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A structure theory for a class of lattice ordered semirings *

by

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Introduction. A *semiring* S is a set of elements which is closed under two binary associative commutative operations $(+)$ and $(\cdot)$ such that $a(b+c) = ab+ac$ for all $a, b, c \in S$, containing elements 0 and 1 such that $s+0 = s$ and $s\cdot 1 = s$ for all $s \in S$. A semiring S will be called *positive* if $1+s$ has a multiplicative inverse for all $s \in S$.

The theory of semirings is relatively new. Bourne in [1] and Bourne and Zassenhaus in [2] have presented some partial results generalizing the Jacobson structure theory for semirings. In 1955, Słowikowski and Zawadowski in [7] studied the structure space of commutative positive semirings. Although they did not attempt to present an algebraic structure theory, their work seemed to indicate that a structure theory was possible for this class of semirings.

If S is a semiring, let $T(S) = \{x \in S\colon x+x = x\}$ and $K(S) = \{x \in S\colon x+a = x+b \text{ implies } a = b\}$. These elements will be called respectively the *a-idempotent* and *a-cancellable* elements. In Section 1 we prove that every positive semiring is a-idempotent or contains a copy of the non-negative rational numbers.

On every semiring S there is a natural quasi-order defined by letting $a \leqslant b$ if $a+x = b$ is solvable in S. A semiring S will be called an *l-semiring* if S is lattice ordered under the natural quasi-order and $a+(b \vee c) = (a+b) \vee (a+c)$ and $a+(b \wedge c) = (a+b) \wedge (a+c)$ for all $a, b, c \in S$. A semiring S will be called *archimedian* if $nx \leqslant a$ for $n = 1, 2, \ldots$ implies $x \in T$. In Section 3, we show that in a positive archimedian l-semiring S, then $K(S) = \{x \in S\colon x \wedge T = \{0\}\}$ and that if $\{1 \wedge k\colon k \in K\}$ has a supremum in K, then $S = K+\{x \in S\colon x \wedge K = \{0\}\}$. In Section 4, we show that both T and K are an intersection of prime l-ideals but that even under very strong hypothesis, the same is not true for $T+K$.

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In Section 5, we consider semirings which are subdirect sums of subdirectly irreducible archimedian semirings. We show that every positive l-semiring satisfies the f-ring condition of Birkhoff and Pierce [4], and using some of their results prove that every semiring which is a subdirect sum of subdirectly irreducible archimedian semirings is a subdirect sum of a-idempotent semirings and subsemirings of the non-negative real numbers.

1. Preliminaries.

DEFINITION 1.1. A *semiring* is a set of elements S which is closed under two binary operations, addition $(+)$ and multiplication $(\cdot)$, such that the following properties are satisfied:

(i) *Both addition and multiplication are associative and commutative.*

(ii) *Multiplication is distributive over addition*; *i.e.* $a(b+c) = ab+ac$ *for all* $a, b, c \in S$.

(iii) *There are elements* 0 *and* 1 *in* S *such that for every* $s \in S$ *we have*

$$s+0 = s \quad \textit{and} \quad 1\cdot s = s\,.$$

EXAMPLES 1.2. (i) Any commutative ring with identity is a semiring.

(ii) Any distributive lattice with maximal and minimal elements is a semiring.

(iii) The set $C^+(X)$ of all non-negative continuous real-valued functions on a topological space X with the usual pointwise operations is a semiring.

Let S be a semiring. We will say that an element $s \in S$ has an *inverse* if there is an element $s^{-1} \in S$ such that $s\cdot s^{-1} = 1$. An element that has an inverse will be called a *unit*. The set of units of S will be denoted by $\Omega(S)$ or, more simply, by Ω.

DEFINITION 1.3. A semiring S will be called *positive* if $1+s$ is a unit for every $s \in S$.

EXAMPLES 1.3. (ii) and (iii) are positive but (i) is not.

Since we are primarily interested in positive semirings, we will consider only positive semirings unless more generality can easily be maintained.

PROPOSITION 1.4. *If* S *is a positive semiring, then* 0 *is multiplicative annihilator.*

Proof. Let $x \in S$. Then $0x = 0x+0 = 0x+01 = 0(1+x)$, so

$$0 = 0\cdot 1 = 0[(1+x)^{-1}+x(1+x)^{-1}] = 0(1+x)^{-1}+0x(1+x)^{-1}$$
$$= 0(1+x)^{-1}+0(1+x)(1+x)^{-1} = 0(1+x)^{-1}+0\cdot 1 = 0(1+x)^{-1}\,.$$

Hence

$$0 = 0\cdot 1 = 0(1+x)^{-1}(1+x) = 0(1+x) = 0x\,.$$

In [7], Corollary 5, p. 219, structure theory is used to prove a weaker result than the preceding.

A non empty subset I of a semiring S will be called an *ideal* if for all $a, b \in I$ and $s \in S$, $s(a+b) \in I$. An ideal P will be called *prime* if $ab \in P$ is equivalent to $a \in P$ or $b \in P$. An ideal I will be called a *k-ideal* if a, $a+b \in I$ implies $b \in I$. An ideal I will be called an *l-ideal* if $a+b \in I$ is equivalent to $a \in I$ and $b \in I$.

A mapping F from a semiring S into a semiring S' will be called a *homomorphism* if for all $a, b \in S$, $F(a\cdot b) = F(a)\cdot F(b)$, $F(a+b) = F(a)+F(b)$ and $F(0) = 0$. An equivalence relation θ on a semiring S is called a *congruence relation* if $a \equiv b(\theta)$ implies $a+s \equiv b+s(\theta)$ and $as \equiv bs(\theta)$ for all $s \in S$.

Let S be a semiring and θ a congruence relation on S. Let $[a] = \{x \in S\colon x \equiv a(\theta)\}$ and let $S_\theta = \{[a]\colon a \in S\}$. If we define $[a]+[b] = [a+b]$ and $[a]\cdot[b] = [ab]$, then S_θ is a semiring and the mapping $F\colon S \to S_\theta$ is a homomorphism of S onto S_θ. This homomorphism will be called the *natural homomorphism* of S onto S_θ.

Let I be an ideal of S. We will say that $a \equiv b(\varphi_I)$ or, more simply, $a \equiv b(I)$ if there are elements $i, j \in I$ such that $a+i = b+j$. This is easily seen to be a congruence relation. Thus $S/I = S_{\varphi_I}$ is a semiring. Note that $[0]$ is a k-ideal; in fact $[0]$ is the smallest k-ideal containing I. We have just proved the following proposition.

PROPOSITION 1.5. *If* $F\colon S \to S'$ *is a homomorphism, then* $\{s \in S\colon F(s) = 0\}$ *is a* k*-ideal. Conversely, if* θ *is a congruence relation, then there is a semiring* S_θ *and a homomorphism* F *such that* $F(a) = F(b)$ *if and only if* $a \equiv b(\theta)$.

