Thus, f is the desired quasi-isomorphism and the Theorem is proved.

We observe that the construction of s_i^j except for the condition $i \leqslant j$, which is most essential for the proof of the Theorem, resembles a similar construction given in [1] of [2].

As an immediate consequence of the above theorem we have [3]:

COROLLARY. *Every partial order in a set P can be extended to a simple order in the same set P preserving the original order among the elements of P.*

References

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THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO, USA

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On embedding curves in surfaces

by

K. Borsuk (Warszawa)

I. Preliminaries

1. Elementary properties of surfaces. By a *bounded surface* we understand here a continuum N such that every point of it has a neighborhood which is a disk, i.e. a topological image of the square. In particular, the disk, the circular ring (annulus) and the Möbius band are bounded surfaces. The points of a bounded surface N for which no neighborhood is homeomorphic to the Euclidean plane E^2 constitute a set $N^\bullet$ called the *boundary* of N. The set $N - N^\bullet$ is said to be the *interior* of the bounded surface N; it will be denoted by N°. The set $N^\bullet$ is the union of a finite number of simple closed curves disjoint with one another. If we match each of these curves with the boundary of a disk, then we obtain from N another bounded surface M with an empty boundary, i.e. a *closed surface*, or simply a *surface*. Hence every bounded surface is homeomorphic to a subset of a surface.

A bounded surface is said to be *orientable* if it does not contain topologically the Möbius band. All other bounded surfaces are said to be *non-orientable*.

A subset X_0 of a space X is said *to have arbitrarily small neighborhoods* (in X) with a property (α) provided every neighborhood of X_0 contains a neighborhood of X_0 with property (α). If there exists a neighborhood U_0 of X_0 such that every neighborhood of X_0 contained in U_0 has property (α), then we say that the property (α) holds for all sufficiently small neighborhoods of X_0.

Let us formulate some elementary properties of surfaces:

(1.1) *Each closed subset of a surface M has arbitrarily small neighborhoods (in M) which are bounded surfaces.*

(1.2) *Each arc (and also each disk) lying on a surface M has arbitrarily small neighborhoods which are disks.*

(1.3) *If C is a simple closed curve lying on a surface M, then only the following two cases are possible:* (i) *C has arbitrarily small neighborhoods homeomorphic to the Möbius band.* (ii) *C has arbitrarily small neighborhoods homeomorphic to the annulus.*

In case (i) the curve C is said to be *one-sided* (on M), in case (ii) *two-sided*. If M is orientable, then each simple closed curve $C \subset M$ is two-sided on M. If M is non-orientable, then M contains also one-sided curves.

We say that a subset X_0 of a space X *decomposes* a subset Y of X (globally) if the set $Y - X_0$ is not connected. In the case where X is connected, we say that X_0 *locally decomposes* Y provided for every sufficiently small neighborhood U of X_0 the set X_0 decomposes the set $U \cap Y$. Moreover, we say that X_0 *decomposes* Y *at a point* $y_0 \in Y$ provided X_0 decomposes every sufficiently small neighborhood of y_0. It is well known that

(1.4) *If M is an orientable surface, then every simple closed curve $C \subset M$ locally decomposes M.*

(1.5) *If M is a surface, then every simple closed curve $C \subset M$ decomposes M at every point of C.*

(1.6) *If M is a surface, then every arc $L \subset M$ decomposes M at each point belonging to the interior of L.*

To every surface M corresponds an integer $\gamma(M)$, called the *genus* of M, defined as the maximal number of mutually disjoint simple closed curves in M which together do not decompose M. Let us recall the following propositions:

(1.7) *Two surfaces M and M' are homeomorphic if and only if they are both orientable or both non-orientable and $\gamma(M) = \gamma(M')$.*

(1.8) *Two surfaces M and M' are homeomorphic if and only if $p_i(M) = p_i(M')$ for $i = 1, 2$, where $p_i(X)$ denotes the i-th Betti number of X.*

The fundamental theorem on surfaces gives (see, for instance, [8], p. 141) an explicit enumeration of all topological types of them. Let P denote the *perforated tore*, i.e. the bounded surface which we get from the surface of a tore by removing the interior of a disk, and let Q denote the Möbius band. Then

(1.9) *Each surface M of genus m is homeomorphic to the space which is obtained from the sphere S^2 by replacing m disjoint disks $D_1, D_2, \ldots, D_m$ lying in S^2 by m bounded disjoint surfaces $N_1, N_2, \ldots, N_m$ in the following manner:*

(i) *If M is orientable, then N_i is homeomorphic to P for every $i = 1, 2, \ldots, m$.*

(ii) *If M is non-orientable, then N_1 is homeomorphic to Q and all other N_i are homeomorphic to P.*

(iii) *$N_i \cap S^2 = N_i^\bullet \cap S^2 = D_i^\bullet$ for every $i = 1, 2, \ldots, m$.*

Moreover, let us recall that

(1.10) *Every bounded surface is triangulable.*

2. Plane sets. A set A is said to be *plane* if it is homeomorphic to a subset of the Euclidean plane E^2, or, which is the same, to a proper subset of the sphere S^2. Let us recall that

(2.1) *A bounded surface $N \neq S^2$ is plane if and only if every simple closed curve $C \subset N^\circ$ decomposes N.*

In particular

(2.2) *A bounded surface N with a non-empty and connected boundary $N^\bullet$ and such that every simple closed curve $C \subset N^\circ$ decomposes N is a disk.*

Moreover, let us recall that

(2.3) *The interior of the perforated tore P and also of the Möbius band Q contain simple closed curves which do not decompose them.*

The following theorem of Kuratowski ([4], p. 272) characterizes 1-dimensional plane ANR's:

(2.4) THEOREM. *A 1-dimensional* ANR *is plane if and only if it does not contain topologically any of the following two graphs:*

K, which is the union of all edges of a tetrahedron and of a segment joining two points lying in the interiors of two opposite edges of it.

