O$ is connected. Hence $(B + O) - O$ is a subset either of H_1 or of H_2. For each $i (1 \leqq i \leqq 2)$ let H_i^* denote the point set obtained by adding to H_i every point of $K_n - O$ which lies in a complementary domain of k_m which contains no point of D^* and whose boundary is, except for the point O, a subset of H_i. Every point of $K_n - O$ belongs either to H_1^* or to H_2^*. For suppose this is not the case. Then d_m, that complementary domain of k_m which contains D^*, contains a point P belonging to $K_n - O$. Since the point P belongs neither to D^* nor to its boundary k, it is separated from D^* by k. It follows, with the help of Theorem 6, that P is separated from D^* by some point set k_i of the collection g. Clearly k_i must be distinct from k_m. But it has been shown above that this is impossible. It follows that $K_n - O$ is the sum of the two point sets H_1^* and H_2^*. But clearly these two point sets are mutually separated. Hence $K_n - O$ is not connected. Thus the supposition that $k_m^* - O$ is not connected has led to a contradiction.

Now let A denote some definite point of D^* and subject the plane S to an inversion about the point A. Let $\bar{k}$ and $\bar{k_{\bar{n}}}$ denote the point sets obtained by adding A to the images of k^* and $k^*_{\bar{n}}$ respectively and, for every n, except $\bar{n}$, let $\bar{k}_n$ denote the image of k^*_n and let $\overline{O}$ denote the image of O, under this inversion. Then $\bar{k}$ is a bounded continuum, $\bar{k}-\overline{O}$ is connected, and $\bar{k}$ is the sum of the countable collection of continua $\bar{k_1}, \bar{k_2},...$ Furthermore, every one of these continua contains $\overline{O}$, no two of them have in common, any point except $\overline{O}$, no one of them separates the plane S and no one of them is disconnected by the omission of the point O. But this is contrary to Theorem 10. Thus in Case 2 we have a contradiction.

The supposition that Theorem 11 is false has led to a contradiction, both in Case 1 and in Case 2. The truth of this theorem is therefore established.

———————