DEFINITION 1.6. A semiring S will be called *semisimple* if the intersection of all the maximal ideals is $\{0\}$.

In [7], Theorem 5, p. 219, Słowikowski and Zawadowski proved the following:

PROPOSITION 1.7. *A positive semiring* S *is semisimple if and only if for every* $x \neq 0$ *in* S, *there is a non-unit* y *in* S *such that* $x+y \in \Omega$.

We will now introduce two subsets of a semiring that will play an important role.

DEFINITION 1.8. If S is a semiring, let

(i) $T(S) = \{x \in S\colon x+x = x\}$,

(ii) $K(S) = \{x \in S\colon x+a = x+b \text{ implies } a = b\}$.

We will call $T = T(S)$ the set of *a-idempotent* elements of S, and $K = K(S)$ the set of *a-cancellable* elements of S. If $S = T$, then S will be called an *a-idempotent* semiring, and if $S = K$, then S will be called an *a-cancellable* semiring.

It is easily seen that T is an ideal.

We will denote by N the set of all positive integers.

EXAMPLES 1.9. (i) For each $n \in N$, let $S_n = \{0, x^0, x^1, \ldots, x^n\}$ and let $a+a = a$, $x^0+a = a+x^0 = x^0$, $0\cdot a = a\cdot 0 = 0$ for all $a \in S_n$, $x^ix^j = x^{\min[n,i+j]}$. Clearly S_n is a positive semiring with identity element x^0 for each $n \in N$.

(ii) For each $n \in N$, let $S^n = \{0, x^0, x^1, \ldots, x^n\}$ with addition as in (i) and multiplication defined by $x^ix^j = x^{i+j}$ if $i+j \leqslant n$, $x^ix^j = 0$ if $i+j > n$, and $0\cdot a = a\cdot 0 = 0$ for all $a \in S^n$. Clearly S^n is a positive semiring with identity element x^0, for each $n \in N$.

(iii) Let $S = \{0, x^0, x^1, \ldots, x^n, \ldots\}$ with addition defined as in (i) and multiplication defined by $x^i\cdot x^j = x^{i+j}$ and $0\cdot a = a\cdot 0 = 0$ for all $a \in S$. Clearly S is a positive semiring with identity element x^0. Notice that S^n and S_n are homomorphic images of S for each $n \in N$.

PROPOSITION 1.10. *If S is a positive semiring, then K is an l-ideal.*

Proof. Clearly K is closed under addition. Let $k \in K$ and $s \in S$. If $ks+a = ks+b$, then $k(1+s)+a = k(1+s)+b$. Thus $k+a(1+s)^{-1} = k+b(1+s)^{-1}$, so $a(1+s)^{-1} = b(1+s)^{-1}$, whence $a = b$. Thus K is an ideal. Let $x+y \in K$. If $x+a = x+b$, then $x+y+a = x+y+b$, whence $a = b$, so $x \in K$. Thus K is an l-ideal.

If S fails to be positive, K need not be an ideal. For, let S be the non-negative integers together with a distinguished point $\{p\}$. Add and multiply non-negative integers as usual, let $n+p = p$ for every $n \in S$, let $np = p$ for every $0 \neq n \in S$, and let $0p = 0$. It is easily verified that S is a semiring which is not positive since $1+p = p$ is not a unit. Now $K(S)$ is the set of non-negative integers, which is not an ideal.

LEMMA 1.11. *Let S be a positive semiring with multiplicative identity element e. If there exist positive integers $n, m \in N$ such that $ne+me = ne$, then S is a-idempotent.*

Proof. We may assume that $m > 1$ since $ne = ne+me = ne+2me$ and $2m > 1$. If $p \geqslant n$ is an integer, then $pe = pe+qme$ for every non-negative integer q. Let $r > 1$ be an integer such that $m^r > n$. Now $m^re = m^re+m^{r-1}me = 2m^re$ since $m^r > n$. But S is positive, hence m^re is unit, whence $e = 2e$, so S is a-idempotent.

Let Q^+ denote the semiring of non-negative rational numbers with the usual operations.

THEOREM 1.12. *If S is a positive semiring, then either S is a-idempotent or S contains a copy of Q^+.*

Proof. Let e be the multiplicate identity element of S. Define $F\colon Q^+ \to S$ by $F(p/q) = pe(qe)^{-1}$. It is easily seen that F is a homomorphism. If $F(p/q) = F(r/s)$, then $(pe)(qe)^{-1} = (re)(se)^{-1}$, so $pse = qre$. Now either $ps = qr$, whence $p/q = r/s$ and F is an isomorphism, or $ps \neq qr$, so by Lemma 1.11, S is a-idempotent.

COROLLARY 1.13. *Every finite positive semiring is a-idempotent.*

2. l-semirings.

Every semiring S has a *natural quasi-order* defined by letting $a \leqslant b$ if $a+x = b$ is solvable in S.

DEFINITION 2.1. A semiring S is said to be *partially ordered* if S is partially ordered under the natural quasi-order.

PROPOSITION 2.2. *A semiring S is partially ordered if and only if $a+x = a$ implies $a+y = a$ for all $y \leqslant x$.*

Clearly every subsemiring S' of a partially ordered semiring S is partially ordered but the partial order on S' need not be the same as the partial order on S.

EXAMPLE 2.3. Let $S = \{f\colon [0,1] \to R^+\}$ and $S' = \{f\colon [0,1] \to R^+\colon f(0) \neq 0\} \cup \{0\}$. Clearly S' is a subsemiring of S and S' is partially ordered, however 1 and $1+x$, where x denotes the identity function, are comparable in S but incomparable in S'.

PROPOSITION 2.4. *An ideal I in a partially ordered semiring is an l-ideal if and only if $a \leqslant b$ and $b \in I$ imply $a \in I$.*

DEFINITION 2.5. If A is a subset of a semiring S, then $\alpha(A) = \{x \in S\colon xA = \{0\}\}$ will be called the *annihilator* of A.

If A is a subset of a partially ordered semiring S, then $\alpha(A)$ is an l-ideal.

DEFINITION 2.6. A partially ordered semiring S will be called *archimedian* if $nx \leqslant a$ for every $n \in N$ implies $x \in T$.

PROPOSITION 2.7. *If S is an archimedian semiring, then T is an l-ideal.*

The following example shows that T need not be an l-ideal if S fails to be archimedian.

EXAMPLE 2.8. Let $S = Q_1^+ \cup Q_2^+$ where Q_1^+ and Q_2^+ are disjoint copies of Q^+, with the following operations $\oplus$, $\odot$. If $a_i, b_i \in Q_1^+$ for $i = 1, 2$, then $a_i \oplus b_i = a_i + b_i$, $a_1 \oplus b_2 = b_2 \oplus a_1 = a_1$, $a_1 \odot b_1 = a_1b_1$, $a_2 \odot b_1 = b_1 \odot a_2 = (ab)_2$, and $a_2 \odot b_2 = 0_2$. S is easily seen to be totally ordered. But S is not archimedian since $n(1_2) = n_2 \leqslant 0_1$ for all $n \in N$. Now $T = \{0_1, 0_2\}$ is not an l-ideal since $1_2 \leqslant 0_1 \in T$ and $1_2 \notin T$.