K', which is the union of all edges of a tetrahedron and of all segments joining its barycentre with its vertices.

In the sequel we need also the following elementary fact ([4], p. 282)

(2.5) *The graph K is homeomorphic to a subset of the perforated tore P and also to a subset of the Möbius band Q.*

Let us observe that for every surface M there exists a graph which is not homeomorphic to any subset of M. In order to see it, let us denote by H_m (for every $m = 1, 2, \ldots$) the graph which is the union of m disjoint graphs homeomorphic to K. Then

(2.6) *A surface M contains topologically the graph H_m if and only if $\gamma(M) \geqslant m$.*

Proof. It follows by (1.9) and (2.5) that $\gamma(M) \geqslant m$ implies that H_m is topologically contained in M. On the other hand, if M contains m disjoint copies $K_1, K_2, \ldots, K_m$ of the graph K, then (1.1) implies that there exists in M a system of m disjoint bounded surfaces $N_1, N_2, \ldots, N_m$ such that $K_i \subset N_i^\circ$. Let $C_{i,1}, C_{i,2}, \ldots, C_{i,n_i}$ be simple closed curves which are components of $N_i^\bullet$, and let $D_{i,1}, D_{i,2}, \ldots, D_{i,n_i}$ be disjoint disks. By matching $C_{i,j}$ with $D_{i,j}^\bullet$ we get from N_i a surface M_i which contains topologically K. Since S^2 does not contain K, we infer by (2.1) that $\gamma(M_i) > 0$. Hence there is on M_i a simple closed curve C_i which does not decompose M_i. Let us consider a disk $D'_{i,j} \subset D^{\circ}_{i,j} - C_i$. Evidently there is a homeomorphism h_i mapping M_i onto itself such that $h_i(D'_{i,j}) = D_{i,j}$. Then $h_i(C_i)$ is a simple closed curve on N_i° which does not decompose M_i, and consequently does not decompose N_i either. It follows at once that the system of disjoint curves $h_1(C_1), h_2(C_2), \ldots, h_m(C_m)$ does not decompose M, whence $\gamma(M) \geqslant m$.

(2.7) Problem. *Let M be an orientable and M' a non-orientable surface with $\gamma(M) = \gamma(M')$. Is it true that there exist two graphs $G \subset M$ and $G' \subset M'$ such that G is not homeomorphic to any subset of M' and G' is not homeomorphic to any subset of M?*

Some results concerning similar problems are given by E. Vázsonyi [9].

3. Moore decompositions. By a *Moore decomposition* of a space M we understand any upper semicontinuous decomposition Σ of M such that each element A of Σ is a continuum having arbitrarily small neighborhoods (in M) homeomorphic to the plane E^2. It follows by (1.2) that each upper semicontinuous decomposition Σ of a surface M into elements which are disks, arcs and individual points is necessarily a Moore decomposition. Evidently every element of a Moore decomposition is acyclic in all dimensions.

By a classical theorem of R. L. Moore ([6], p. 427), the decomposition space of a Moore decomposition of E^2 is homeomorphic to E^2. Let us observe that an analogous proposition holds also for every surface M, i.e.:

(3.1) *If M is a surface, then for every Moore decomposition Σ of M the decomposition space M_Σ is homeomorphic to M.*

Proof. Consider an element A of the decomposition Σ and let U be its neighborhood homeomorphic to E^2. Since the decomposition Σ is upper semicontinuous, there exists a compact neighborhood V and A (in M) such that every element B of Σ such that $B \cap V \neq 0$ is a subset of U. Now let us denote by Σ' the decomposition of U whose non-degenerate elements are all those elements B of Σ for which $B \cap V \neq 0$. Evidently Σ' is a Moore decomposition of U, whence its decomposition space $U_{\Sigma'}$ is homeomorphic to E^2. Moreover, $U_{\Sigma'}$ is locally homeomorphic to $M_{\Sigma'}$ at the point $A \in \Sigma$. It follows that M_Σ is a continuum locally homeomorphic to E^2, i.e. it is a surface. Moreover, by the classical theorem of L. Vietoris ([10], p. 470) the Betti numbers of M_Σ are the same as the corresponding Betti numbers of M. By (1.8), the surface M_Σ is homeomorphic to M.

4. An elementary lemma. Now let us establish a simple lemma concerning locally connected continua, which we need in the sequel:

(4.1) Lemma. *Let X be a locally connected subcontinuum of a space M and $\mathfrak{T}$ a collection of indices. Assume that to each index $\tau \in \mathfrak{T}$ corresponds an open subset G_τ of M and a locally connected continuum $F_\tau \subset M$ satisfying the following conditions:*

(α) *For every $\varepsilon > 0$ the inequality $\delta(F_\tau) < \varepsilon$ holds for almost all indices $\tau \in \mathfrak{T}$.*

(β) *$\tau \neq \tau'$ implies $G_\tau \cap G_{\tau'} = 0$.*

(γ) *$0 \neq X \cap (\overline{G}_\tau - G_\tau) \subset F_\tau$ for every index $\tau \in \mathfrak{T}$.*

Then the set

$$X' = \Big(X - \bigcup_{\tau \in \mathfrak{T}} G_\tau\Big) \cup \bigcup_{\tau \in \mathfrak{T}} F_\tau$$

is a locally connected continuum.

Proof. Since X is a locally connected continuum, there is a continuous map f of the circle S^1 onto X. It follows by (β) and (γ) that the sets $f^{-1}(G_\tau)$ are open, disjoint and non-empty proper subsets of S^1. Hence each component $J_{\tau,\nu}$ of the set $f^{-1}(G_\tau)$ is an open subarc of S^1. Moreover, it follows by (β) that all sets $J_{\tau,\nu}$ are disjoint with one another. Let $a_{\tau,\nu}$ and $b_{\tau,\nu}$ be the end-points of $J_{\tau,\nu}$. By (γ), the points $f(a_{\tau,\nu})$, $f(b_{\tau,\nu})$ both belong to F_τ.

