A short proof of a stronger version of Popruženko's Theorem

by

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Using a rather elaborate method and proof, J. Popruženko has shown (see Fund. Math. 53 (1962), pp. 13-19) that every partially ordered set is *quasi-isomorphic* to a set of sequences of 0 and 1, each with a last non-zero term and ordered by the principle of first differences.

By a *quasi-isomorphism* between two partially ordered sets $(P, \leqslant)$ and $(S, \precsim)$ we mean a one-to-one mapping from P onto S such that $x \leqslant y$ implies $f(x) \precsim f(y)$.

Below we offer a very short and constructive proof of a stronger version of J. Popruženko's remarkable Theorem.

THEOREM. *Every partially ordered set of power $\aleph_\tau$ is quasi-isomorphic to a set S of sequences of type ω_τ made up of $0, 1$, each with a last non-zero term and ordered by the principle of first differences $\precsim$.*

Moreover, for every ordinal $\lambda < \omega_\tau$ there exists an element (s_i) of S such that $s_\lambda = 1$ and $s_i = 0$ for every $i > \lambda$.

Proof. Let $(p_i)_{i<\omega_\tau}$ be a well ordering of a partially ordered set $(P, \leqslant)$ of power $\aleph_\tau$. For $p_j \in P$ let $f(p_j) = (s_i^j)_{i<\omega_\tau}$ such that

$$s_i^j = \begin{cases} 1 & \text{if} \quad p_i \leqslant p_j \text{ and } i \leqslant j, \\ 0 & \text{otherwise}. \end{cases}$$

For $\lambda < \omega_\tau$, clearly, $f(p_\lambda) = (s_i^\lambda)$ is such that $s_\lambda^\lambda = 1$ and $s_i^\lambda = 0$ for $i > \lambda$. Thus, f maps P onto S (the range of f) and S satisfies conditions set forth in the Theorem.

Next, we prove that f is a one-to-one mapping.

Let $f(p_j) = (s_i^j)_{i<\omega_\tau} = (s_i^k)_{i<\omega_\tau} = f(p_k)$. Then $s_j^j = 1$ implies $s_j^k = 1$ which implies $p_j \leqslant p_k$. Similarly, by symmetry, $s_k^k = 1$ implies $p_k \leqslant p_j$. Hence, $p_j = p_k$, as desired.

Finally, we prove that f preserves order.

Let $j, k < \omega_\tau$ and $p_j < p_k$. Then for every $i \leqslant k$ we see that $s_i^j = 1$ implies $p_i \leqslant p_j \leqslant p_k$ and consequently $s_i^k = 1$. However, $s_k^j = 0$ and $s_k^k = 1$ and therefore $f(p_j) \precnsim f(p_k)$.

Thus, f is the desired quasi-isomorphism and the Theorem is proved.

We observe that the construction of s_i^j except for the condition $i \leqslant j$, which is most essential for the proof of the Theorem, resembles a similar construction given in [1] of [2].

As an immediate consequence of the above theorem we have [3]:

COROLLARY. *Every partial order in a set P can be extended to a simple order in the same set P preserving the original order among the elements of P.*

References

[1] E. Mendelson, *Appendix* [2].

[2] W. Sierpiński, *Cardinal and Ordinal Numbers*, Monografie Matematyczne Vol. 34, Warszawa 1958.

[3] E. Szpilrajn-Marczewski, *Sur l'extension de l'ordre partiel*, Fund. Math. 16 (1930), pp. 386-389.

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On embedding curves in surfaces

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I. Preliminaries

1. Elementary properties of surfaces. By a *bounded surface* we understand here a continuum N such that every point of it has a neighborhood which is a disk, i.e. a topological image of the square. In particular, the disk, the circular ring (annulus) and the Möbius band are bounded surfaces. The points of a bounded surface N for which no neighborhood is homeomorphic to the Euclidean plane E^2 constitute a set $N^\bullet$ called the *boundary* of N. The set $N - N^\bullet$ is said to be the *interior* of the bounded surface N; it will be denoted by N°. The set $N^\bullet$ is the union of a finite number of simple closed curves disjoint with one another. If we match each of these curves with the boundary of a disk, then we obtain from N another bounded surface M with an empty boundary, i.e. a *closed surface*, or simply a *surface*. Hence every bounded surface is homeomorphic to a subset of a surface.

A bounded surface is said to be *orientable* if it does not contain topologically the Möbius band. All other bounded surfaces are said to be *non-orientable*.

A subset X_0 of a space X is said *to have arbitrarily small neighborhoods* (in X) with a property (α) provided every neighborhood of X_0 contains a neighborhood of X_0 with property (α). If there exists a neighborhood U_0 of X_0 such that every neighborhood of X_0 contained in U_0 has property (α), then we say that the property (α) holds for all sufficiently small neighborhoods of X_0.

Let us formulate some elementary properties of surfaces:

(1.1) *Each closed subset of a surface M has arbitrarily small neighborhoods (in M) which are bounded surfaces.*

(1.2) *Each arc (and also each disk) lying on a surface M has arbitrarily small neighborhoods which are disks.*

(1.3) *If C is a simple closed curve lying on a surface M, then only the following two cases are possible:* (i) *C has arbitrarily small neighborhoods homeomorphic to the Möbius band.* (ii) *C has arbitrarily small neighborhoods homeomorphic to the annulus.*