Geometrically this example can be visualized as two disjoint copies of the non-negative rational number with the upper one absorbing the lower one under addition and the lower one absorbing the upper one under multiplication (see Fig. 1).

Definition 2.9. A semiring S will be called an *l-semiring* if S is a lattice under the natural quasi-order and both $a+(b \vee c) = (a+b) \vee (a+c)$ and $a+(b \wedge c) = (a+b) \wedge (a+c)$ hold for all $a, b, c \in S$.

Consider the following lattice S (see Fig. 2). Define $x+y = x \vee y$, $x \cdot y = 0$ unless x or $y = 1$ and $1 \cdot x = x$ for all $x, y \in S$. S is easily seen to be a positive semiring, and $(S, \leqslant)$ is clearly a lattice. But $a+(b \wedge c) = a$, while $(a+b) \wedge (a+c) = d$, so S is not an l-semiring.

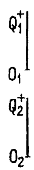

Fig. 1

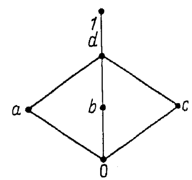

Fig. 2

Proposition 2.10. *If S is an l-semiring and $a, b, c \in S$, then:*

(i) $(a \wedge b)+(a \vee b) = a+b$.

(ii) *If* $a \wedge b = 0$, *then* $a \wedge (b+c) = a \wedge c$.

(iii) *If u is a unit, then* $u(a \wedge b) = ua \wedge ub$ *and* $u(a \vee b) = ua \vee ub$.

(iv) *If* $t \in T$, *then* $t \wedge (a+b) = (t \wedge a)+(t \wedge b)$.

(v) *If* $a \equiv b(I)$, *where I is an l-ideal, then* $a \vee b \equiv a \wedge b(I)$.

Let $\{S_a\colon a \in \Gamma\}$ be a family of semirings. We construct the complete direct sum and the subdirect sum of the family $\{S_a\}$ in the usual fashion and denote them by $\times\{S_a = S_a\colon a \in \Gamma\}$ and $\sum_s S_a = \sum_s\{S_a\colon a \in \Gamma\}$.

Remark 2.11.

1) If each S_a is positive, archimedian, a-cancellable, a-idempotent or an l-semiring, then so is $\times S_a$.

2) If each S is a-idempotent, a-cancellable or archimedian, then so is $\sum_s S_a$.

Example 2.3 is a semiring which is a subdirect sum of totally ordered semirings, but is not an l-semiring.

Definition 2.12. A semiring S is said to be *subdirectly irreducible* if every isomorphism θ of S onto a subdirect sum $\sum_s\{S_a\colon a \in \Gamma\}$ of a family of semirings is such that the mapping $s \to [\theta(s)]_a$ is an isomorphism for at least one $a \in \Gamma$.

If S is not subdirectly irreducible, it will be called *subdirectly reducible*.

The proofs of the following properties are similar to the ones used in ring theory and are thus omitted.

Proposition 2.13. *If $\{\theta_a, a \in \Gamma\}$ is a collection of non-trivial congruence relations on a semiring S such that* $\bigwedge_{a \in \Gamma} \theta_a = 0$, *then S is subdirectly reducible.*

Theorem 2.14. *If S is a subdirectly irreducible l-semiring, then the intersection of every finite collection of non-zero l-ideals is a non-zero l-ideal.*

Corollary 2.15. *If S is an l-semiring and $I_1, I_2, \ldots, I_N$ are non-zero l-ideals such that* $\bigcap_{a=1}^{N} I_a = \{0\}$, *then S is subdirectly reducible.*

If S is a positive archimedian semiring, then T and K are l-ideals, so we have:

Corollary 2.16. *If S is a subdirectly irreducible positive archimedian l-semiring, then* $T = \{0\}$ *or* $K = \{0\}$.

Remark 2.17. If a and b are distinct elements of a semiring S such that for every non-zero congruence relation θ on S we have $a \equiv b(\theta)$, then S is subdirectly irreducible.

Definition 2.18. If I and J are ideals in a semiring S, then by $I+J$ we mean the *smallest ideal containing both I and J*.

Theorem 2.19. *If S is an archimedian positive l-semiring, then $T+K$ is an l-ideal.*

For the remainder of this paper, we will be concerned only with positive l-semirings. Thus whenever it will be inconvenient to do otherwise, we will state the results only for that case.

3. A decomposition theorem.

Definition 3.1. If S is an l-semiring and A is a subset of S, then $\mathfrak{o}(A) = \{x \in S\colon x \wedge a = 0 \text{ for all } a \in A\}$ is called the *orthogonal complement* of A.

Proposition 3.2. *If S is a positive l-semiring and $A \subset S$, then $\mathfrak{o}(A)$ is an l-ideal.*

Proof. We need only to show that $\mathfrak{o}(A)$ is closed under multiplication. If $x \in \mathfrak{o}(A)$, $s \in S$ and $a \in A$, then $sx \wedge a \leqslant x(1+s) \wedge a = (1+s)[x \wedge a(1+s)^{-1}] \leqslant (1+s)(x \wedge a) = 0$, whence $sx \in \mathfrak{o}(A)$.

Proposition 3.3. *If S is a positive l-semiring, then $x \wedge y = 0$ implies $xy = 0$.*

Proof. With no loss of generality we may assume $x \leqslant 1$ and $y \leqslant 1$ since, by Proposition 2.10, $x \wedge y = 0$ if and only if $[(1+x)^{-1}(1+y)^{-1}x] \wedge [(1+x)^{-1}(1+y)^{-1}y] = 0$, and $xy = 0$ if and only if $(1+x)^{-2}(1+y)^{-2}xy$

$= 0$. If $x \leqslant 1$ and $y \leqslant 1$, then $xy \leqslant x$ and $xy \leqslant y$, whence $0 \leqslant xy \leqslant x \wedge \wedge y = 0$. Thus $xy = 0$.

COROLLARY 3.4. *If S is a positive l-semiring and $A \subset S$, then* $\mathfrak{d}(A) \subset \mathfrak{a}(A)$.