Since F_τ is a locally connected continuum, there exists a continuous function $f_{\tau,\nu}$ which maps the closure $\overline{J}_{\tau,\nu}$ of $J_{\tau,\nu}$ into F_τ and satisfies the conditions:

$$f_{\tau,\nu}(a_{\tau,\nu}) = f(a_{\tau,\nu}), \quad f_{\tau,\nu}(b_{\tau,\nu}) = f(b_{\tau,\nu}).$$

Moreover, we may assume that

(4.2) *For every index $\tau \in \mathfrak{T}$ there is an index ν_τ such that $f_{\tau,\nu_\tau}(\overline{J}_{\tau,\nu_\tau}) = F_\tau$; for every $\varepsilon > 0$ and for every index τ the inequality $\delta[f_{\tau,\nu}(\overline{J}_{\tau,\nu})] < \varepsilon$ holds for almost all indices ν.*

Now let us set

$$f'(x) = f(x) \quad \text{for every point} \quad x \in S^1 - \bigcup_{\tau \in \mathfrak{T}} f^{-1}(G_\tau),$$

$$f'(x) = f_{\tau,\nu}(x) \quad \text{for every point } x \in J_{\tau,\nu}.$$

We infer by (α) and (4.2) that f' is a continuous map of S^1 onto X'. Hence X' is a locally connected continuum.

II. On locally plane curves

5. Embedding of 1-dimensional ANR's in surfaces. A set X is said to be *locally plane* if every point $x \in X$ has a plane neighborhood. As has been shown by T. Ważewski ([11], p. 57), every dendrite is plane. We infer that every space in which each point has a neighborhood which is a dendrite is locally plane. Since 1-dimensional ANR's (compact) are the same as compacta locally homeomorphic to dendrites, it follows that

(5.1) *Every 1-dimensional ANR is locally plane.*

Now let us prove the following

(5.2) THEOREM. *If X is an ANR-space such that $\dim X < 2$ and $p_1(X) < n$ and if M is a surface with $\gamma(M) > n-4$, then X is homeomorphic to a subset of M.*

We start with the following

(5.3) LEMMA. *For every $X \epsilon$ ANR with $\dim X = 1$ and with $p_1(X) < 0$ there is a decomposition $X = X_1 \cup X_2$ such that X_1 is a dendrite, X_2 is an ANR with $p_1(X_2) = p_1(X)-1$ and $X_1 \cap X_2$ consists of two points.*

Proof. The hypothesis $p_1(X) > 0$ implies that X contains a simple closed curve C. Since the set of ramification points of X is at most countable ([5], p. 229), there is a point $a \epsilon C$ of order 2 in X. We infer that there exists a dendrite $X_1 \subset X$ which is a neighborhood of a in X and is such that set $X_1 \cap \overline{X-X_1}$ consists of exactly two points b, c. Setting $X_2 = \overline{X-X_1}$, we have $X = X_1 \cup X_2$, and since X and $X_1 \cap X_2$ are ANR's, we infer ([1], p. 226) that $X_2 \epsilon$ ANR. Moreover, X_1, as a dendrite, does not contain C. It follows that $b, c \epsilon C$ and that $X_1 \cap \overline{X-X_1}$ is a carrier of a 0-dimensional true cycle which is homologous to zero in the sets X_1 and X_2. It follows (as a special case of the theorem of Mayer-Vietoris) that

$$p_1(X) = p_1(X_1)+p_1(X_2)+1,$$

and since $p_1(X_1) = 0$, we get $p_1(X_2) = p_1(X)-1$. Thus the proof of Lemma (5.3) is finished.

Now we prove Theorem (5.2) by induction (¹). Let $X \epsilon$ ANR and $\dim X < 2$. Evidently for the graphs K and K' of Kuratowski we have $p_1(K) = 4$ and $p_1(K') = 6$. If $p_1(X) < n = 4$, then X contains neither K nor K', and we infer by (2.4) that X is homeomorphic to a subset of each surface. Thus proposition (5.2) is true for $n \leqslant 4$.

Let us assume now that for an $m \geqslant 4$ proposition (5.2) holds if $n \leqslant m$, and let us consider the case $n = m+1$. Let M be a surface with $\gamma(M) > n-4 = m-3 \geqslant 1$. Thus $\gamma(M) > 1$ and, the projective plane being a surface of genus 1, we conclude that M is not homeomorphic to the projective plane. It follows by (1.9) that M may be represented as the union of two bounded surfaces N_1 and N_2 having a simple closed curve C as their common boundary and such that

(a) $N_1^\circ \cap N_2^\circ = 0$,

(b) N_1 is homeomorphic to the perforated tore P.

(¹) The present form of this proof is due to J. J. Charatonik.

Let us observe that if we add to N_2 a disk D with the boundary C and the interior disjoint with N_2, then we get a surface M_2 such that

$$\gamma(M_2) = \gamma(M)-1 > n-5 = m-4.$$

By Lemma (5.3) there exists a decomposition

$$X = X_1 \cup X_2,$$

where X_1 is a dendrite and X_2 an ANR-set with $p_1(X_2) < n-1$ and where $X_1 \cap X_2$ consists of two points a and b. Moreover, the hypothesis of induction implies that there exists a homeomorphism h_2 mapping X_2 onto a subset of M_2. Since each point of a 1-dimensional ANR-set lying in the Euclidean plane is accessible from its complement ([1], p. 233), we infer that there are in M_2 two disjoint disks D_a and D_b such that $D_a \cap h_2(X_2) = (h_2(a))$ and $D_b \cap h_2(X_2) = (h_2(b))$. On the other hand, there exists a homeomorphism h_1 mapping X_1 onto a subset of S^2 and there are on S^2 two disjoint disks D'_a and D'_b such that

$$D'_a \cap h_1(X_1) = (h_1(a)) \quad \text{and} \quad D'_b \cap h_1(X_1) = (h_1(b)).$$

