On continuous functions, commuting functions, and fixed points

by

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It has been conjectured that if two continuous functions f and g map the interval $[0,1]$ into itself and if they commute, i.e. $f(g(x)) = g(f(x))$ for all x in $[0,1]$, then the two functions possess a common fixed point. The conjecture is trivially verified if one of the two functions, say f, has the property that it has a fixed point in any non-empty closed subset G (of $[0,1]$) which is mapped into itself by f, or if f has the property that the repeated application of f to any point x in the interval produces a convergent sequence. However, these results have the disadvantage that, in practice, given any particular f, it is difficult to verify whether f has either of these properties.

The purpose of this paper is to present a theorem which, we feel, is of interest in itself and in which it is shown that the properties mentioned above are equivalent to each other and to several other properties which are more verifiable.

First, let $f^1(x) = f(x)$ and $f^k(x) = f(f^{k-1}(x))$, $k = 2, 3, \ldots$ and $x \in [0,1]$. We then have:

THEOREM 1. *Let f be a continuous mapping of the interval $[a,b]$ into itself. Then the following conditions are equivalent:*

(i)-(a) *for each $x \in [a,b]$ such that $f(x) \neq x$, we have $f^2(x) \neq x$;*

(b) *for each $x \in [a,b]$ such that $f(x) > x$, we have $f^2(x) > x$, and for each $x \in [a,b]$ such that $f(x) < x$, we have $f^2(x) < x$;*

(ii) *If G is any non-empty closed subset of $[a,b]$ mapped into itself by f, then f has a fixed point in G;*

(iii)-(a) *for each $x \in [a,b]$ such that $f(x) \neq x$, we have $f^k(x) \neq x$ for every $k > 1$.*

(b) *for each $x \in [a,b]$ such that $f(x) > x$, we have $f^k(x) > x$ for all $k > 1$, and for each $x \in [a,b]$ such that $f(x) < x$, we have $f^k(x) < x$ for all $k > 1$;*

(iv) *$\{f^k(x)\}_{k=1}^{\infty}$ is a convergent sequence for every x in $[a,b]$.*

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* The work of the second author was carried out while he was at Bellcomm, Inc., Washington, D. C.

Proof. We first show that (iii)-(a) is equivalent to (iii)-(b). Notice that (iii)-(b) trivially implies (iii)-(a). To show that (iii)-(a) implies (iii)-(b), let $f(x) > x$ for some x. By (iii)-(a), $f^k(x) \neq x$ for all $k > 1$. Assume that there exists $k > 1$ such that $f^k(x) < x$.

Case I. $f(y) > y$ for all $y \in [a, x]$. Since f maps $[a, b]$ into $[a, b]$, $f^k(a) \geqslant a$. But by (iii)-(a) $f^k(a) \neq a$; hence $f^k(a) > a$. Therefore, by continuity of f^k on the connected set $[a, x]$, there exists $x_0 \in [a, x]$ such that $f^k(x_0) = x_0$, but $f(x_0) > x_0$. This is contrary to (iii)-(a).

Case II. There exists a fixed point of f in $[a, x]$. Let x_0 be the largest fixed point of f in $[a, x]$. Note that $f(y) > y$ for all $y \in (x_0, x)$. By the continuity of f, there exists an $h_1 > 0$ such that f maps (x_0, x_0+h_1) into (x_0, x). Hence, for $y \in (x_0, x_0+h_1)$, $f(f(y)) = f^2(y) > y$. We claim that there exists an h_j such that $f^{j+1}(y) > y$ for $y \in (x_0, x_0+h_j)$, $j = 1, 2, \ldots$ We have shown it to be true for $j = 1$. Assume true for $j-1$; then there exists an h_j such that f maps (x_0, x_0+h_j) into (x_0, x_0+h_{j-1}). Hence, if $y \in (x_0, x_0+h_j)$, then

$$f^{j+1}(y) = f^j(f(y)) > f(y) > y .$$

By induction this holds for every j. In particular, $f^k(y) > y$ for $y \in (x_0, x_0+h_{k-1})$. By assumption, $f^k(x) < x$. Thus, f^k must have a fixed point in (x_0, x) which is contrary to (iii)-(a) and the choice of x_0. A similar argument can be made when $f(x) < x$.

By setting $k = 2$ in the above argument, we immediately have that (i)-(a) is equivalent to (i)-(b).

Next we show that (i) (in particular (i)-(b)) implies (ii). Assume (ii) to be false. Then there exists a non-empty closed subset G of $[a, b]$ which is mapped into itself by f and is such that $f(x) \neq x$ for $x \in G$. In particular, there must exist either a largest $x \in G$ such that $f(x) > x$ or a smallest $y \in G$ such that $f(y) < y$. For definiteness assume the former, and call this largest element x_0. (A similar argument can be made for the other case.)