THEOREM 3.5. *If S is a positive l-semiring, then S is the direct sum of K and $\mathfrak{d}(K)$ if and only if $\{1 \wedge k\colon k \in K\}$ has a supremum in K.*

Proof. If $S = K \oplus \mathfrak{d}(K)$, then there exists $k_0 \in K$ and $c_0 \in \mathfrak{d}(K)$ such that $1 = k_0 + c_0$. Now by Proposition 2.10, $1 \wedge k = (k_0 + c_0) \wedge k = k_0 \wedge k \leqslant k_0$, since $c_0 \wedge k = 0$. Thus $\sup\{1 \wedge k\colon k \in K\} = k_0$ since $k_0 = k_0 \wedge 1 \in \{1 \wedge k\colon k \in K\}$. Conversely, let $k_0 = \sup\{1 \wedge k\colon k \in K\}$. Since $k_0 \leqslant 1$, there is an element $c_0 \in S$ such that $k_0 + c_0 = 1$. Let $k \in K$ and $k' = c_0 \wedge k$. Since $k' \leqslant c_0$, there is an x such that $c_0 = k' + x$. Hence $1 = k_0 + c_0 = k_0 + k' + x$. Thus $k_0 + k' \leqslant 1$, so $k_0 + k' \in \{1 \wedge k\colon k \in K\}$. It follows that $k_0 + k' \leqslant k_0$. Hence, since k_0 is a-cancellable, $k' = 0$, so $c_0 \in \mathfrak{d}(K)$. Thus for any $x \in S$, $x = xc_0 + xk_0$, so $S = \mathfrak{d}(K) \oplus K$.

THEOREM 3.6. *If S is a positive l-semiring and $\{1 \wedge k\colon k \in K\}$ has a supremum in K, then $\mathfrak{a}(K) = \mathfrak{d}(K)$.*

Proof. By the preceding theorem $S = K \oplus \mathfrak{d}(K)$, so $1 = c_0 + k_0$. For some $c_0 \in \mathfrak{d}(K)$ and $k_0 \in K$. Let $x \in \mathfrak{a}(K)$. Now $x = xc_0 + xk_0 = xc_0 \in \mathfrak{d}(K)$, so $\mathfrak{d}(K) \subset \mathfrak{a}(K)$. Then by Corollary 3.4 $\mathfrak{d}(K) = \mathfrak{a}(K)$.

COROLLARY 3.7. *If S is a positive l-semiring, then $S = K \oplus \mathfrak{a}(K)$ if and only if $\{1 \wedge k\colon k \in K\}$ has a supremum in K.*

The following example shows that $\{1 \wedge k\colon k \in K\}$ can have a supremum not in K. In this case $\mathfrak{a}(K)$ may contain K.

EXAMPLE 3.8. Let S be the semiring in Example 2.8. As was noted before, S is a totally ordered positive semiring and hence an l-semiring. Clearly $K = Q_2^+$, $\mathfrak{d}(K) = \{0_2\}$, $\mathfrak{a}(K) = Q_2^+ \cup \{0_1\}$ and $\sup\{1 \wedge k\colon k \in K\} = 0_1 \notin K$.

The following example shows that the analogue for T of Theorem 3.5 and Corollary 3.7 do not hold.

EXAMPLE 3.9. Let S be the set of all sequences $(x_0, x_1, x_2, \ldots)$ such that $x_0 \in Q^+$, $x_i \in I$ if $i > 0$, and $\{i\colon x_i \neq 0$ is finite$\}$ where I is the two element lattice, together with the sequence $(0, 0, 0, \ldots)$ under coordinatewise addition and multiplication. It is easily seen that S is an archimedian, semisimple, positive l-semiring. Now $T = \{(x_i)\colon x_0 = 0\}$, whence $\sup\{1 \wedge t\colon t \in T\} = (0, 1, 1, 1, \ldots) \in T$. But $\mathfrak{d}(T) = \mathfrak{a}(T) = K = \{(0)\}$, so $S \neq T \oplus \mathfrak{d}(T) = T \oplus \mathfrak{a}(T)$.

The more complicated Example 4.8 shows the same thing. In that example $T = \{[(a_i), (0)]\}$ and $\sup\{1 \wedge t\colon t \in T\} = [(1), (0)] \in T$. But $\mathfrak{d}(T) = \mathfrak{a}(T) = K = \{[(0), (a_i)]\}\colon (a_i) \in l_1^+\}$ so $S \neq T \oplus \mathfrak{d}(T) = T \oplus \mathfrak{a}(T)$.

Theorem 3.5 and Corollary 3.7 show that a condition on the a-cancellable elements yields a decomposition of the semiring. We will now show that under very reasonable hypothesis the a-idempotent elements completely determine the a-cancellable elements.

PROPOSITION 3.10. *If S is an archimedian semiring and $x \in T$, then $x + a = a$ for all $a \geqslant x$. Conversely, if $a + x = a$ for some $a \in S$ then $x \in T$.*

THEOREM 3.11. *If S is a positive archimedian l-semiring, then* $\mathfrak{d}(T) = K$.

Proof. Since S is a positive archimedian semiring, both T and K are l-ideals. Thus $\{t \wedge k\colon t \in T, k \in K\} \subset T \cap K = \{0\}$, so $\mathfrak{d}(T) \supset K$. Let $x \in \mathfrak{d}(T)$ and suppose $x + a = x + b$. Let z be such that $(a \wedge b) + z = a$. Now $x + (a \wedge b) = (x + a) \wedge (x + b) = x + a = x + (a \wedge b) + z$, whence by the previous proposition $z \in T$. But by Proposition 2.10 $u = z \wedge a = z \wedge (x + a) = z \wedge (x + b) = z \wedge b$ since $z \wedge x = 0$. Thus $z \leqslant b$, whence $z \leqslant a \wedge b$, so by the previous proposition $a \wedge b = (a \wedge b) + z = a$. Dually $a \wedge b = b$, so $a = b$, whence $x \in K$. Thus $\mathfrak{d}(T) = K$.

COROLLARY 3.12. *If S is a positive archimedian l-semiring and $x \notin K$, then there is an element $t_x \in T$ such that $x \geqslant t_x \neq 0$.*

COROLLARY 3.13. *If S is a positive archimedian l-semiring, then S is a direct sum of $\mathfrak{d}(K)$ and $\mathfrak{d}(T)$ if and only if $\{1 \wedge k\colon k \in K\}$ has a supremum in K.*

Proof. Theorem 3.5 and Theorem 3.11.

The following example shows that the analogue of Theorem 3.11 for K does not hold.

EXAMPLE 3.14. Let B be the Boolean algebra of finite and cofinite subsets of the integers. Let $S = \{(a, b)\colon a \in Q^+, b \in B$ and b finite implies $a = 0\}$. Now $K = \{(0, 0)\}$ since if $(a, b) \in K$, then b is the empty set, whence $a = 0$. But $T = \{(0, b)\colon b \in B\} \neq S$. Thus $\mathfrak{a}(K) = \mathfrak{d}(K) \neq T$.

4. a-idempotent and a-cancellable ideals.

In this section we will show that under reasonable hypothesis both the a-idempotent and a-cancellable ideals are the intersection of prime l-ideals and present a very important example.