Now let us consider a disk $D \subset M_2$ containing the disks D_a and D_b in its interior. Let us fix a positive orientation on D and also on S^2. They induce positive orientations on the boundaries $D_a^\bullet$, $D_b^\bullet$, $D_a'^\bullet$ and $D_b'^\bullet$ of the disks D_a, D_b, D'_a and D'_b. Manifestly there exists a homeomorphism h mapping $D_a^\bullet \cup D_b^\bullet$ onto $D_a'^\bullet \cup D_b'^\bullet$ which preserves the positive orientation and maps $h_2(a)$ onto $h_1(a)$ and $h_2(b)$ onto $h_1(b)$. If we identify every point $y \epsilon D_a^\bullet \cup D_b^\bullet$ with the point $h(y) \epsilon D_a'^\bullet \cup D_b'^\bullet$, we get from the sets $M_2 = (D_a^\circ \cup D_b^\circ)$ and $S^2-(D_a'^\circ \cup D_b'^\circ)$ a surface M' homeomorphic to M, and the homeomorphisms h_1 and h_2 give together a homeomorphism of the set $X = X_1 \cup X_2$ into M'. Thus the proof of Theorem (5.2) is finished.

(5.4) PROBLEM. *Is it true that every ANR-space of dimension $<n$ such that each of its points has a neighborhood homeomorphic to a subset of the Euclidean n-space E^n is homeomorphic to a subset of an n-dimensional manifold?*

6. Locally plane curve which cannot be embedded in any surface.

Let us prove the following

(6.1) THEOREM. *There exists a locally plane and locally connected curve which is not homeomorphic to any subset of any surface.*

Proof. Consider in the Euclidean 3-space E^3 a tetrahedron T with vertices a, b, c, d and let c_n denote the point dividing the segment $\overline{ac}$ at the ratio $1:(n-1)$, and d_n denote the point dividing the segment $\overline{bd}$

at the same ratio $1:(n-1)$. The points a, b, c_n, d_n are vertices of a tetrahedron T_n and the segment $\overline{c_{n+1}d_{n+1}}$ joins two inner points of two opposite edges of it. It follows that the set

$$X_n = \overline{ab} \cup \overline{ac_n} \cup \overline{ad_n} \cup \overline{bc_n} \cup \overline{bd_n} \cup \overline{c_n d_n} \cup \overline{c_{n+1}d_{n+1}}$$

is homeomorphic to the graph K, whence it is not plane. Moreover, it follows by our construction that

$$\lim_{n\to\infty} X_n = \overline{ab}\,.$$

Setting

(6.2) $$X = \bigcup_{n=1}^{\infty} X_n\,,$$

we get a curve which is the union of two sets

$$Z = X - \overline{bd} \quad \text{and} \quad Z' = X - \overline{ac}\,,$$

open in X. In order to prove that X is locally plane, it suffices to show that Z and Z' are plane, and since Z is homeomorphic with Z', it suffices to construct a homeomorphism h mapping Z onto a subset of E^2.

In order to do that, let us consider in E^2 an orthogonal system of coordinates x, y and let

$$a' = (0,0)\,, \quad b' = (1,0)\,,$$

$$c_n' = \left(0, \frac{1}{n}\right), \quad b_n' = \left(1, \frac{1}{n}\right), \quad d_n' = \left(1, \frac{2}{2n+1}\right), \quad d_n'' = \left(1, -\frac{1}{n}\right)$$

$$\text{for} \quad n = 1, 2, \ldots$$

Now let us consider the linear maps f, g, f_n, f_n', f_n'' given by the following conditions:

$$\begin{aligned}
f&: \overline{ab} \to \overline{a'b'} && \text{with} \quad f(a) = a',\ f(b) = b';\\
g&: \overline{ac} \to \overline{a'c'} && \text{with} \quad g(a) = a',\ g(c) = c';\\
f_n&: \overline{c_n d_n} \to \overline{c_n' d_n'} && \text{with} \quad f_n(c_n) = c_n',\ f_n(d_n) = d_n';\\
f_n'&: \overline{c_n b} \to \overline{c_n' b_n'} && \text{with} \quad f_n'(c_n) = c_n',\ f_n'(b) = b_n';\\
f_n''&: \overline{ad_n} \to \overline{a'd_n''} && \text{with} \quad f_n''(a) = a',\ f_n''(d_n) = d_n'';
\end{aligned}$$

Setting

$$h(x) = \begin{cases} f(x) & \text{for} \quad x \in \overline{ab} - (b)\,,\\ g(x) & \text{for} \quad x \in \overline{ac}\,,\\ f_n(x) & \text{for} \quad x \in \overline{c_n d_n} - (d_n)\,,\\ f_n'(x) & \text{for} \quad x \in \overline{c_n b} - (b)\,,\\ f_n''(x) & \text{for} \quad x \in \overline{ad_n} - (d_n), \end{cases}$$

we get a homeomorphism h mapping Z onto the set

$$Z_0 = \overline{a'b'} \cup \overline{a'c'} \cup \bigcup_{n=1}^{\infty} (\overline{b_n' c_n'} \cup \overline{c_n' d_n'} \cup \overline{a' d_n''}) - \overline{d_1' d_1''} \subset E^2\,.$$

Thus the proof that X is a locally plane curve is concluded.

Now let us prove that X is not homeomorphic to any subset of any surface. Suppose, on the contrary, that there exists a homeomorphism h_0 mapping X onto a subset $h_0(X)$ of a surface M. Then $h_0(\overline{ab})$ is an arc in M and we infer by (1.2) that there is a disk $D \subset M$ containing $h_0(\overline{ab})$ in its interior. It follows by (6.2) that for almost all indices n the set $h_0(X_n)$ is a subset of D. But this is impossible, because the set X_n is not plane.