We first show that

$$(1) \qquad f\colon [x_0, f(x_0)] \to (x_0, b]$$

(and thus the image of $[x_0, f(x_0)]$ under f is a compact subset of $(x_0, b]$). Let x_1 be the smallest fixed point of f that is greater than x_0 (which exists since $f(x_0) > x_0$ and $f(b) \leqslant b$). Then $f(y) > y \geqslant x_0$ for all $y \in [x_0, x_1]$. Thus, if $f(x_0) \leqslant x_1$, then (1) is verified. So we assume that $f(x_0) > x_1$. Since f is continuous, we must have $[x_1, f(x_0)] \subset f([x_0, x_1])$. Therefore,

$$f([x_1, f(x_0)]) \subset f^2([x_0, x_1]) .$$

But by (i)-(b) we have, in turn,

$$f^2([x_0, x_1]) \subset (x_0, b] .$$

Hence we have

$$(2) \qquad f\colon [x_1, f(x_0)] \to (x_0, b] ,$$

and (1) holds. We next show, by induction, that

$$(3) \qquad f(x_0) \geqslant \ldots \geqslant f^k(x_0) \geqslant f^{k+1}(x_0) \geqslant \ldots > x_0$$

for all $k = 1, 2, \ldots$ (Notice that since $f(G) \subset G$, we have $f^k(x_0) \in G$ for all $k = 1, 2, \ldots$). Inequality (3) is certainly true for $k = 1$, since $f^2(x_0) > x_0$ by (i)-(b) and $f(f(x_0)) \leqslant f(x_0)$ by choice of x_0. Assume now that (3) holds for $k = N$; then we have

$$x_0 < f^{N+1}(x_0) \leqslant f^N(x_0) \leqslant f(x_0) .$$

Since x_0 is chosen to be the largest $x \in G$ such that $f(x) > x$, we must have $f^{N+2}(x_0) = f(f^{N+1}(x_0)) \leqslant f^{N+1}(x_0)$. On the other hand, $f^{N+1}(x_0) \in [x_0, f(x_0)]$, and by relation (1), we must have

$$x_0 < f(f^{N+1}(x_0)) = f^{N+2}(x_0) .$$

Therefore, the induction is complete and (3) is verified for all k. Hence the sequence $\{f^k(x)\}_{k=1}^{\infty}$ is a monotonic sequence which converges to some limit point x_2. Since G is closed, $x_2 \in G$; since f is continuous, $f(x_2) = x_2$, which is contrary to the assumption that f has no fixed points in G. Part (ii) is now verified.

We now show that (ii) implies (iii) (in particular (iii)-(a)). Assume that there exists $x \in [a, b]$ and $k > 1$ such that $f(x) \neq x$, but $f^k(x) = x$. Then the (closed) set $\{x, f(x), \ldots, f^{k-1}(x)\}$ is invariant under f. By (ii), f has a fixed point in that set. By the assumption that $f(x) \neq x$, there must exist l, $1 \leqslant l \leqslant k-1$, such that $f(f^l(x)) = f^l(x)$. We have therefore

$$x = f^{k-l}(f^l(x)) = f^{k-l}(f^{l+1}(x)) = f(f^k(x)) = f(x) ,$$

which is contrary to assumption. Part (iii) is verified.

We now show that (iii) (in particular (iii)-(b)) implies (iv). Let $x \in [a, b]$, and let

$$A^+ = \{k\colon f^{k+1}(x) > f^k(x)\} ,$$
$$A^- = \{k\colon f^{k+1}(x) < f^k(x)\} .$$

If for some i, $f^i(x) = f^{i+1}(x)$, then (iv) is verified. Hence we assume that all the iterates are distinct from their successors, in which case $A^+ \cup A^-$ is the set of all positive integers. Let

$$x^+ = \varliminf_{k\to\infty} \{f^k(x)\colon k \in A^+\} .$$

Since, by (iii), for any $k \in A^+$ and for all $l > k$,

$$f^l(x) > f^k(x) ,$$

we must have $x^+ \geqslant f^k(x)$, $k \in A^+$. Therefore,

$$x^+ \geqslant \overline{\lim_{k\to\infty}} \{f^k(x):\ k \in A^+\},$$

implying that x^+ is the unique limit of $\{f^k(x):\ k \in A^+\}$. Similarly, the sequence $\{f^k(x):\ k \in A^-\}$ has a unique limit; call it x^-.