PROPOSITION 4.1. *If S is an l-semiring, I is an l-ideal, and A is a multiplicative system disjoint from I then there is a prime l-ideal P disjoint from A with $I \subset P$.*

PROPOSITION 4.2. *If S is an l-semiring and I is an l-ideal of S, then I is an intersection of prime l-ideals if and only if $x^2 \in I$ implies $x \in I$.*

THEOREM 4.3. *If S is an archimedian positive l-semiring, then T is an intersection of prime l-ideals.*

Proof. By the previous proposition it is enough to show $x^2 \in T$ implies $x \in T$. Let $x^2 \in T$ and assume first that $x \leqslant 1$. Now there is an element $a \in S$ such that $x+a = 1$. Since T is an ideal, $x^n \in T$ for all $n \geqslant 2$. For $n \geqslant 4$, we have

$$1 = (a+x)^{n-2} = a^{n-2}+(n-2)a^{n-3}x+\sum_{i=2}^{n-2} a^{n-2-i}x^i,$$

so

$$ax = a^{n-1}x+(n-2)a^{n-2}x^2+\sum_{i=2}^{n-2} a^{n-1-i}x^{i+1} = a^{n-1}x+\sum_{i=2}^{n-1} a^{n-i}x^i.$$

Thus

$$nax = na^{n-1}x+\sum_{i=2}^{n-1} a^{n-1}x^i \leqslant a^n+na^{n-1}x+\sum_{i=2}^{n} a^{n-i}x^i = (a+x)^n = 1.$$

Hence $nax \leqslant 1$ for $n = 1, 2, \ldots$, so $ax \in T$ since S is archimedian. Thus $x = x(x+a) = x^2+ax \in T$. For the general case, note that $x^2 \in T$ if and only if $[x(1+x)^{-1}]^2 \in T$ and that $x(1+x)^{-1} \leqslant 1$.

COROLLARY 4.4. *Every nilpotent element of an archimedian positive l-semiring is a-idempotent.*

COROLLARY 4.5. *An a-cancellable positive l-semiring has no non-zero nilpotent elements.*

THEOREM 4.6. *If S is an archimedian positive l-semiring, then K is an intersection of prime l-ideals if and only if S has no non-zero nilpotent elements.*

Proof. Suppose that S has no non-zero nilpotent elements and let $x \in S$ with $x \notin K$. By Corollary 3.12, there is an element $t \in T$ such that $x \geqslant t \neq 0$. Thus, since t is not a-cancellable, $x^2 \geqslant t^2 \neq 0$. Hence by Proposition 3.10, $x^2+t^2 = x^2$, so $x^2 \notin K$. By Propositional 4.2, K is an intersection of prime l-ideals. Conversely, if $x^2 = 0$ and $x \notin K$, then by Proposition 4.2, K is not an intersection of prime l-ideals, so $x \in K$. But by Corollary 4.4, $x \in T$, so S has no non-zero nilpotent elements.

COROLLARY 4.7. *If S is a semisimple archimedian positive l-semiring, then K is an intersection of prime l-ideals.*

Proof. Let $x^2 = 0$. Clearly every maximal ideal is prime, so $x \in M$ for every maximal ideal M. But S is semisimple, so $x = 0$ and S has no non-zero nilpotent elements. Thus by the previous theorem, K is an ntersection of prime l-ideals.

The preceding theorems show that, under reasonable hypotheses, both T and K are intersections of prime l-ideals, and Theorem I. 3.12 shows that $T+K$ is an l-ideal. Unfortunately $T+K$ need not be an intersection of prime l-ideals as is shown by

EXAMPLE 4.8. The following is an example of a semisimple archimedian positive l-semiring in which $T+K$ is not an intersection of prime l-ideals.

Before presenting the example, it will be necessary to present several other semirings which will be used in the main example.

Let S_2 and I be as in Example 1.9 and let S' be the sub-semiring of $S_2 \oplus \bigtimes_{i=1}^{\infty} I$ in which $x_i = 1$ for all but finitely many i's.

It is clear that S' is a positive semiring.

If $a_i \leqslant b_i$ for all i, then $(a_i)+(b_i) = (b_i)$. Thus the natural order coincides with the coordinatewise order. Hence, since $(S', \leqslant)$ is a lattice and S_2 and I are l-semirings, S' is also an l-semiring.

S' is archimedian since it is a-idempotent.

Let $l_2^+ = \{(x_i)_{i>0}\colon x_i \in Q^+$ and $\sum_{i>0} x_1^2 < \infty\}$, let $l_1^+ = \{(x_i)_{i>0}\colon x_i \in Q^+$ and $\sum_{i>0} x_1 < \infty\}$, and let S^* be the sub-semiring of $\bigtimes_{i=1}^{\infty} Q^+$ in which $(a_i) = (c+x_i-z_i)$ where $c \in Q^+$ and $(x_i), (z_i) \in l_2^+$. Let $(\alpha_i) = (c+x_i-z_i)$, $(\beta_i) = (c'+x_i'-z_i')$, and (γ_i) be elements of S^*.

It is clear that S^* is a positive semiring and the natural order on S^* coincides with the coordinatewise order.

Thus S^* is an l-semiring since $(\alpha_i) \wedge (\beta_i) = (\alpha_i \wedge \beta_i) \in S^*$, and $(\alpha_i) \vee (\beta_i) = (\alpha_i \vee \beta_i) \in S^*$, $(\alpha_i)+[(\gamma_i) \wedge (\beta_i)] = [(\alpha_i)+(\gamma_i)] \wedge [(\alpha_i)+(\beta_i)]$ and $(\alpha_i)+[(\beta_i) \vee (\gamma_i)] = [(\alpha_i)+(\beta_i)] \vee [(\alpha_i)+(\gamma_i)]$ since these properties hold in each coordinate.

By Remark 2.11, S^* is archimedian since each coordinate is.

We are now ready to define a semisimple archimedian positive l-semiring in which $T+K$ is not an intersection of prime l-ideals.

Let $S = \{[(a_i), (\alpha_i)]\colon (a_i) \in S, (\alpha_i) \in S^*, (\alpha_i) \notin l_1^+$ implies $a_0 \geqslant x$, and $(\alpha_i) \notin l_2^+$ implies $a_0 = 1\}$ with coordinatewise operations. We will denote elements of S by the letters a, b, c where $a = [(a_i), (\alpha_i)]$, $b = [(b_i), (\beta_i)]$ and $c = [(c_i), (\gamma_i)]$.