In order to conclude the proof of Theorem (6.1), it suffices to show that X is a subset of a locally plane and locally connected curve. Let us consider, for every $i = 1, 2, \ldots, n$ and for $n = 1, 2, \ldots$, the point $a_{i,n}$ decomposing the segment $\overline{a'b'}$ at the ratio $i:(n+1-i)$ and let $L_{n,i}$ denote the segment which is parallel to the segment $\overline{a'c_1'}$ and has $a_{i,n}$ as one of its end-points, with the other end-point lying on the segment $\overline{c_n' d_n'}$. Moreover, let $L_{n,i}'$ denote the segment parallel to $\overline{a'c_1'}$ with one end-point in $a_{i,n}$ and the other on the segment $\overline{a'd_n''}$. We easily verify that the set

$$Y' = Z_0 \cup \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n} (L_{n,i} \cup L_{n,i}')$$

is locally connected and it is easy to show that the homeomorphism $h^{-1}\colon Z_0 \to Z$ can be extended to a homeomorphism φ of the set Y' onto a subset of $E^3 - \overline{bd}$ so that for every $\varepsilon > 0$ there exists an index $n(\varepsilon)$ such that for $n \geqslant n(\varepsilon)$ the diameters of the sets $\varphi(L_{n,i} \cup L_{n,i}')$ are less than ε. Setting

$$Y = \varphi(Y') \cup \overline{bd}\,,$$

we easily see that Y is a locally connected curve containing Z and such that

$$Y = \varphi(Y') \cup [\overline{bd} \cup \varphi(Y') - \overline{ac}]\,,$$

where each of the sets $\varphi(Y')$ and $\overline{bd} \cup \varphi(Y') - \overline{ac}$ is open in Y and homeomorphic to the set $Y' \subset E^2$. Hence the set Y is locally plane and (since it contains Z) it is not homeomorphic to any subset of any surface. Thus the proof of Theorem (6.1) is concluded.

III. S-curves in surfaces

7. A_M-curves. Let M be a surface. A locally connected curve $A \subset M$ is said to be an *S-curve* in M (cf. G. T. Whyburn [13], p. 321) if there exists a sequence $\{D_i\}$ of mutually disjoint disks in M such that

(7.1) $$A = M - \bigcup_{i=1}^{\infty} D_i^{\circ}\,.$$

We can easily see that the hypothesis that A is locally connected implies

(7.2) $$\lim_{i\to\infty}\delta(D_i)=0\,.$$

On the other hand, if $\{D_i\}$ is a sequence of mutually disjoint disks in M, satisfying (7.2), and if the set $\bigcup_{i=1}^{\infty} D_i$ is dense in M, then the set A given by formula (7.1) is a locally connected curve.

By a theorem of G. T. Whyburn ([13], p. 322), any two S-curves in the sphere S^2 are homeomorphic. More exactly:

(7.3) *If $A=S^2-\bigcup_{i=1}^{\infty} D_i$ and $A'=S^2-\bigcup_{i=1}^{\infty} D_i'$ are two S-curves in S^2 and if h_1 is a homeomorphism mapping D_1 onto D_1', then h_1 can be extended to a homeomorphism $h\colon A\to A'$.*

Let us apply (7.3) to prove the following generalization of the theorem of Whyburn:

(7.4) *Any two S-curves in a given surface M are homeomorphic.*

Proof. Let A be an S-curve in the surface M, given by (7.1). By (1.10) there exists a triangulation T_0 of M. Consider the upper semi-continuous decomposition Σ of M whose non-degenerate elements are disks D_i. It follows by (1.2) that Σ is a Moore decomposition of M, whence by (3.1) the decomposition space M_Σ is homeomorphic to M. Since the subset of M_Σ consisting of all points d_i corresponding to disks D_i is only countable, we see at once that there exists a triangulation T_1 of M isomorphic to the triangulation T_0 and such that no point d_i belongs to the 1-dimensional skeleton Z of the triangulation T_1, i.e. to the union of all 1-dimensional simplexes of T_1. This skeleton Z may be considered as lying in the set $A-\bigcup_{i=1}^{\infty} D_i$ and thus we get a triangulation T of M isomorphic to T_0 and such that every disk D_i lies in the interior of a triangle $\Delta\in T$. Now let us consider another S-curve $A'=M-\bigcup_{i=1}^{\infty} D_i'^{\circ}$ in M. By the same argument we get another triangulation T' of M isomorphic to T_0 and such that every disk D_i' lies in the interior of a triangle $\Delta'\in T'$. Let Z' denote the 1-dimensional skeleton of T'.

Since T and T' are isomorphic, there exists a homeomorphism $h\colon M\to M'$ such that each triangle $\Delta\in T$ is mapped by h onto a triangle $\Delta'\in T'$. Evidently the common part of the curve A with the triangle Δ (and also the common part of A' with Δ') may be considered as an S-curve on a sphere, which we obtain if we match the boundary of the triangle with the boundary of another triangle. It follows by (7.3) that the partial homeomorphism

$$h/\Delta^\bullet\colon\ \Delta^\bullet\to\Delta'^\bullet$$

can be extended to a homeomorphism

$$h_\Delta\ :\ A\cap\Delta\to A'\cap\Delta'.$$

Setting

$$h'(x)=h_\Delta(x)\quad\text{ for every point } x\in A\cap\Delta,\text{ where }\Delta\in T,$$

we get a homeomorphism h' mapping A onto A'. Thus the proof of (7.4) is finished.

It follows by (7.4) that, from the topological point of view, *there exists only one S-curve in each surface M*; it will be denoted by A_M.