Now, either for sufficiently large k one of the sets A^+ or A^- contains all the successors of k, or there are infinitely many k in each of $A^\pm$ such that $k+1 \in A^\mp$. The former case implies that the sequence $\{f^k(x)\}$ eventually becomes monotonic and, therefore, converges, verifying (iv). In the latter case, $f(x^+) = x^-$ and $f(x^-) = x^+$ and, by (iii), we must have $x^+ = x^-$, again verifying (iv).

Finally, we observe that (iv) trivially implies (i)-(a), since if there exists x such that $f(x) \neq x$ but $f^2(x) = x$, the sequence $\{f^k(x)\}$ cannot converge.

The proof of Theorem 1 is now complete.

We remark here that although some of the properties trivially follow from others, we have listed them all for the sake of interest, since most of them are seemingly unrelated.

Definition. Let f be a continuous self-mapping of an interval $[a, b]$ into itself. If $f(x) \neq x$ implies $f^2(x) \neq x$ for all $x \in [a, b]$, then f is called *non-cyclic*.

The choice of the term *non-cyclic* was motivated by the fact that the condition in the definition is equivalent to condition (iii)-(a) of Theorem 1.

As an illustration, the following two kinds of functions are non-cyclic if they are continuous self-mappings of some interval $[a, b]$ into itself:

(a) $\quad f$ non-decreasing on $[a, b]$,

(b) $\quad f(x) \geqslant x \quad$ for all $\quad x \in [a, b]$.

A third example is given by:

(c) $\quad f(x) = a - \beta x, \quad x \in [0, 1]$

where $\beta \neq 1$ and $0 \leqslant \beta \leqslant a \leqslant 1$.

We are now able to give a simple verification of the commuting functions conjecture for a special case, and we state it as:

Theorem 2. *Let f and g be two continuous self-mappings of $[0, 1]$ into itself such that f and g commute on $[0, 1]$. Suppose that there exists a subinterval $[a, b]$ on which one of the functions, say f, is non-cyclic, and in which g has a fixed point. Then f and g possess a common fixed point in $[a, b] \subset [0, 1]$.*

Proof. Let G be the set of all fixed points of g in $[a, b]$. G is non-empty by hypothesis. $f(G) \subset G$, since f and g commute. G is closed, since g is continuous. Apply Theorem 1.

We remark here that if one attempts to verify the commuting functions conjecture by utilizing the commutivity property only at the fixed points (what we might call the "invariant subset" approach), then Theorem 1 suggests that the non-cyclic assumption, restrictive though it might be, is one of the most general possible.

Finally, we wish to compare our results with those of Cohen [2] who has verified the conjecture for the case of full functions, where a function is full if the interval $[0, 1]$ can be partitioned into a finite number of subintervals each of which is mapped homeomorphically onto $[0, 1]$. (See also Baxter and Joichi [1].) Theorem 2 above gives a verification of the conjecture for a class of functions which in general are not full. In fact, the only functions which are both full and non-cyclic on $[0, 1]$ are those which are monotonically increasing, and are zero at the origin and one at $x = 1$. Hence, in a sense, Theorem 2 is complementary to the result of Cohen, but by no means exhausts all the remaining possibilities.

As an illustration, we give the following two examples:

(d) $$f(x) = \begin{cases} 2x-1+(\frac{1}{3})^i & 1-(\frac{1}{3})^i \leqslant x \leqslant 1-\frac{1}{2}(\frac{1}{3})^i, \\ 3-(\frac{1}{3})^i-2x & 1-\frac{1}{2}(\frac{1}{3})^i \leqslant x \leqslant 1-(\frac{1}{3})^{i+1}, \\ 1 & x = 1, \end{cases}$$

for $i = 0, 1, 2, \ldots$;

(e) $$f(x) = \begin{cases} 3x & 0 \leqslant x \leqslant \frac{1}{3}, \\ 2-3x & \frac{1}{3} \leqslant x \leqslant \frac{1}{2}, \\ 3x-1 & \frac{1}{2} \leqslant x \leqslant \frac{2}{3}, \\ 3-3x & \frac{2}{3} \leqslant x \leqslant 1. \end{cases}$$

Example (d) gives a function which is "nowhere full", but since it belongs to the class of functions in example (b) above, it is non-cyclic on $[0, 1]$. On the other hand, (e) exhibits a function which is neither full nor non-cyclic.

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References

[1] G. Baxter and J. T. Joichi, *On functions that commute with full functions*, Nieuw Archief voor Wiskunde (3), XII (1964), pp. 12-18.

[2] H. Cohen, *On fixed points of commuting functions*, Proc. Amer. Math. Soc. 15 (1964), pp. 292-296.

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