S is a semiring since if $a, b \in S$, then $a+b = [(a_i+b_i), (\alpha_i+\beta_i)]$, and $a \cdot b = [(a_i b_i), (\alpha_i \beta_i)]$ are elements of S. For if $(\alpha_i+\beta_i) \notin l_1^+$, then $(\alpha_i) \notin l_1^+$ or $(\beta_i) \in l_1^+$, so $a_0 \geqslant x$ or $b_0 \geqslant x$, whence $a_0+b_0 \geqslant x$ and $a+b \in S$. If $(\alpha_i+\beta_i) \notin l_2^+$, then $(\alpha_i) \notin l_2^+$ or $(\beta_i) \notin l_2^+$, so $a_0 = 1$ or $b_0 = 1$, whence $a_0+b_0 = 1$ and $a+b \in S$. If $(\alpha_i\beta_i) \notin l_1^+$, then $(\alpha_i) \notin l_1^+$ and $(\beta_i) \notin l_2^+$ or $(\beta_i) \notin l_1^+$ and $(\alpha_i) \notin l_2^+$, since if $(\alpha_i), (\beta_i) \in l_2^+$ then $(\alpha_i\beta_i) \in l_1^+$. Thus $a_0 \geqslant x$ and $b_0 = 1$ or $b_0 \geqslant x$ and $a_0 = 1$, so $a_0 b_0 \geqslant x$, whence $a, b \in S$. If $(\alpha_i\beta_i) \notin l_2^+$ then $(\alpha_i) \notin l_2^+$ and $(\beta_i) \notin l_2^+$, so $a_0 = 1$ and $b_0 = 1$, whence $a_0 \cdot b_0 = 1$ and $a \cdot b \in S$.

The commutative, associative and distributive laws clearly hold in S, whence S is a positive semiring.

S is archimedian since both S' and S^* are.

S is semisimple. For by Proposition 1.8, it suffices to show that if $a \neq 0$, there is a non-unit s such that $a+s$ is a unit. Now if $a \neq 0$, then $(a_n) \neq (0)$ or $(\alpha_n) \neq 0$. If $(a_n) \neq 0$, then there is an n_0 with $a_{n_0} \neq 0$, so let $s = [(s_i), (1)]$ where $s_i = 1$, $i \neq n_0$ and $s_{n_0} = 0$. Now $a+s \geqslant [(1), (1)] \in \Omega$, whence $a+s \in \Omega$, but $s \notin \Omega$ since $s_{n_0} = 0$. If $(a_n) \neq 0$, then there is an n_0 with $a_{n_0} \neq 0$, so let $s = [(1), (\sigma_i)]$ where (σ_i) is defined by $\sigma_i = 1$, $i \neq n_0$ and $\sigma_{n_0} = 0$. Now $a+s \geqslant [(1)]$, $(a_{n_0} \wedge 1)] \in \Omega$, so $a+s \notin \Omega$, but $s \notin \Omega$ since $\sigma_{n_0} = 0$.

Clearly $T = \{[(a_i), (0)]\}$, $K = \{[(0), (a_i)]: (a_i) \in l_1^+\}$ and $T+K = \{[(a_i), (a_i)]: (a_i) \in l_1^+\}$. Let $s = [(1), (1/i)]$, $s \notin T+K$ since $(1/i) \notin l_1^+$, but $s^2 = [(1), (1/i^2)] \in T+K$, since $(1/i^2) \in l_1^+$. Hence by Proposition 4.2, $T+K$ is not the intersection of prime l-ideals. Thus S has all the desired properties.

In the next sections we will present a partial structure theory for a class of l-semirings. Birkhoff has proved that every algebra can be written as a subdirect sum of subdirectly irreducible algebras. With this in mind, we will examine the subdirectly irreducible semirings.

5. l-homomorphisms and l-congruence relations.

DEFINITION 5.1. If S and S' are l-semirings and $f\colon S \to S'$ is a homomorphism, then f will be called an *l-homomorphism* if $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$.

DEFINITION 5.2. A congruence relation θ of an l-semiring S will be called an *l-congruence* relation if $a \equiv b(\theta)$ implies $a \vee x \equiv b \vee x(\theta)$ and $a \wedge x \equiv b \wedge x(\theta)$ for all $x \in S$.

Remark 5.3. It is easily seen that the kernel of an l-homomorphism is an l-ideal.

LEMMA 5.4. *If I is an l-ideal of an l-semiring S and $a \equiv a+x(I)$, then $a \equiv b(I)$ for all $b \in S$ such that $a \leqslant b \leqslant a+x$.*

Proof. If $a \leqslant b \leqslant a+x$, then $b = a+y$. Now with no loss of generality we may assume $y \leqslant x$ since $b = a+y = (a+y) \wedge (a+x) = a+(x \wedge y)$ and $x \wedge y \leqslant x$. Now since $a \equiv a+x(I)$, there are elements $i_1, i_2 \in I$ such that $a+i_1 = a+x+i_2$. Thus $a+y+i_2 \leqslant a+x+i_2 = a+i_1$, whence $a+y+i_2 = (a+y+i_2) \wedge (a+i_1) = a+[(y+i_2) \wedge i_1]$ and $(y+i_2) \wedge i_1 \in I$ since I is an l-ideal. Hence $a+y \equiv a(I)$.

COROLLARY 5.5. *If I is a proper l-ideal of an l-semiring S, then S/I is a partially ordered semiring.*

PROPOSITION 5.6. *If I is an l-ideal of an l-semiring S, then $a \equiv b(I)$ if and only if $a \vee b \equiv a \wedge b(I)$.*

LEMMA 5.7. *If I is an l-ideal in an l-semiring S, then φ_I is an l-congruence relation.*

Proof. Let $a, b \in S$ with $a \equiv b(I)$ and let $x \in S$. By the previous proposition, $a \equiv b(I)$ if and only if $a \wedge b \equiv a \vee b(I)$, so we may assume $a \leqslant b$. Since $a \equiv b(I)$, there are elements $i_1, i_2 \in I$ such that $a+i_1 = b+i_2$. Now $a \vee x \leqslant b \vee x \leqslant (b+i_2) \vee (x+i_1) = (a+i_1) \vee (x+i_1) = (a \vee x)+i_1 \equiv a \vee x(I)$. Thus by Lemma 5.4, $a \vee x \equiv b \vee x(I)$. Dually $a \wedge x \equiv b \wedge x(I)$. Hence I is an l-congruence relation.

THEOREM 5.8. *If I is a proper l-ideal in an l-semiring, then S/I is an l-semiring and the natural homomorphism $f\colon S \to S/I$ is an l-homomorphism.*

Proof. Let $a, b \in S$ and for all $x \in S$ let $[x] = \{y\colon y \equiv x(I)\}$. By Corollary 1.5, S/I is partially ordered. Clearly $[a \vee b] \geqslant [a]$ and $[a \vee b] \geqslant [b]$. Suppose $x \in S$ and $[x] \geqslant [a]$, $[x] \geqslant [b]$, i.e., there are elements $y, z \in S$ with $[x] = [a]+[y]$, $[x] = [b]+[z]$. Now $x \equiv a+y(I)$ and $x \equiv b+z(I)$ so $x \equiv (a+y) \vee (b+z) \geqslant a \vee b(I)$, whence $[x] \geqslant [a \vee b]$. Thus $[a] \vee [b] = [a \vee b]$. Dually $[a \wedge b] = [a] \wedge [b]$. Let $a, b, c \in S$. Now $[a]+([b] \vee [c]) = [a+(b \vee c)] = [(a+b) \vee (a+c)] = [(a+b)] \vee [(a+c)] = ([a]+[b]) \vee ([a]+[c])$. Dually $[a]+([b] \wedge [c]) = ([a]+[b]) \wedge ([a]+[c])$, and hence S/I is an l-semiring and f is an l-homomorphism.