8. The boundary and the interior of A_M. Consider now in a surface M an S-curve A given by formula (7.1). Let us prove that

(8.1) *None of the curves $D_i^\bullet$ decomposes A at any point $a\in D_i^\bullet$.*

It suffices to prove that for every $\varepsilon>0$ there exists a neighborhood U of a in A such that every two points $a_1,a_2\in U-D_i$ can be joined in the set $A-D_i$ by a continuum with diameter less than ε. First let us observe that (7.2) implies that there exists a positive number $\eta<\frac{1}{3}\varepsilon$ such that the distance between the point a and every disk D_j with $j\neq i$ and with diameter $\geqslant\frac{1}{3}\varepsilon$ is greater than η. Moreover, since the set $M-D_i\supset A-D_i$ is uniformly locally connected, there exists a neighborhood U of a in A such that every two points $a_1,a_2\in U-D_i$ can be joined in $M-D_i$ by an arc L with diameter less than η. Let R denote the set of all indices k such that $L\cap D_k^\bullet\neq 0$. Then lemma (4.1) implies that the set

$$X'=(L-\bigcup_{k\in R}D_k^\circ)\cup\bigcup_{k\in R}D_k^\bullet$$

is a continuum joining the points a_1,a_2 in the set $A-D_i$ and its diameter is less than ε, because $\delta(L)<\eta<\frac{1}{3}\varepsilon$ and $\delta(D_k)<\frac{1}{3}\varepsilon$ for every index $k\in R$. Thus the proof of (8.1) is finished.

By an analogous argument we show that

(8.2) *$D_i^\bullet$ does not locally decompose A.*

Now let us prove that

(8.3) *If $a\in A-\bigcup_{i=1}^{\infty} D_i$, then every simple closed curve $C\subset A$ containing the point a decomposes A at a.*

Proof. Evidently there exists for every $\varepsilon>0$ a disk $D\subset M$ with diameter less than ε such that $a\in D^\circ$ and that the set $D^\circ-C$ is the union of two regions G_1 and G_2. In order to prove that C locally decomposes A at the point a it suffices to show that each of the sets $A \cap G_1$ and $A \cap G_2$ is non-empty. Suppose, on the contrary, that $A \cap G_1 = 0$. Then there is an index i_0 such that $G_1 \subset D^\circ_{i_0}$. Since $a \in \overline{G}_1$, we conclude that $a \in D^\bullet_{i_0}$, which contradicts the relation $a \in A - D^\circ_{i_0}$. By the same argument we infer that $A \cap G_2 \neq 0$. Thus the proof of (8.3) is concluded.

It follows by (8.1) and (8.3) that the subset $A^\bullet$ of A given by the formula

(8.4) $$A^\bullet = \bigcup_{i=1}^{\infty} D_i^\bullet$$

is topologically marked out in the curve A. This set $A^\bullet$ will be said to be the *boundary* of the S-curve A, and the set

$$A^\circ = A - A^\bullet$$

will be said to be the *interior* of A. We can easily see that the set A° coincides with the subset of A consisting of all points x such that every arc $L \subset A$ containing x in its interior locally decomposes A at the point x.

Now let us consider an arbitrarily given subcompactum Y of M of dimension less than 2. Evidently there exists in the set $M - Y$ a sequence $\{D_i\}$ of mutually disjoint disks satisfying (7.2) and such that the set $\bigcup_{i=1}^{\infty} D_i$ is dense in M. It follows that the set A, given by the formula (7.1), is an S-curve in M and that $Y \subset A^\circ$. Hence

(8.5) *If Y is a subcompactum of M and* $\dim Y \leqslant 1$, *then there exists an S-curve in M containing Y in its interior.*

9. Two lemmas. In this section we establish two lemmas which we need for the proof of the principal theorem in Section 11. In both lemmas A denotes an S-curve in a surface M given by the formula (7.1).

(9.3) LEMMA. *Let $L \subset A^\circ$ be an arc and let x_0 be a point of its interior. Then there exist in A:*

(i) *A sequence $\{C_n\}$ of simple closed curves with diameters converging to zero such that the set $L_n = C_n \cap L$ is an arc containing x_0 in its interior for every $n = 1, 2, \ldots$*

(ii) *An arc L' starting from x_0 and disjoint with all sets $C_n - (x_0)$.*

Proof. Consider a disk $D' \subset M$ containing x_0 in its interior and such that $D'^\circ - L$ is the union of two distinct regions G_1 and G_2. It follows by (7.2) that for every natural number n there exists a disk $D'_n \subset D'$ with the following properties:

(9.2) $$x_0 \in D'^\circ_n .$$

(9.3) $$\delta(D'_n) < 1/3n .$$

(9.4) $$\text{If } D_i \cap D'_n \neq 0 \quad \text{then} \quad D_i \subset D'^\circ \text{ and } \delta(D_i) < 1/3n .$$

Let F_n denote the union of the set $D'^\bullet_n \cap A$ and of all curves $D_i^\bullet$ such that $D_i^\circ \cap D'^\bullet_n \neq 0$. By Lemma (4.1) we infer that F_n is a local continuum with diameter less than $1/n$ which decomposes the disk D' between its boundary D'° and the point x_0. It follows that F_n contains a simple closed curve C'_n which is the boundary of a disk $D''_n \subset D'$ containing x_0 in its interior. Evidently there is an arc $L'_n \subset C'_n$ with the interior in G_1 and with end-points a, b belonging to different components of the set $L - (x_0)$. We infer that a and b are end-points of another arc $L_n \subset L$ containing the point x_0. Setting

$$C_n = L_n \cup L'_n \quad \text{for} \quad n = 1, 2, \ldots,$$

we get a sequence of simple closed curves satisfying condition (i).

Let us denote by $G_{2,\varepsilon}$, for every $\varepsilon > 0$, the set of all points $x \in G_2$ with $\varrho(x, x_0) < \varepsilon$. Since $x_0 \in A^\circ$, we infer by (7.2) that there exists a positive number ε_0 so small that $D_i \cap D_{2,\varepsilon_0} \neq 0$ implies $D_i \subset G_2$. Now let us consider an arc L_0 starting from x_0 and such that $L_0 - (x_0) \subset G_{2,\varepsilon_0}$. Since $x_0 \in A^\circ$, there is a point $x_1 \in (L_0 - A) - (x_0)$. Let R denote the set of all indices i such that $D_i^\bullet \cap L_0 \neq 0$. It follows by (4.1) that the set

$$F = \Big(L_0 - \bigcup_{i \in R} D_i^\circ\Big) \cup \bigcup_{i \in R} D_i^\bullet$$

is a locally connected subcontinuum of A containing the points x_0 and x_1 and such that $F - (x_0) \subset G_2$. This continuum contains an arc L' with end-points x_0 and x_1. It is clear that this arc satisfies condition (ii). Thus the proof of Lemma (9.1) is finished.

(9.5) LEMMA. *If a compactum $X \subset A^\circ$ does not decompose the surface M, then the set $A - X$ is connected.*

Proof. Let $a, b \in A - X$. Then there is an arc $L \subset M - X$ with end-points a and b. Let Z denote the union of L and of all curves $D_i^\bullet$ such that $L \cap D_i^\bullet \neq 0$. It follows by Lemma (4.1) that the set $Z \cap A = Z - \bigcup_{i=1}^{\infty} D_i^\circ$ is a subcontinuum of the set $A - X$ containing the points a and b. Hence the set $A - X$ is connected.

10. Embedding S-curves in surfaces. Let us prove the following

(10.1) THEOREM. *If M and M' are surfaces and either M is orientable or M and M' are both non-orientable, then $\gamma(M) < \gamma(M')$ implies that A_M is homeomorphic to a subset of $A_{M'}$.*

Proof. First let us assume that M is orientable and that M' is another surface, its genus $m' = \gamma(M')$ being greater than the genus $m = \gamma(M)$ of M. It follows by (1.9) that M' is a surface which we get from M by replacing $k = m'-m$ disjoint disks $D_1, D_2, \ldots, D_k$ lying on M by k disjoint bounded surfaces $N_1, N_2, \ldots, N_k$, each of them homeomorphic either to the perforated tore P or to the Möbius band Q, and that $N_i \cap M = N_i^\bullet \cap M = D_i^\bullet$ for $i = 1, 2, \ldots, k$. By (7.4) and (7.1) we may assume that the system of disks $D_1, D_2, \ldots, D_k$ may be completed to the sequence of disjoint disks $\{D_i\}$ such that $A_M = M - \bigcup_{i=1}^{\infty} D_i^\circ$. It follows that $A_M \subset M'$, whence in the case where M is orientable, the proof of (10.1) is finished.

The proof in the case where M and M' are non-orientable is analogous, but in that case we can assume that each of the bounded surfaces $N_1, N_2, \ldots, N_k$ (where $k = m'-m$) is homeomorphic to the perforated tore P.

11. S-curves which cannot be embedded in a given surface. As a complement to Theorem (10.1) let us prove the following

(11.1) THEOREM. *Let M and M' be two surfaces. The curve A_M is not homeomorphic to any subset of M' in the following three cases*:

Case 1. *M is non-orientable and M' is orientable.*

Case 2. $\gamma(M) > \gamma(M')$.

Case 3. *M is orientable and M' is non-orientable with $\gamma(M) = \gamma(M')$.*

Proof. In case 1, the surface M contains a Möbius band. Evidently the equator of this band is a simple closed curve C such that if we cut M along it, then we get from C another simple closed curve B, and from $M-(a)$ bounded surface N having B as its boundary. Then there exists a map φ of N onto M which is a homeomorphism on the set $N^\circ = N - B$. If we identify B with the boundary of a disk D, we get from N and D a surface M^*.

By (8.5) we can assume that C is a subset of the interior A° of the S-curve $A = A_M$. By cutting M along C and by matching B with the boundary of the disk D, we get from A a curve A^* on the surface M^*. Manifestly, if

$$A = M - \bigcup_{i=1}^{\infty} D_i^\circ$$

then

$$A^* = M^* - \Big(\bigcup_{i=1}^{\infty} \varphi^{-1}(D_i^\circ) \cup D^\circ\Big),$$

and consequently A^* is an S-curve on M^* containing B in its boundary. It follows by (8.1) that B has arbitrarily small neighborhood U^* in A^* with U^*-B connected. Since the partial map $\hat{\varphi} = \varphi/(N-B)$ is a homeomorphism of the set $N-B$ onto $M-C$, and since $A^*-B \subset N-B$, we infer that the set $U = C \cup \hat{\varphi}(U^*-B)$ is a neighborhood of C in A such that the set $U-C$ is connected. Moreover, if the neighborhood U^* of B in the space A^* is sufficiently small, then the neighborhood U of C in the space A is arbitrarily small. Thus we have shown that

(11.2) *There exists in A a simple closed curve C having arbitrarily small neighborhoods U in A with $U-C$ connected.*

Suppose now that there exists a homeomorphism h mapping the curve A onto a subset of an orientable surface M'. Then $C' = h(C)$ is a simple closed curve on M' and (1.3) implies that C' has a neighborhood W (in M') homeomorphic to an annulus. Then $C' \subset W^\circ$ and $W^\circ - C'$ is the union of two regions G_1, G_2 with $\overline{G}_1 \cap \overline{G}_2 = C'$. Consider an arc $L \subset C$ and a point $x_0 \in L^\circ$. Since $C \subset A^\circ$, we infer by Lemma (9.1) that there exists in A a sequence of simple closed curves $\{C_n\}$ and an arc L' satisfying conditions (i) and (ii) of this Lemma. Then at least one of the sets $\overline{G}_1, \overline{G}_2$ (say, the set $\overline{G}_1$) contains an infinite collection of simple closed curves $h(C_n)$. Then $x_0' = h(x_0)$ is an accumulation point for the set $h(A) \cap G_1$. Moreover, the set $h(C_n) \cap h(L)$ is a subarc of the arc $h(L)$ for every $m = 1, 2, \ldots$ If we recall that x_0' is an end-point of the arc $h(L')$ and that $h(C_n) \cap h\big(L' - (x_0')\big) = 0$, we infer that $h\big(L' - (x_0')\big) \subset G_2$. Hence x_0' is also an accumulation point of the set $h(A) \cap G_2$. It follows that for every neighborhood $V \subset W$ of the curve C' in the set $h(A)$, the set $V - C'$ is not connected. But this contradicts proposition (11.2) since h is a homeomorphism. Thus in case 1 the proof is concluded.