THEOREM 5.9. *If S is a subdirectly irreducible archimedian l-semiring, then S is a-idempotent or a-cancellable.*

Proof. For each $s \in S$, let $B[s] = \{t \in T\colon t \leqslant s\}$. Define $a \equiv b(\theta)$ if for all $N \geqslant 0$ and all sequences $\{(x_i, \alpha_i)\}_{i=1}^N$ such that $x_i \in S$ and α_i is one of the operations $+, \cdot, \vee, \wedge$, we have

$$B\Big[\Big(\ldots\big((a\alpha_1 x_1)\,\alpha_2 x_2\big)\ldots\Big)\alpha_N x_N\Big] = B\Big[\Big(\ldots\big((b\alpha_1 x_1)\,\alpha_2 x_2\big)\ldots\Big)\alpha_N x_N\Big].$$

It is easily seen that θ is an l-congruence relation since the above equality holds for all such finite sequences. Now let $a, b \in S$ and assume $a \equiv b(\theta)$ and $a \equiv b(\varphi_T)$. By Proposition 5.6, we may assume $a \geqslant b$, so there is an x with $a = b+x$. Now, since $a \equiv b(\varphi_T)$, there is an element t such that $a+t = b+t$, hence $b+t = a+x+t = b+x+t$, so $x \in T$ by Proposition II.1.10. Now $x \leqslant a$, whence $x \in B[a] = B[b]$, so $x \leqslant b$. Hence by Proposition 3.10, $a = b+x = b$. Thus $\theta \wedge \varphi_T = 0$. Now, since S is subdirectly irreducible and θ, T are congruence relations, by Proposition 2.13, either θ or φ_T is trivial. If $\varphi_T = 0$, then $T = \{0\}$, so by Theorem 3.11, $S = K$. Suppose that θ is the trivial congruence relation and let $\{(x_i, \alpha_i)\}_{i=1}^N$ be an arbitrary sequence of elements and operations. Now

$$\begin{aligned}
B\Big[\Big(\ldots\Big(\big((2\alpha_1 x_1)\,\alpha_2 x_2\big)\,\alpha_3 x_3\Big)\ldots\Big)\alpha_N x_N\Big] &\leqslant B\Big[\Big(\ldots\Big(\big(\{2\,(1\alpha_1 x_1)\}\,\alpha_2 x_2\big)\,\alpha_3 x_3\Big)\ldots\Big)\alpha_N x_N\Big]\\
&\leqslant B\Big[\Big(\ldots\big(2\,\{(1\alpha_1 x_1)\,\alpha_2 x_2\}\,\alpha_3 x_3\big)\ldots\Big)\alpha_N x_N\Big] \leqslant \ldots\\
&\leqslant B\Big[2\Big(\ldots\Big(\big((1\alpha_1 x_1)\,\alpha_2 x_2\big)\,\alpha_3 x_3\Big)\ldots\Big)\alpha_N x_N\Big]\\
&= B\Big[\Big(\ldots\Big(\big((1\alpha_1 x_1)\,\alpha_2 x_2\big)\,\alpha_3 x_3\Big)\ldots\Big)\alpha_N x_N\Big]\\
&\leqslant B\Big[\Big(\ldots\Big(\big((2\alpha_1 x_1)\,\alpha_2 x_2\big)\,\alpha_3 x_3\Big)\ldots\Big)\alpha_N x_N\Big].
\end{aligned}$$

Thus $2 \equiv 1(\theta)$, whence $2 = 1$, so S is a-idempotent.

The following example shows that a totally ordered subdirectly irreducible positive semiring need not be a-idempotent or a-cancellable.

EXAMPLE 5.10. Let S be the semiring in Example 2.8. As was previously noted, S is totally ordered and not archimedian. We will now show that S is subdirectly irreducible. Let θ be a non-trivial congruence relation. Now there are elements $a, b \in S$ with $a \neq b$ and $a \equiv b(\theta)$.

Case I. Suppose $a, b \in Q_1^+$ and $a \neq 0_1 \neq b$. Then $ab^{-1} \equiv 1_1 \equiv ba^{-1}(\theta)$ where $ab^{-1} > 1$, or $ba^{-1} > 1_1$. Assume $ab^{-1} > 1_1$. Now there is an integer n with $(ab^{-1})^n > 2_1$. Thus $1_1 \leqslant 2_1 \leqslant (ab^{-1})^n \equiv 1_1(\theta)$, so by Lemma 1.4, $1_1 \equiv 2_1(\theta)$. Thus $1_2 = 1_2 \cdot 1_1 \equiv 1_2 \cdot 2_1 \equiv 2_2(\theta)$.

Case II. Suppose $a, b \in Q_2^+$ and $a \neq 0_2 \neq b$. Thus $a_1^{-1}b_2 \equiv 1_2 \equiv b_1^{-1}a_2$ where $a_1^{-1}b_2 > 1_2$ or $a_2b_1^{-1} > 1_2$. Assume $a_1^{-1}b_2 > 1_2$. Now $a_1^{-1}b_2 = 1_2 + c_2$, so since $1_2, c_2 \in Q_2^+$, there is an integer n with $nc_2 > 1_2$. Thus $1_2 < 2_2 < 1_2 + nc_2 \equiv 1_2(\theta)$, so by Lemma 1.4, $1_2 \equiv 2_2(\theta)$.

Case III. Suppose $a \in Q_1^+$, $b \in Q_2^+$ and $a \neq 0_1$. Now $2 \cdot b_2 < a_1 < 2a_1 \equiv 2b_2(\theta)$, whence by Lemma 1.4, $a_1 \equiv 2a_1(\theta)$ so by Case I, $1_2 \equiv 2_2(\theta)$.

Case IV. Suppose $a = 0_1$ and $b \in Q_2^+$ such that $b \neq 0_2$. Now $b_2 < 2b_2 < 0_1 \equiv b_2(\theta)$, whence by Lemma 1.4, $b_2 \equiv 2b_2(\theta)$ so by Case II, $1_2 \equiv 2_2(\theta)$.

Case V. Suppose $a = 0_1$ and $b \in Q_1^+$ such that $b_1 \neq 0_1$. Now $b_1^2 \equiv 0 \equiv b_1(\theta)$, so by Case I, $1_2 \equiv 2_2(\theta)$.