Passing to case 2, consider in M a system of $m = \gamma(M)$ mutually disjoint simple closed curves $C_1, C_2, \ldots, C_m$, which together do not decompose M. By (8.5), we can assume that there exists in M an S-curve A such that $C_1 \cup C_2 \cup \ldots \cup C_m \subset A^\circ$. It follows by Lemma (9.5) that

(11.3) $$\textit{The set } A - \bigcup_{i=1}^{\infty} C_i \textit{ is connected.}$$

Suppose now that there is a homeomorphism h mapping A onto a subset $h(A)$ of the surface M'. Then $h(C_1), h(C_2), \ldots, h(C_m)$ are simple closed curves in M', disjoint with one another. Since $m' - \gamma(M')$ is less that m, we infer that the set $M' - \bigcup_{i=1}^{m} h(C_i)$ contains two distinct components G_1 and G_2 such that for an index i_0 the curve $h(C_{i_0})$ is contained in the boundary of G_1 and also in the boundary of G_2. Then the set $G_1 \cup G_2 \cup h(C_{i_0})$ is a neighborhood in M' of the curve $h(C_{i_0})$ decomposing this neighborhood into two regions G_1 and G_2. Applying Lemma (9.1), we infer that for every point $x_0' \in h(C_{i_0})$ there exists in $h(A)$ a sequence $\{C_n'\}$ of simple closed curves with diameters converging to zero and such that $L_n = C_n' \cap h(C_{i_0})$ is an arc containing the point

x_0' in its interior for each $n = 1, 2, \ldots$ Moreover, there exists an arc $L' \subset h(A)$ starting from x_0' and disjoint with all sets $C_n' - (x_0')$. We infer, as in the proof in case 1, that x_0' is an accumulation point for the sets $h(A) \cap G_1$ and $h(A) \cap G_2$. Since G_1 and G_2 are distinct components of the set $M - \bigcup_{i=1}^{m} h(C_i)$, we infer that the set $\bigcup_{i=1}^{m} h(C_i)$ decomposes the set $h(A)$, which by (11.3) is impossible, because h is a homeomorphism.

It remains to consider case 3. Then $\gamma(M) = \gamma(M') = m$ and let $C_1, C_2, \ldots, C_m$ be a system of mutually disjoint simple closed curves on M which together do not decompose M. By (8.5), the S-curve A on M (given by formula (7.1)) can be found so that $\bigcup_{i=1}^{m} C_i \subset A^\circ$. It follows by Lemma (9.5) that the set $W = A - \bigcup_{i=1}^{m} C_i$ is connected.

Now let us suppose that there exists a homeomorphism

$$h\colon A \to A' \subset M'.$$

Consider a simple closed curve C, which is the boundary of one of the disks D_i. Then the system of curves $h(C), h(C_1), \ldots, h(C_m)$ decomposes the surface M'. Consequently there are at least two distinct components G' and G'' of the set $M' - h(C) - h(C_1) - \ldots - h(C_m)$, both containing $h(C)$ is their boundaries and such that the whole (connected) set $h(W)$ lies in one of them (say in G''). Then the set $N = \overline{G'}$ is a bounded surface, having the curve $h(C)$ as its boundary. If G' contains a simple closed curve C' which does not decompose the bounded surface N, then the mutually disjoint curves $C', h(C_1), \ldots, h(C_m)$ together do not decompose the surface M'. But this contradicts the hypothesis that $\gamma(M') = m$. It follows by (2.2) that N is a disk.

Thus we have shown that each of the curves $h(D_i^\bullet)$ is the boundary of a disk D_i' such that $D_i'^\circ \subset M' - h(A)$. Since the diameters of the sets $h(D_i^\bullet)$ converge to zero, the diameters of the disks D_i converge to zero. It follows that if we extend the homeomorphism $h/D_i^\bullet$ to a homeomorphism of the whole disk D_i onto D_i' for every $i = 1, 2, \ldots$, we get a homeomorphism h' of the whole surface M onto a subset of M'. But this is possible only if $h'(M) = M'$. If we recall that M is assumed to be orientable, we infer that M' must also be orientable. Thus the proof of theorem (11.1) is concluded.

12. Final remarks. It follows by Theorem (11.1) that two S-curves A_M and $A_{M'}$ are homeomorphic if and only if the surfaces M and M' are homeomorphic. Thus the correspondence between the topological types of surfaces M and of the curves A_M is one-to-one. Since the decomposition of an n-dimensional manifold into the Cartesian product of 1-dimensional and 2-dimensional factors is topologically unique

([2], p. 296), the question arises whether a decomposition of finite-dimensional compacta into the Cartesian products of S-curves is topologically unique. In particular the following case seems to be interesting. Let P be the perforated tore, R the annulus and N a 2-sphere with three holes. By a remark due to J. H. C. Whitehead ([12], p. 827) the Cartesian products $P \times R$ and $N \times R$ are homeomorphic. But P may be considered as the first approximation of the S-curve A_T, where T denotes the tore, and N is an approximation of the S-curve A_{S^2} (the usual universal plane curve of Sierpiński). Thus the following problem arises

(12.1) *Are the Cartesian products $A_T \times A_{S^2}$ and $A_{S^2} \times A_{S^2}$ homeomorphic?*

If the answer is positive, then the problem concerning the uniqueness of the decomposition of finite-dimensional continua into Cartesian products of curves ([3], p. 110) is solved negatively. Let us observe that for finite-dimensional ANR-spaces the uniqueness of the Cartesian decomposition into 1-dimensional factors has recently been proved by H. Patkowska ([7]).

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