Case VI. Suppose $a = 0_1$, $b = 0_2$. Now $0_2 < 1_2 < 2_2 < 0_1 \equiv 0_2(\theta)$, so by Lemma 5.4, $1_2 \equiv 2_2(\theta)$.

Case VII. Suppose $a \in Q_2^+$ and $b = 0_2$. Now $0_2 < a_2 < 2a_2 \equiv 0_2(\theta)$, hence by Lemma 5.4, $a_2 \equiv 2a_2(\theta)$, so by Case II, $1_2 \equiv 2_2(\theta)$.

Thus for any non-trivial congruence relation θ, $1_2 \equiv 2_2(\theta)$, so by Remark 2.17, S is subdirectly irreducible.

DEFINITION 5.11. An l-semiring S will be called an *f-semiring* if $a \wedge b = 0$ implies $a \wedge bc = 0$ for all $c \in S$.

THEOREM 5.12. *If S is a positive l-semiring, then S is an f-semiring.*

Proof. Let $a, b \in S$ such that $a \wedge b = 0$ and let $c \in S$. Now $a \wedge bc = (1+c)\{[(1+c)^{-1}a] \wedge [bc(1+c)^{-1}] \leqslant (1+c)(a \wedge b)\} = 0$ since $(1+c)^{-1} \leqslant 1$ and $c(1+c)^{-1} \leqslant 1$. Thus $a \wedge bc = 0$, so S is an f-semiring.

A ring A will be called a *partially ordered ring* if A is partially ordered and $a \geqslant b$ implies $a+c \geqslant b+c$ for each $c \in A$ and $a \geqslant 0$, $b \geqslant 0$ implies $ab \geqslant 0$. If A is lattice ordered, then it will be called a *lattice ordered ring*. If A is totally ordered, then it will be called an *ordered ring*. A lattice ordered ring A will be called an *f-ring* if $a \wedge b = 0$ implies $a \wedge bc = a \wedge cb$ for all $c \geqslant 0$.

In [5], Theorem 2, p. 106, Fuchs has shown that any a-cancellable l-semiring S can be embedded in a partially ordered ring A consisting of all formal differences $x - y$ where $x, y \in S$ such that $S = \{x \in A: x \geqslant 0\}$. Now since $|a-b| = (a \vee b) - (a \wedge b) \in S$ for all $a, b \in A$, by a theorem of Birkhoff's ([3], Theorem 2, p. 215), A is a lattice ordered ring. Moreover, since S is positive, by the previous theorem, A is an f-ring. Thus we have proved

THEOREM 5.13. *If S is an a-cancellable positive l-semiring, then S can be embedded in an f-ring A such that $S = \{x \in A: x \geqslant 0\}$.*

In [4], p. 56, Birkhoff and Pierce have proved

PROPOSITION 5.14. *Any f-ring can be written as a subdirect sum of subdirectly irreducible f-rings and every subdirectly irreducible f-ring is an ordered ring.*

THEOREM 5.15. *If S is a subdirectly irreducible a-cancellable positive l-semiring, then S is totally ordered. Moreover if S is archimedian, then S is a subsemiring of the non-negative real numbers.*

Proof. By Theorem 5.13, S can be embedded in an f-ring A such that $S = \{x \in A: x \geqslant 0\}$. Now by Proposition 5.14, there is a family of subdirectly irreducible, hence totally ordered, f-rings $\{A_\alpha: \alpha \in \Gamma\}$ such that $A = \sum_s \{A_\alpha: \alpha \in \Gamma\}$. For each α let $A_\alpha^+ = \{x \in A_\alpha: x \geqslant 0\}$. Clearly A_α^+ is an a-cancellable positive l-semiring and $S = \{x \in A: x \geqslant 0\} = \sum_s \{A_\alpha^+: \alpha \in \Gamma\}$. But S is subdirectly irreducible, so there is an $\alpha_0 \in \Gamma$ such that S is isomorphic to $A_{\alpha_0}^+$. Thus S is totally ordered. Now if S is archimedian, then A is an ordered archimedian ring so A is a subring of the real numbers. Thus $S = \{x \in A: x \geqslant 0\}$ is a subsemiring of the non-negative real numbers.

THEOREM 5.16. *Let S be a positive l-semiring. If S has a representation as a subdirect sum of subdirectly irreducible archimedian l-semirings, then S is a subdirect sum of a-idempotent l-semirings and subsemirings of the non-negative real numbers.*

Proof. Proposition 5.14, Theorem 5.9, and Theorem 5.15.

It is easily seen that not every positive archimedian l-semiring has a representation as a subdirect sum of subdirectly irreducible archimedian l-semirings; e.g., the semiring of non-negative Lebesgue measurable function modulo the null functions. In ring theory, every such ring is a subdirect sum of subdirectly irreducible ordered rings. This leads us to state the following questions:

(i) Is every subdirectly irreducible a-idempotent positive l-semiring totally ordered?

(ii) Is every positive archimedian l-semiring the subdirect sum of totally ordered semirings?

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Regular iteration of functions with multiplier 1

by

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Let $f(x)$ be a function which is defined, continuous and strictly increasing in an interval (a, b), $a \geqslant -\infty$, $b \leqslant +\infty$. Furthermore

$$a < f(x) < x \quad \text{for} \quad x \in (a, b) .$$

Definition 1. A one-parametr family of functions $f^u(x)$, $u \in (-\infty, +\infty)$, is called an *iteration group* of the function $f(x)$ provided that the following conditions are fulfilled (see [6], [4]):

(I) *for every* $u \in (-\infty, +\infty)$ *the function* $f^u(x)$ *is defined, continuous and strictly increasing in an interval* (a, b_u), *where* $a < b_u \leqslant b$;

(II) *for every pair of* $u, v \in (-\infty, +\infty)$

$$f^u[f^v(x)] = f^{u+v}(x)$$

holds for every x *for which both sides are meaningful*;

(III) $f^1(x) = f(x)$ *for* $x \in (a, b)$;

(IV) *for every fixed* $x \in (a, b)$, $f^u(x)$ *is a continuous function of* u.

It follows from conditions (II) and (III) that, for integral u, $f^u(x)$ are identical with the natural iterates $f^n(x)$ of the function $f(x)$ defined by

$$f^0(x) = x, \quad f^{n+1}(x) = f[f^n(x)], \quad n = 0, \pm 1, \pm 2, \ldots$$

We note also that it follows from (II) that for every fixed iteration group and for every u and v the functions $f^u(x)$ and $f^v(x)$ are commutative:

$$f^u[f^v(x)] = f^v[f^u(x)] . \tag{1}$$

It is known (see [1], [2], [9]) that every iteration group $f^u(x)$ of the function $f(x)$ is given by the formula

$$f^u(x) = a^{-1}[a(x)+u] , \tag{2}$$

where $a(x)$ is a continuous and strictly monotonic solution of the Abel equation

$$a[f(x)] = a(x)+1 , \tag{3}$$